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ONSET CONDITION AND TRANSIENT BEHAVIORS OF NOISE-INDUCED BIFURCATION

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Abstract

To efficiently determine critical condition of noise-induced bifurcation in nonlinear dynamical systems, a stochastic sensitivity function (SSF) around a deterministic periodic attractor is approximated based on stroboscopic mapping. Then the confidence ellipses are constructed and used to judge if it collides with some invariant sets of system in order to obtain the critical noise intensity of noise-induced transition phenomena. Furthermore, to effectively capture the larger stochastic transient behaviors over the critical noise intensity, an idea of evolving probabilistic vector (EPV) is introduced into the Generalized Cell Mapping method (GCM) in order to enhance the computation efficiency of the numerical method. The feasibility of the proposed methods is demonstrated through the study of a Duffing oscillator under external periodic excitation and additive noise.

Key words

Stochastic sensitivity function; Stroboscopic map, Generalized cell mapping; Evolving probabilistic vector; Noise-induced transition.

1 Introduction

Noise is ubiquitous in nature and engineering systems that are all inherently nonlinear. Uncertain disturbances or noises on nonlinear dynamical systems often evoke some unexpected and even coherent responses. Various noise-induced behaviors have been found, such as noise-induced chaos [Zhang, et al., 2011; Tél and Lai, 2010], stochastic bifurcation [Malick and Marcq, 2003; Xu, He and Fang, 2003], noise-induced intermittency [Suso and Ulrike, 2003; Bashkirtseva and Ryashko, 2013], noise-induced hopping [Arecchi, Badii and Politi, 1985; Suso and Ulrike, 2002] and so on.

In [Thompson, Stewart and Udea, 1994], the bifurcations of deterministic (dissipative) nonlinear dynamical systems are classified into three categories: safe, explosive and dangerous. The explosive bifurcations are defined as catastrophic global bifurcations with an abrupt enlargement of the attracting set but with no jump to remote disconnected attractor. The dangerous bifurcations are catastrophic bifurcations with blue-sky disappearance of the attractor with a sudden fast jump to a distant unrelated attractor. Undoubtedly, these two kinds of bifurcations have very important engineering meaning since they imply that abrupt and great change in the operation state of a machine or a system takes place with a continuous variation of a parameter that may even induce possible damage or destruction of the system. Since uncertain disturbance is usually unavoidable in real engineering environment, it is thus of great interest to exploit the condition when the bifurcations are induced by noise and the quantitative prediction that captures the transient responses of noise-induced large transition. Thees are the two purposes of the present paper.

It is well known that Monte-Carlo simulation (MCS) is a direct method to obtain the probabilistic distribution of a stochastic system, but it is too expensive in computations to be used for a systematic investigation. For the case of excitation under Gaussian white noise, the probabilistic description of the stochastic responses is governed by Fokker-Planck-Kolmogorov (FPK). Several approximate methods on solving FPK equation have been developed, including Finite Element Method [Spencer and Bergman, 1993], exponential-polynomial closure method [Zhu, 2012], stochastic averaging procedure [Gu and Zhu, 2014], path integral method [Wehner and Wolfer,1983; Di Paola and Santoro, 2009], etc.

Based on the quasipotential theory, the stochastic sensitivity function, proposed by Bashkirtseva, can give an approximate analytical description of the probabilistic distribution. This method is easier than other FPK equation-based methods and has successfully applied to analysis the sensitivity of stationary point, 2D cycle, 3D cycle in differential dynamical systems. For discrete systems, the sensitivity of fixed point and periodic solution can also be analyzed using SSF [Bashkirtseva, Ryashko and Tsvetkov, 2010]. Moreover, SSF can help to stabilize the equilibrium in noise disturbed chaotic system [Bashkirtseva, Chen and Ryashko, 2012]. In this paper, a non-autonomous dynamical system is discretized into a discrete map by 1/*N*-period stroboscopic map. Through solving stochastic sensitivity functions of periodic attractors in maps, confidence ellipses were constructed to describe the distributions of the random attractors. In this way, boundary value problems of matrix differential equations were avoided, while there were and only matrix algebra equations need to be solved. Thus, probabilistic distribution of periodic attractors can be analytically predicted when the explosive bifurcation occurs.

Generalized Cell Mapping method, which was pioneered by Hsu [Hsu, 1981 and 1987] in 1980s, may effectively deal with the global analysis of stochastic dynamic systems [Sun and Hsu, 1988] and capture noise-induced probability evolution for a invariant sets to another, especially for the dangerous bifurcations. But, like many other numerical methods for stochastic dynamics, the computational efficiency is still a crucial problem faced by GCM that needs to be solved with effort. In the study, we are interest in the probability distribution of the initial states localizes near a given deterministic attracting set. Thus, the traditional GCM that always deals with a priori defined sufficient large chosen region in the state space is not quite efficient for the analysis of the above problem. Therefore, the idea of evolving probabilistic vector (EPV) is introduced in this paper in order to enhance the efficiency of the GCM. By using EPV, only the one-step transition probability of the cells in the chosen region, whose probabilities are within a given fiducial probability, will be calculated, instead of all the cells within the chosen region in the state space. In this way, the dimension of the probabilistic vector in the present GCM method (GCM with EPV), which varies with the evolution of the stochastic response, is greatly reduced and usually much smaller than that of the corresponding fix-sized probabilistic vector.

This paper is organized as follows: In Section 2, the algorithm to obtain SSF of periodic attractors in non-autonomous nonlinear system is proposed by constructing 1/N-period stroboscopic map. Section 3, the idea of evolving probabilistic vector is proposed in order to enhance the efficiency of the GCM, and the corresponding algorithm is devised. In Section 4, the proposed methods are applied to a Duffing system under external periodic excitation and additive noise. Finally, conclusions are drawn in Section 5.

2 Stochastic sensitivity function of periodic attractors

Consider a continuous non-autonomous dynamical system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}\left(\boldsymbol{x}, t\right) \tag{1}$$

When there is a periodic attractor with period T in the deterministic system (1), stroboscopic map at discrete times $t=t_0+k\Delta t$ (k is positive integer) is often used to investigate the character of the attractor, which can be defined as

$$\boldsymbol{x}_{k+1} = \varphi_{\Delta t}\left(\boldsymbol{x}_k\right) \tag{2}$$

However, though the algorithm to get SSF of fixed point of maps is raised in [Roy, 1995], for most of the nonlinear dynamical systems, the explicit expression of the 1-period stroboscopic map cannot be obtained. Note that, if $\Delta t \to 0$, the linear approximation of map (2) can be taken in the interval $[t_0+k\Delta t,\ t_0+(k+1)\Delta t]$

$$\boldsymbol{x}_{k+1} = \exp\left(\boldsymbol{J}_k \Delta t\right)\boldsymbol{x}_k \tag{3}$$

where

$$\boldsymbol{J}_k = \partial \boldsymbol{f} / \partial \boldsymbol{x}\,|_{\boldsymbol{x}=\boldsymbol{x}_k,\, t=t_0+k\Delta t}$$

is Jacobian matrix at point x_k and time $t_0+k\Delta t$.

So, the sampling time interval Δt of stroboscopic map can be set to

$$\Delta t = T/N, \quad N \gg 1 \tag{4}$$

and a new stroboscopic map can be written in the form (3). This new map is named a 1/N-period stroboscopic map.

Through this new map, the original periodic attractor Γ is discretized into a period-N cycle $\Gamma^*=\{x_1, \ldots, x_N\}$ by N stroboscopic sections $\{\Sigma_1, \ldots, \Sigma_N\}$. (see Fig. 1)

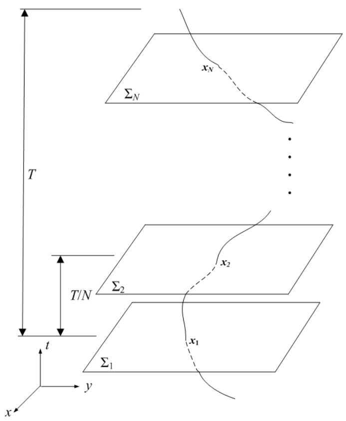


Fig. 1 1/N-period stroboscopic map of a 2-dimensional non-autonomous system

Now consider system (1) subjects to stochastic disturbance

$$\dot{\boldsymbol{x}} = \boldsymbol{f}\left(\boldsymbol{x}, t\right) + \varepsilon\boldsymbol{\sigma}\left(\boldsymbol{x}\right)\boldsymbol{\xi}\left(t\right) \tag{5}$$

where ξ is n-dimensional Gaussian white noise, σ is $n\times n$ matrix which defines the relation between the noise and the system state, ε is the noise intensity.

The 1/N-period stroboscopic map of system is written as

$$\boldsymbol{x}_{k+1} = \exp\left(\boldsymbol{J}_k \Delta t\right)\boldsymbol{x}_k + \varepsilon\boldsymbol{\sigma}\left(\boldsymbol{x}_k\right)\Delta\boldsymbol{w} \tag{6}$$

where

$$\Delta\boldsymbol{w} = \sqrt{\Delta t}\,\boldsymbol{\xi}$$

is an increment of Wiener process during time interval $[t_0+k\Delta t,\ t_0+(k+1)\ \Delta t]$.

According to [[Bashkirtseva, Ryashko and Tsvetkov, 2010], if the deterministic period-N cycle in (3) is exponentially stable in its neighborhood, one can define

$$\begin{aligned}
&\boldsymbol{S}_k = \boldsymbol{\sigma}\left(\boldsymbol{x}_k\right),\ \boldsymbol{Q}_k = \boldsymbol{S}_k\boldsymbol{S}_k^T,\ \boldsymbol{F}_k = \exp\left(\boldsymbol{J}_k\Delta t\right) \qquad k = 1, \ldots, N \\
&\boldsymbol{B} = \boldsymbol{F}_N\boldsymbol{F}_{N-1}\cdots\boldsymbol{F}_2\boldsymbol{F}_1 \\
&\boldsymbol{Q} = \boldsymbol{Q}_N + \boldsymbol{F}_N\boldsymbol{Q}_{N-1}\boldsymbol{F}_N^T + \boldsymbol{F}_N\boldsymbol{F}_{N-1}\boldsymbol{Q}_{N-1}\boldsymbol{F}_{N-1}^T\boldsymbol{F}_N^T + \cdots + \boldsymbol{F}_N\cdots\boldsymbol{F}_2\boldsymbol{Q}_1\boldsymbol{F}_2^T\cdots\boldsymbol{F}_N^T
\end{aligned} \tag{7}$$

If the period-N cycle is attractor, it is always exponentially stable. The SSF of point $\boldsymbol{x}_1$ is then the unique solution of matrix equation

$$\boldsymbol{W}_1 = \boldsymbol{B}\boldsymbol{W}_1\boldsymbol{B}^T + \boldsymbol{Q} \tag{8}$$

and $W_2, W_3, \ldots W_N$ can be calculated by the recurrence relation below

$$\boldsymbol{W}_{k+1} = \boldsymbol{F}_k\boldsymbol{W}_k\boldsymbol{F}_k^T + \boldsymbol{Q}_k \qquad k = 1, \cdots, N-1 \tag{9}$$

After $\boldsymbol{W}_k$ is calculated, a confidence ellipse that represents the spatial distribution of stochastic states concentrated near point $\boldsymbol{x}_k$ in stroboscopic section Σ_k can be obtained using the following equation

$$\left(\boldsymbol{x} - \boldsymbol{x}_k\right)^T \boldsymbol{W}_k^{-1}\left(\boldsymbol{x} - \boldsymbol{x}_k\right) = 2\varepsilon^2\Delta t\left(-\ln\left(1-P\right)\right) \tag{10}$$

where $P \approx 1$ is the probability, with which the points in the stochastic attractor are contained in the ellipse.

3 GCM with evolving probabilistic vector

In this part, GCM with evolving probabilistic vector will be developed to capture large stochastic transition of a nonlinear system under noise.

The response of a N-dimensional nonlinear system subjected to additive and/or multiplicative Gaussian white noise excitations is well known to be a diffusion Markov process. Based on the Generalized Cell Mapping method, the probability evolution of the stochastic system is described by a homogeneous Markov chain in the cell space as

$$\mathbf{P} \cdot \mathbf{p}(n) = \mathbf{p}(n+1) \tag{11}$$

where $p(n)$ denotes the probabilistic vector describing the probability of each cell at nth step, and P the one-step transition probability matrix of the stochastic system. The element P_{ij} and $p_i(n)$ can be determined by following formulae

$$\left.\begin{aligned} P_{ij} &= \int_{C_i} p(\mathbf{x}, t|\mathbf{x}_j, t_0)d\mathbf{x} = \int_{C_i} p(\mathbf{x}, \tau|\mathbf{x}_j, 0)d\mathbf{x} \\ p_i(n) &= \int_{C_i} p(\mathbf{x}, n\tau)d\mathbf{x} \end{aligned}\right\} \tag{12}$$

where τ=t-t_0 denotes a mapping time step; C_i is the domain occupied by ith cell in $\mathbf{R}^N$, and $p(\mathbf{x}, \tau|\mathbf{x}_j, 0)$ and $p(\mathbf{x}, n\tau)$ represent the one-step transition probability and the probability under n-steps mapping in $\mathbf{R}^N$, respectively.

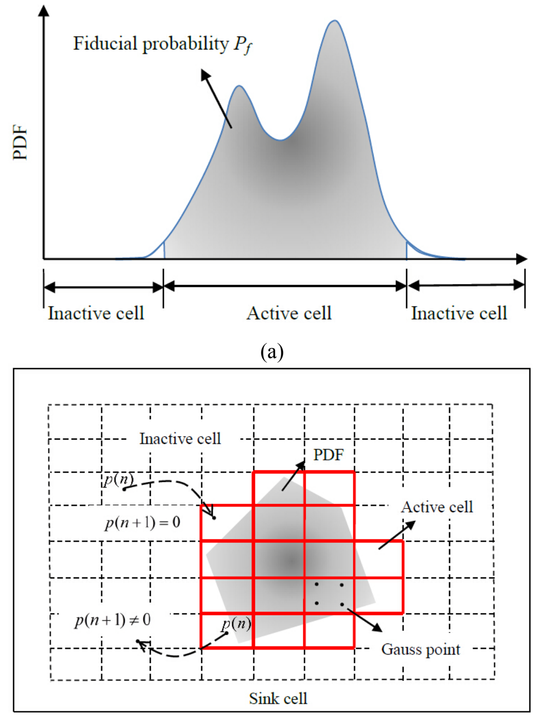
A Gauss-Legendre quadrature is applicable to estimate the above integral in domain C_i. This means that probabilities in ith cell are discretely expressed by that at Gauss quadrature points in the cell. Therefore, based on this rule

$$P_{ij} = \sum_{k=1}^{s_i} A_k p(\mathbf{x}^k, \tau|\mathbf{x}_j, 0), \qquad p_i(n) = \sum_{k=1}^{s_i} A_k p(\mathbf{x}^k, n\tau) \tag{13}$$

where $\mathbf{x}_j$ is the geometrical center of jth cell; $\mathbf{x}^k$ is the kth Gauss quadrature point, s_i is the number of Gauss quadrature points in ith cell, and A_j is the quadrature factor.

To release the difficulty of huge time-consumption in solving nonlinear stochastic equations based on sampling methods, like straightforward MCS to estimate the one-step transition probability matrix P_{ij}, a short-time Gaussian approximation approach proposed in [Sun and Hsu, 1990] is adopted. Thus, the distribution can be approximately specified by the mean and the variance, which can be evaluated by integrating moment equations from t=0 to t=τ.

Borrowing the idea from Point Mapping under Cell Reference method [Jiang, 2011 and 2012], the cells in the chosen region will be classified into active cells and inactive cells. An *active cell* represents the cell whose probability density function (PDF) is within the prescribed fiducial probability, and an *inactive cell* is the cell whose PDF is outside the prescribed fiducial probability, as shown in Fig. 2. In simulation, the inactive cells can be neglected in the computation of the short-time mapping, that is, $P_{ir}p_r$=0 when rth cell is an inactive cell (see Fig. 2(b)).



(b)

Fig. 2 Schematic representations of fiducial probability and active cell, inactive cell, sink cell: (a) fiducial probability; (b) Active cell, inactive cell, sink cell.

So the probabilistic vector $\mathbf{p}(n)$ in the present work is no longer a vector with a fixed length N as in the traditional GCM, rather its length will vary and equal the number of active cells. Then the evolving probabilistic vector is governed by

$$\begin{cases} P_{ij}\ p_j(n) = 0 & when\quad j = r \\ P_{ij}\ p_j(n) = p_i(n+1) & when\quad j \neq r \end{cases} \quad j = 1, 2, 3, \ldots N \tag{14}$$

4 Stochastic responses in noise-induced Duffing oscillator

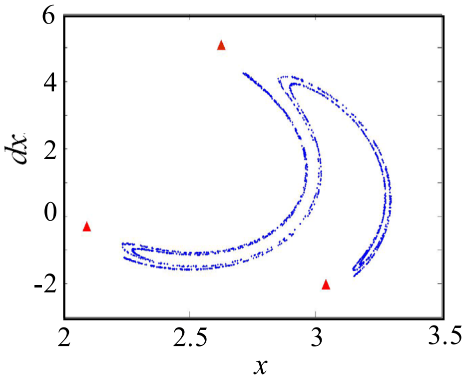
To demonstrate the capability of above proposed methods, a Duffing system under external periodic excitation and additive noise

$$\ddot{x} + c\dot{x} + (k + x^2)x = B\cos t + w(t) \tag{15}$$

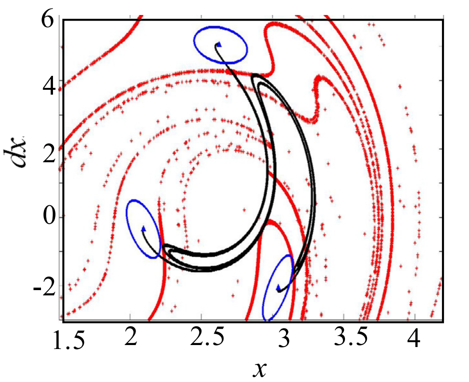
where $w(t)$ a Gaussian white noise stochastic process as defined as

$$E[w(t)] = 0\ ,\ E[w(t)w(t+\tau)] = \sigma_w^2\delta(t) \tag{16}$$

where σ_w is the noise intensity. Let us fix the parameters c=0.25, B=8.50 and k= -0.12. For the deterministic case, namely σ_w =0, the forced Duffing system has a stable period-3 motion and a chaotic saddle, as shown in Fig. 2(a).



(a)



(b)

Fig. 2 (a) Global dynamical structure of Duffing system when c=0.25, B=8.50, k=-0.12, σ_w=0.0, triangular stands for period-3 attractor, dots for chaotic saddle; (b) σ_w=0.05, blue curves stands for confidence ellipses, black and red curves for stable and unstable manifolds of chaotic saddle.

4.1 Sensitivity analysis

It is not hard to imagine that the stochastic responses will mainly concentrate around the deterministic attractors when the noise intensity is sufficient small. The method proposed in Section 2 is employed to investigate the confidence ellipse of the three attractors. Let N=300, Δt =T/N=0.02π and the fiducial probability P=99.99%. By increasing the noise intensity, the size of confidence ellipses increases. When σ_w=0.05, the ellipses begin to touch the unstable manifold of the chaotic saddle (see Fig.2b). To check our prediction, let us first take noise intensity σ_w=0.03, and MCS are used to show its validity. The confidence ellipse is found to be in very good agreement with MCS results (see Fig. 3).

However, as the noise intensity increases, say to σ_w=0.05, noise-induced intermittency occurs that most of response realizations are still concentrated around the deterministic attractors, but a portion of the response realizations go around the structure of chaotic saddle (Fig. 4). Now, the quantification of the probabilistic distribution of noise-induced bifurcation can not be well predicted by the stochastic sensitivity function technique.

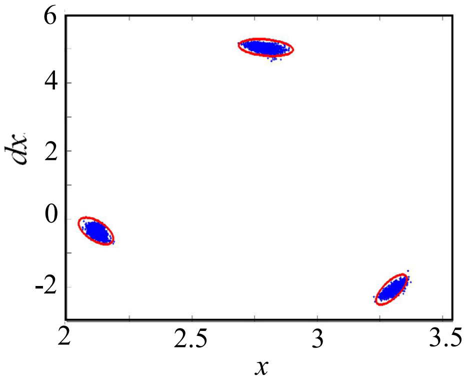


Fig. 3 The confidence ellipses predicted by SSF when noise intensity σ_w=0.03.

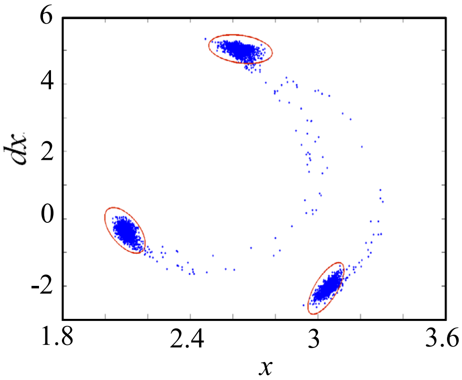


Fig. 4 The confidence ellipses predicted by SSF when noise intensity σ_w=0.05.

The interesting domain of x and $\dot{x}$ is taken to be [1.5, 4.2]×[-3.0, 6.0], and covered by 500×500 cells with 0.0054×0.018 resolution on the chose region. In the subsequent discussion, the initial condition is taken at point (3, 1) with probability one in this section.

In this part, we are interested in noise-induced transition responses. Fig. 5 shows that response PDF corresponding to noise-induced intermittency is more accurately illustrated contrasting with Fig. 4. It is interesting to note from Fig. 6 that when σ_w=0.15, the stochastic responses around P-3 attractor have a significant qualitative change, by which the most of response realizations start to flee away from the vicinity of P-3 attractor and evolve along the chaotic saddle and then go back to P-3 attractor repeatedly.

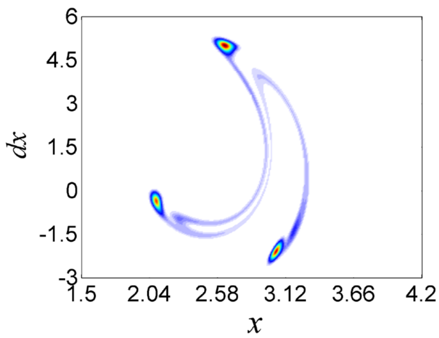


Fig. 5 The stable-state PDFs predicted by GCM-EPV when noise intensity σ_w=0.05.

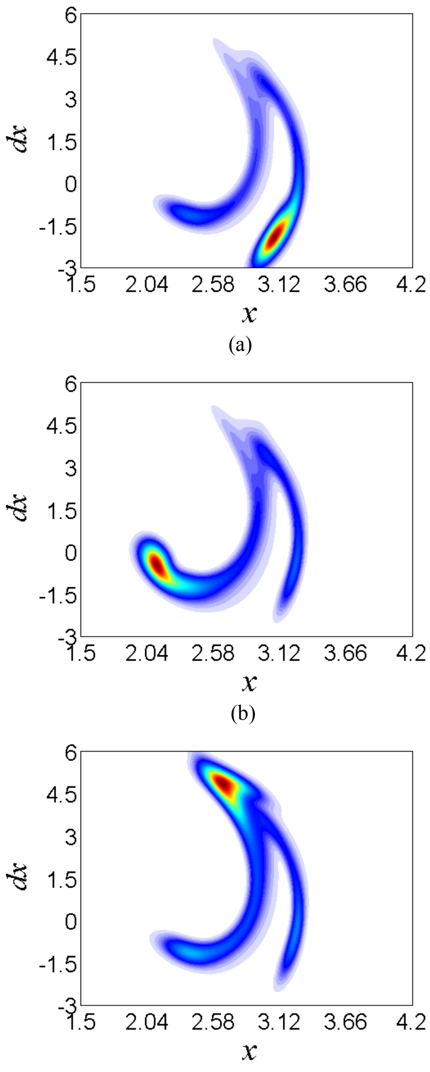


Fig. 6 The stable-state PDFs predicted by GCM-EPV when noise intensity σ_w=0.15: (a) at t=T; (a) at t=2T; (a) at t=3T.

When the noise intensity is further increased, say to σ_w=0.20, the stable-state PDF of the stochastic response will fill into the structure of chaotic saddle gradually to form a stochastic chaotic attractor as shown by Fig. 7.

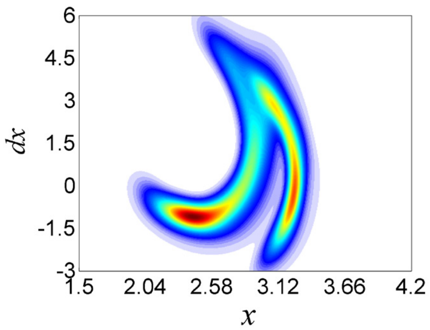


Fig. 7 The stable-state PDFs predicted by GCM-EPV when noise intensity σ_w=0.20.

5 Conclusions

To efficiently capture critical condition of periodic attractors in noise-induced nonlinear dynamical systems, the sensitivity of the periodic attractors is analyzed by discretize the non-autonomous system into a discrete 1/N-period stroboscopic map. In order to obtain the critical noise intensity of noise-induced transition phenomena, SSF is used to judge if the corresponding confidence ellipse is in touch with the manifolds of certain saddle-typed invariant sets. In this way, boundary value problems of matrix differential equations were avoided by solving only matrix algebra equations. SSF can give an approximate analytical description of the distribution, while its implementation is easy. The effectiveness of this method is verified by comparing the confidence ellipses with the stochastic attractors through the Monte Carlo simulation.

To validity investigate the lager stochastic transition and bifurcation of nonlinear dynamical systems after the critical condition, an idea of evolving probabilistic vector is introduced into the Generalized Cell Mapping method to enhance the computation efficiency of the numerical method. By using EPV, both computation consumption and memory storage are much more reduced to make the method even more suitable for detection of large stochastic transition in stochastic systems. Final, a Duffing oscillator under external periodic excitation and additive noise is studied as an example of application.

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