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BIRTH OF THE SHAPE OF A REACHABLE SET

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Abstract
We address a linear control system under geometric constraints on control and study its reachable sets starting at zero time from the origin. The main result is the existence of a limit shape of the reachable sets as the terminal time tends to zero. Here, a shape of a set stands for the set regarded up to an invertible linear transformation. Both autonomous and non-autonomous cases are considered.

Key words
Linear control dynamic system, reachable sets.

1 Problem Statement

Consider the following linear control system

$$\dot{x} = Ax + Bu,\ x \in \mathbb{V} = \mathbf{R}^n,\ u \in U \subset \mathbb{U} = \mathbf{R}^m, \tag{1}$$

where the set U is a central symmetric convex body in $\mathbf{R}^m$, i.e., U is assumed to be a convex compact set with the non-empty interior, and $U = -U$. For any $T > 0$ we get the reachable set $\mathcal{D}(T)$ being the set of the ends $\{x(T)\}$ at time T of all admissible trajectories of system (1) such that $x(0) = 0$.

We study the asymptotic behavior of the reachable sets $\mathcal{D}(T)$ as $T \to 0$. The problem is not as silly as it may seem. Of course, $\mathcal{D}(T)$ is small with a small T. Still we can look at it through a kind of microscope and see fine details of its development, in particular, its shape. For instance, consider a rather trivial system (1), where $A = 0$, $B = 1$, and U is the unit ball B_1 in $\mathbf{R}^n = \mathbf{R}^m$. Then $\mathcal{D}(T) = TB_1$, and the reachable set has the shape of a ball at any time $T > 0$.

There is a well developed mathematical concept of shapes [Ovseevich, 1990]. Informally speaking, a shape of a set is the set regarded up to an invertible linear transformation. We will cite some further details in Section 2. In our example for all $T > 0$ the sets $\mathcal{D}(T)$ have the same shape, so that there exists a limit of the shapes as $T \to 0$, and this limit is not zero, but the shape of a ball.

In this paper we will show that, for any completely controllable linear system, shapes of the reachable sets converge as time tends to zero. We will give an estimate for the rate of convergence. The paper can be regarded as the $T \to 0$ counterpart of the results [Ovseevich, 1990] on the large-time dynamics of the reachable sets, where the existence of a limit shape has been established.

This study of the fine structure of reachable sets $\mathcal{D}(T)$ as $T \to 0$ is interesting not only by itself. Another face of the problem is the study of minimum-time control bringing a given state vector to the origin, e.g. the optimal damping of a pendulum. The problem also arises naturally in any attempt to devise a bounded feedback control steering a linear control system to the equilibrium in finite time [Ananievskii et al., 2010].

2 Shapes of Reachable Sets

Consider the metric space $\mathbb{B}$ of central symmetric convex bodies with the Banach-Mazur distance ρ:

$$\begin{aligned}&\rho(\Omega_1, \Omega_2) = \log(t(\Omega_1, \Omega_2)t(\Omega_2, \Omega_1)),\ \text{where}\\ &t(\Omega_1, \Omega_2) = \inf\{t \geq 1 : t\Omega_1 \supset \Omega_2\}.\end{aligned} \tag{2}$$

The general linear group $\mathrm{GL}(\mathbb{V})$ naturally acts on the space $\mathbb{B}$ by isometries. The factorspace $\mathbb{S}$ is called the space of shapes of central symmetric convex bodies, where the shape $\mathrm{Sh}\,\Omega \in \mathbb{S}$ of a convex body $\Omega \in \mathbb{B}$ is the orbit $\mathrm{Sh}\,\Omega = \{g\Omega : \det g \neq 0\}$ of the point Ω with respect to the action of $\mathrm{GL}(\mathbb{V})$. The Banach-Mazur factormetric

$$\rho(\mathrm{Sh}\,\Omega_1, \mathrm{Sh}\,\Omega_2) = \inf_{g \in GL(\mathbb{V})} \rho(g\Omega_1, \Omega_2)$$

makes $\mathbb{S}$ into a compact metric space. In what follows, the convergence of the reachable sets $\mathcal{D}(T)$ and their shapes is understood in the sense of the Banach-Mazur metric. The convergence of convex bodies may be also understood in the sense of convergence of their support functions. Remind that the support function of a convex compact set Ω is given by formula $H_\Omega(\xi) = \sup_{x\in\Omega}\langle x,\xi\rangle$, where $\xi \in \mathbb{V}^*$, and uniquely defines the set Ω. The equivalence of the two definitions of convergence of convex bodies is established by the following easy lemma [Figurina and Ovseevich, 1999]:

Lemma 1. *A sequence $\Omega_i \in \mathbb{B}$ converges to $\Omega \in \mathbb{B}$ in the sense of the Banach-Mazur metric if and only if the corresponding sequence of the support functions $H_{\Omega_i}(\xi)$ converges to the support function $H_\Omega(\xi)$ pointwise and is uniformly bounded on the unit sphere in the dual space $\mathbb{V}^*$.*

If $\rho(\Omega_1(T),\Omega_2(T)) \to 0$ as $T \to 0$, we say that the convex bodies Ω_1 and Ω_2 are asymptotically equal, and we write $\Omega_1(T) \sim \Omega_2(T)$. The asymptotic equivalence of shapes is defined similarly.

3 Main Result: Autonomous Case

We assume that system (1) is time-invariant and the Kalman controllability condition holds. The Kalman condition ensures that the reachable sets $\mathcal{D}(T)$ to system (1) are central symmetric convex bodies in $\mathbf{R}^n$.

Theorem 1. *The shapes $\mathrm{Sh}\,\mathcal{D}(T)$ have a limit Sh_0 as $T \to 0$. The Banach–Mazur distance $\rho(\mathrm{Sh}\,\mathcal{D}(T),\mathrm{Sh}_0)$ is $O(T)$.*

This means that there exists a time independent convex body Ω such that

$$\mathcal{D}(T) \sim C(T)\Omega,$$

where $C(T)$ is a matrix function. The Banach-Mazur distance between the left- and the right-hand sides of the latter formula is $O(T)$.

Note that the initial reachable set $\mathcal{D}(0) = \{0\}$ does not belong to the space $\mathbb{B}$ of symmetric convex bodies. The Banach-Mazur distance between $\mathrm{Sh}\,\mathcal{D}(T)$ and $\mathrm{Sh}\,\mathcal{D}(0)$ equals infinity.

Proof of Theorem 1 is based on two easy lemmas, and the use of the Brunovsky normal form of a controllable system.

Lemma 2. *Consider the linear system*

$$\dot x = \widetilde{A}x + \widetilde{B}u,\ u \in U, \tag{3}$$

obtained from system (1) *by adding a linear feedback, that is, $\widetilde{A} = A + BC$ and $\widetilde{B} = B$. Then the Banach–Mazur distance $\rho(\mathcal{D}(T),\widetilde{\mathcal{D}}(T))$ is $O(T)$ as $T \to 0$, where $\mathcal{D}(T)$ and $\widetilde{\mathcal{D}}(T)$ are the reachable sets to systems* (1) *and* (3)*, respectively.*

Lemma 3. *Consider the linear system*

$$\dot x = \widetilde{A}x + \widetilde{B}u,\ u \in U, \tag{4}$$

obtained from system (1) *by a gauge transformation, where $\widetilde{A} = C^{-1}AC$, $\widetilde{B} = C^{-1}B$, and C is an invertible matrix. Then $\mathrm{Sh}\,\widetilde{\mathcal{D}}(T) = \mathrm{Sh}\,\mathcal{D}(T)$, where $\widetilde{\mathcal{D}}(T)$ is the reachable set to system* (4)*.*

Lemma 3 is obvious. We postpone proving Lemma 2 for a moment, and conclude that applying gauge transformations coupled with adding a linear feedback do not affect the validity of Theorem 1. However, by these transformations one can reduce the general system (1) to the Brunovsky normal form [Brunovsky, 1970], where the matrices A and B are the direct sums $A = \oplus A_i$, $B = \oplus B_i$, and the matrices A_i and B_i of sizes $n_i \times n_i$ and $n_i \times 1$, respectively, take the form

$$A_i = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}, \quad B_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}. \tag{5}$$

Note that the Brunovsky classification is known to be closely related to the Grothendieck theorem [Grothendieck, 1957] on decomposition of vector bundles on $\mathbf{P}^1$ into a sum of line bundles.

We can relate to the Brunovsky system (1), (5) a distinguished matrix function $\delta = \oplus\,\delta_i$, where

$$\delta_i(T) = \mathrm{diag}(T^{-n_i}, T^{-n_i+1}, \ldots, T^{-1}) \tag{6}$$

such that

$$\delta A \delta^{-1} = T^{-1}A, \quad \delta B = T^{-1}B. \tag{7}$$

This immediately implies that for T fixed and $y = \delta x$, we have

$$\dot y = T^{-1}\left(Ay + Bu\right). \tag{8}$$

Equation (8) reveals the geometric meaning of the matrix $\delta(T)$: The corresponding gauge transformation is equivalent to the passage to the new time scale $t \mapsto t/T$ in (1), (5). Since the gauge transformations do not change shapes of the reachable sets (Lemma 3), we conclude that the shapes $\mathrm{Sh}\,\mathcal{D}(T)$ of the reachable sets to the Brunovsky system do not depend on T, and we are done.

It remains to prove Lemma 2. Consider a trajectory $t \mapsto x(t)$ of system (1), and the corresponding trajectory $\widetilde{x}(t)$ of (3). We have

$$\begin{aligned} \dot x(t) &= Ax(t) + Bu(t), \\ \dot{\widetilde{x}}(t) &= A\widetilde{x}(t) + B(u(t) + C\widetilde{x}(t)) \end{aligned}$$

It is clear that $C\widetilde{x}(t) = O(t)$, and therefore for all $t \le T$ the control vector $\widetilde{u}(t) = u(t) + C\widetilde{x}(t)$ belongs to the set $(1+\varepsilon)U$, where $\varepsilon = O(T)$. This means that $\widetilde{\mathcal{D}}(T) \subset (1+\varepsilon)\mathcal{D}(T)$, where $\varepsilon = O(T)$. Since the relation between systems (1) and (3) is symmetric, we similarly have that $\mathcal{D}(T) \subset (1+\varepsilon)\widetilde{\mathcal{D}}(T)$. But this implies Lemma 2 in view of the definition of the Banach–Mazur distance (2).

4 Non-autonomous Case

In fact, the same phenomenon of the existence of a limit shape takes place in the non-autonomous case. We should just assume a kind of a genericity condition generalizing the Kalman one we operated with in the time-invariant case.

We study system (1), where now the data A, B, and U are C^∞-functions of time $t \geq 0$. First, by a standard trick we make (1) into the time-invariant system

$$\dot{\tau} = 1, \tag{9}$$
$$\dot{x} = A(\tau)x + B(\tau)u. \tag{10}$$

Second, consider the Lie algebra $\mathcal{L}$ generated by the vector fields $(1, A(\tau)x)$ and $(0, B(\tau)u)$ in $\mathbf{R} \times \mathbb{V} = \mathbf{R}^{n+1}$, where $u \in \mathbf{R}^m$ is a constant vector. Define $\mathcal{L}(\tau, x)$ as the set of the values at (τ, x) of all vector fields from $\mathcal{L}$.

We use the following Kalman type condition as a standing assumption:

> For each $(\tau, x) \in \mathbf{R} \times \mathbb{V}$ the set $\mathcal{L}(\tau, x)$ coincides with the entire tangent space $\mathbf{R}^{n+1}$. In other words,
>
> $$\dim \mathcal{L}(\tau, x) = n+1. \tag{11}$$

It is well known that in the time-invariant case this assumption coincides with the Kalman controllability condition.

Theorem 2. *Let $\mathcal{D}(T)$ be the reachable set to a non-autonomous system of the form* (1) *and the genericity condition* (11) *hold. The shapes $\operatorname{Sh}\mathcal{D}(T)$ have a limit Sh_0 as $T \to 0$. Moreover, the Banach–Mazur distance $\rho(\operatorname{Sh}\mathcal{D}(T), \operatorname{Sh}_0)$ is $O(T)$.*

Proof. One can easily reduce system (1) to the case $A = 0$. Indeed, make a gauge transformation $x = Cy$, where $\dot{C} = C^{-1}AC$, $C(0) = 1$. The Cauchy problem is, at least locally in time, solvable. Then, $\dot{y} = C^{-1}Bu$, and the shapes of the reachable sets to the new system are the same as those of the old one.

For the new system $\dot{y} = \widetilde{B}u$ the condition (11) takes the form: For any (constant) vector $\xi \in \mathbb{V}^* = \mathbf{R}^n$ the function $t \mapsto \widetilde{B}^*\xi$ is not flat at any time instant τ, i.e., a higher derivative does not vanish at τ.

We associate a flag in $\mathbb{V}^*$ to the matrix function $\widetilde{B}$. With an integer $k \geq 0$ we associate the set

$$F_k^* = \{\xi : \widetilde{B}^*(t)\xi = O(t^k)\}$$

of vectors $\xi \in \mathbb{V}^* = \mathbf{R}^n$. It is obvious that F_k^* form a decreasing sequence of subspaces of $\mathbb{V}^*$ such that $F_0^* = \mathbb{V}^*$ and $F_\infty^* = 0$. The latter equality is a restatement of the nonflatness condition (11).

Consider the graded space $\operatorname{Gr}\mathbb{V}^* = \oplus_{k=0}^\infty F_k^*/F_{k+1}^*$, and choose an isomorphism

$$\phi : \mathbb{V}^* \simeq \operatorname{Gr}\mathbb{V}^*$$

such that for any j the subspace F_j^* maps to $\oplus_{k=j}^\infty F_k^*/F_{k+1}^*$ in such a way that the induced map $F_j^*/F_{j+1}^* \to F_j^*/F_{j+1}^*$ is identical. In other words, for any $\xi \in \mathbb{V}^*$ we have a unique representation of the form $\xi = \sum_{i \in I}\xi_i$, where ξ_i belongs to the subspace

$$V_{k(i)} = \phi^{-1}\left(F_{k(i)}^*/F_{k(i)+1}^*\right)$$

in $F_{k(i)}^*$. Here, the set I of indices can be identified with the set $\{k(i) : i \in I\}$ of jumps in the filtration F^*, i.e., the values of k such that $F_k^* \neq F_{k+1}^*$.

We define a generalization $\Delta^*(T)$ of the transposed matrix $\delta(T)$ from (6) as follows:

$$\Delta^*(T)\xi = \sum_{\in I} T^{-k(i)-1}\xi_i.$$

In other words, $\Delta^*(T)$ is equal to $T^{-k(i)-1}$ on $V_{k(i)}$. If $\xi_i \neq 0$, then $\xi_i \in F_{k(i)}^* \setminus F_{k(i)+1}^*$, and we have that $\widetilde{B}^*(t)\xi_i = t^{k(i)}\eta_i(t)$, where $\eta_i(t)$ is C^∞-smooth in t, and $\eta_i(0) \neq 0$. Thus, we have the C^∞-smooth matrix function $\widetilde{B}_i^*(t)\xi = \eta_i(t)$, and the decomposition $\widetilde{B}^*(t) = \sum t^{k(i)}\widetilde{B}_i^*(t)$. Therefore,

$$\widetilde{B}^*(t)\Delta^*(T) = \frac{1}{T}\sum\left(\frac{t}{T}\right)^{k(i)}\widetilde{B}_i^*(t).$$

Since each $\widetilde{B}_i^*(t) = \widetilde{B}_i^*(0) + O(t)$, the left-hand side coincides with $\frac{1}{T}\widetilde{B}^*\left(\frac{t}{T}\right) + O\left(\frac{t}{T}\right)$ so that

$$\widetilde{B}^*(t)\Delta^*(T) = \frac{1}{T}\widetilde{B}^*\left(\frac{t}{T}\right) + O\left(\frac{t}{T}\right). \tag{12}$$

The support function of the reachable set $\mathcal{D}(T)$ has the form $H_{\mathcal{D}(T)}(\xi) = \int_0^T h_t(\widetilde{B}^*(t)\xi)$, where h_t is the support function of the set U_t of controls at time t. Thus, the support function of the normalized reachable set $\widetilde{\mathcal{D}}(T) = \Delta(T)\mathcal{D}(T)$ is given by

$$H_{\widetilde{\mathcal{D}}(T)}(\xi) = \int_0^T h_t(\widetilde{B}^*(t)\Delta^*(T)\xi)dt,$$

where $\Delta(T)$ is by definition the adjoint of the already introduced operator $\Delta^*(T)$. Because of (12) the integral can be rewritten as

$$H_{\mathcal{D}(T)}(\xi) = \int_0^1 h_{\tau T}(\widetilde{B}^*(\tau)\xi + O(\tau T))d\tau,$$

where $\tau = t/T$. The latter integral equals $\int_0^1 h_0(\widetilde{B}^*(\tau)\xi))d\tau + O(T)$, and surely converges as $T \to 0$ to $\int_0^1 h_0(\widetilde{B}^*(\tau)\xi))d\tau$. If $\xi \neq 0$ the function $\tau \mapsto \widetilde{B}^*(\tau)\xi$ does not vanish identically in any open interval. Thus, the latter integral is positive, what means that it defines the support function of a convex body Ω,

$$H_\Omega(\xi) = \int_0^1 h_0(\widetilde{B}^*(\tau)\xi))d\tau,$$

and we conclude by invoking Lemma 1 that the shapes $\operatorname{Sh}\mathcal{D}(T) = \operatorname{Sh}\widetilde{\mathcal{D}}(T)$ tend to $\operatorname{Sh}_0 = \operatorname{Sh}\Omega$ as $T \to 0$.

5 Conclusion

In the paper we obtained a refined picture of evolution of reachable sets of a linear control system at the vicinity of the initial time. It is shown that there exists a limit shape of the reachable set as the time of motion tends to zero. These results provide a starting point for design of a bounded feedback control bringing the system to equilibrium in a finite time.

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