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DYNAMICS OF PHYSICAL SYSTEMS BY DISCRETE TIME COUPLING

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Abstract
We present here an example of how to synchronize physical systems inspired in biological models using the Poincaré coupling. With this type of coupling one is able to study the synchronization phenomena among coupled systems by means of a detection of a threshold without disrupting the monitored system. The idea is to generate a coupling signal, triggered in discrete periods of time as a response to the crossing events of the monitored systems orbit with the previously defined Poincaré plane. This type of coupling comes to satisfy the needs of forcing a system for some specific intervals of time, for example periodic, chaotic or random events of triggering. In order to detect if the systems are synchronized we use two methods: i) the auxiliary system method measuring the Euclidean distance among the forced systems, ii) the maximum conditional Lyapunov exponent. An illustrative example is given by computer simulations in order to demonstrate the approach proposed.

Key words
Poincaré coupling, multivalued synchronization, coupling signal, synchronization in discrete events, mathematical model of the β cell.

1 Introduction

The idea of forcing a system by means of some coupling signal in order to impose a desired behavior, has been of great interest for the scientific community. The applications may diverge in a vast range of areas, since almost everything in nature and man-made artifacts contain coupled subsystems: the cells in the human body, the planets in the universe, cell phones in communication systems, the metronomes on a boat, among others.

Since the report of chaotic system from Lorenz [Lorenz, 1963], many studies have been made in order to synchronize or suppress the chaotic behavior [Chen and Dong, 1992; Chen and Dong, 1993; Kapitaniak, 1995; Aguilar-López *et. al.*, 2010; Sambuco, Sanjuán and Yorke 2012]. One of the most common method to do so was the OGY method [Ott, Grebogi and Yorke, 1990]. In which the dynamics of a chaotic system is monitored, and each time that the orbit of this system diverge form a preestablished vicinity, the system is forced to returned to that vicinity. A recent coupling method was proposed in [Ontañón-García *et. al.*, 2013]. This method is based on the idea that a system is monitored by a Poincaré plane previously defined, and when the orbit of this system crosses the plane a coupling signal is generated and applied to the forced system. The main difference between this two methods is that the OGY monitors and perturbs the same system, while the Poincaré coupling monitors one master system, but perturbs a different forced system.

Two important feature arise from this method: i) The monitored system is never perturbed, as only its dynamic is being detected. ii) Time is considered in the coupling signal, meaning that if the monitored system presents a periodic orbit, the coupling will be applied periodically, if the system is chaotic, then the coupling will be applied chaotically in discrete events of time, respectively. This attends to the needs of some biological systems, in which the time is important to considered in order to induced an specific behavior. For example the pancreatic β cells synchronize at specific periods of time by the increase or decrease of substances such as intracellular calcium [Ontañón-García and Campos-Cantón, 2013]. It has also been discovered that this particular biological system presents chaotic behavior when coupled in the cluster of cells [Lebrun and Atwater, 1985].

Taking this in consideration, we based this work on the Poincaré coupling, and present an approach on how to apply a coupling signal to the biological systems of the β cell described by mathematical equations. To do so we considered the benchmark chaotic systems Rössler and Lorenz as the monitored systems, in order to generate a coupling signal and apply it in chaotic discrete events of time.

This work is organized as follows: In the second section, we make a brief description of the Poincaré coupling method and the benchmark systems used as monitored systems. The third section describes the unidirectional coupling used in the experiment along with the model of the mathematical β cell system. In the fourth sections are the numerical results of the computer simulations. Conclusions are made in the last section.

2 Preliminaries on Poincaré Plane

Here we make a brief description of the Poincaré plane as defined by [Ontañón-García *et. al.*, 2013]. An autonomous system described as:

$$\mathbf{x}' = F(\mathbf{x}),\ F : \mathbf{R}^m \to \mathbf{R}^m \tag{1}$$

is being monitored by a Poincaré plane $\Sigma := \{(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) : \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \alpha_3\mathbf{x}_3 + \alpha_4 = 0\}$ where $\alpha_1, \ldots, \alpha_4 \in \mathbf{R}$ are coefficients of a hyperplane equation whose values are considered arbitrarily according to the following discussion. We are interested in the crossing events of the trajectory of the monitored system Eq. (1) restricted to the projection $\mathcal{A}_x$ with Σ, captured by the points $\{\varphi_m^{t_0}(\mathbf{x}_0), \varphi_m^{t_1}(\mathbf{x}_0), \varphi_m^{t_2}(\mathbf{x}_0), \ldots\} \in \Sigma$ at each crossing event. Where $\varphi_m^{t_i}(\mathbf{x}_0)$ is the flow restricted to $\mathcal{A}_x$. Therefore, we can specify the following time series $\Delta_{\mathbf{x}_0} = \{t_0, t_1, t_2, \ldots\}$, which depends on the initial conditions of the system in Eq. (1). The location of the plane must be located in order to meet the condition $\mathcal{A}_x \bigcap \Sigma \neq \emptyset$, assuming that at least one crossing event at time t_i exists. Throughout this work we have focused on the crossing events of the trajectory of the monitored system with Σ in only one direction. So the time series $\Delta_{\mathbf{x}_0}$ contains each crossing event that satisfy $\frac{dx_1}{dt} > 0$. Following the above discussion, we define a coupling signal as follows:

$$\xi(t) = Ae^{-\tau(t-t_i)}, \tag{2}$$

where $A \in \mathbf{R}$ is the amplitude of the signal and $0 \leq \tau \in \mathbf{R}$ represents an underdamping factor which allows us to modulate the signal. Therefore the underdamped signal is triggered with each crossing event of Eq. (1) with Σ.

2.1 Monitored Systems

Two benchmark chaotic systems are considered in this article as the monitored systems. The Rössler system given by:

$$\begin{aligned}\dot{x}_{1R} &= -x_{2R} - x_{1R},\\ \dot{x}_{2R} &= x_{1R} - 0.2x_{2R},\\ \dot{x}_{3R} &= 0.2 + x_{3R}(x_{1R} - 5.7),\end{aligned} \tag{3}$$

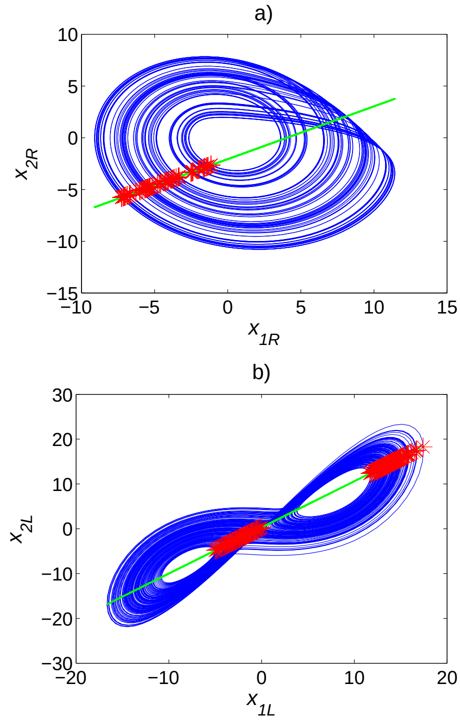

Figure 1. a) Projection of the monitored a) Rössler system onto the plane (x_{1R}, x_{2R}), b) Lorenz system onto the plane (x_{1L}, x_{2L}). Both systems are intersected by the Poincaré plane Σ_R and Σ_L, respectively marked in green. The points of each crossing event $\{\varphi_m^{t_0}(\mathbf{x}_0), \varphi_m^{t_1}(\mathbf{x}_0), \varphi_m^{t_2}(\mathbf{x}_0), \ldots\}$ are marked with asterisk.

where x_{1R}, x_{2R} and x_{3R} are the states of the system. Figure 1 a) shows a projection of the Rössler system onto the plane (x_{1R}, x_{2R}) intersected by the Poincaré plane with $\alpha_1 = 0.5934, \alpha_2 = -1.1636, \alpha_3 = 0, \alpha_4 = -2.4068$ marked with green, and all the crossing events t_i of each intersection of the Rössler system with the plane Σ_R are marked with asterisk. The Lorenz system is given by:

$$\begin{aligned}\dot{x}_{1L} &= 10(x_{2L} - x_{1L}),\\ \dot{x}_{2L} &= 25x_{1L} - x_{2L} - x_{1L}x_{3L},\\ \dot{x}_{3L} &= x_{1L}x_{2L} - 8/3x_{3L},\end{aligned} \tag{4}$$

where x_{1L}, x_{2L} and x_{3L} are the states of the system. Figure 1 b) shows a projection of the Lorenz System onto the plane (x_L, y_L) intersected by the Poincaré plane with $\alpha_1 = -0.179, \alpha_2 = -0.1739, \alpha_3 = 0, \alpha_4 = 0.0246$ marked with green, and all the crossing events t_i of each intersection of the Lorenz system with the plane Σ_L are marked with asterisk.

3 Unidirectional Coupling

Throughout the work we will consider a unidirectional coupling among the monitored system and the forced system. Since some of the states of the monitored system are involved in the coupling, then this system is also a master system. The forced system takes the

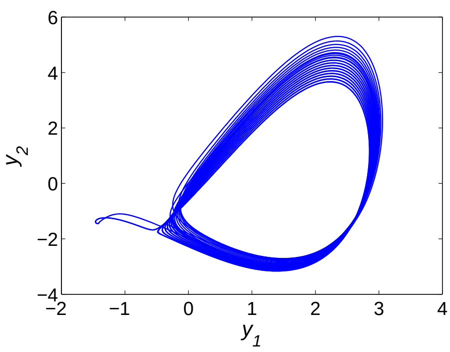

Figure 2. Projection of the β cell system from Eq. (6) onto the (y_1, y_2) plane, without coupling

form:

$$\mathbf{y}' = G(\mathbf{x}, \mathbf{y}),\ G : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^n \qquad (5)$$

where $\mathbf{y} \in \mathbf{R}^n$ stands for the state vector of the slave system, and $G(\cdot)$ is the dynamics of the system (5). Here we considered the mathematical model of a single cell coupled in a cluster of cells described by [Pernarowski, 1998], which is given as follows:

$$\begin{aligned} \dot{y_1} &= ay_1^3 + by_1^2 + cy_1 - y_2 - y_3, \\ \dot{y_2} &= dy_1^3 + by_1^2 + ey_1 - y_2 - 3, \\ \dot{y_3} &= fy_1 + gy_3 + h, \end{aligned} \qquad (6)$$

where y_1 is the membrane potential, y_2 is a channel activation parameter for the voltage-gated potassium channel, and y_3 are concentrations of agents which regulate the BEA, such as intracellular calcium, concentration of calcium in the endoplasmic reticulum and ADP. Here $a = -1/12$, $b = 3/8$, $c = 37/64$, $d = 13/12$, $e = -155/64$, $f = 1/100$, $g = 477/50000$ and $h = -1/400$ are specific values for which the system exhibits square-wave bursting analogous to the BEA in the pancreatic beta cell. Figure 2 depicts the projection onto the (y_1, y_2) plane of the β cell system.

This system is coupled unidirectionally by the master system (1) in the following way:

$$\mathbf{y}' = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} + k \begin{bmatrix} x_1\xi(t) - y_1 \\ 0 \\ 0 \end{bmatrix}. \qquad (7)$$

where $k \in \mathbf{R}$ is the coupling strength. We will consider the following parameters for the numerical simulation: $A = 1, k = 1, \tau = 0.0031$.

4 Numerical Results

To detect synchronization in the scheme proposed, we use the auxiliary system approach presented by [Abarbanel, Rulkov and Sushchik, 1996], in which a system $\mathbf{z}' = \mathbf{y}'$ is coupled in the same way as (7) but initialized with a set of initial conditions $\mathbf{z}_0$ where $\mathbf{y}_0 \neq \mathbf{z}_0$. If both system converge as $t \to \infty$ and the distance between them given by the Euclidean norm $d_n\left(\{\mathbf{y}(i)\}_{i\in\{1,\ldots,n\}}, \{\mathbf{z}(i)\}_{i\in\{1,\ldots,n\}}\right) = \frac{1}{n}\sum_{i=1}^{n} \|\mathbf{y}(i) - \mathbf{z}(i)\| \to 0$, where n stands for the number of iterations in the numerical simulation made with a fourth-order Runge Kutta method, then we say that the master and slave systems are coupled.

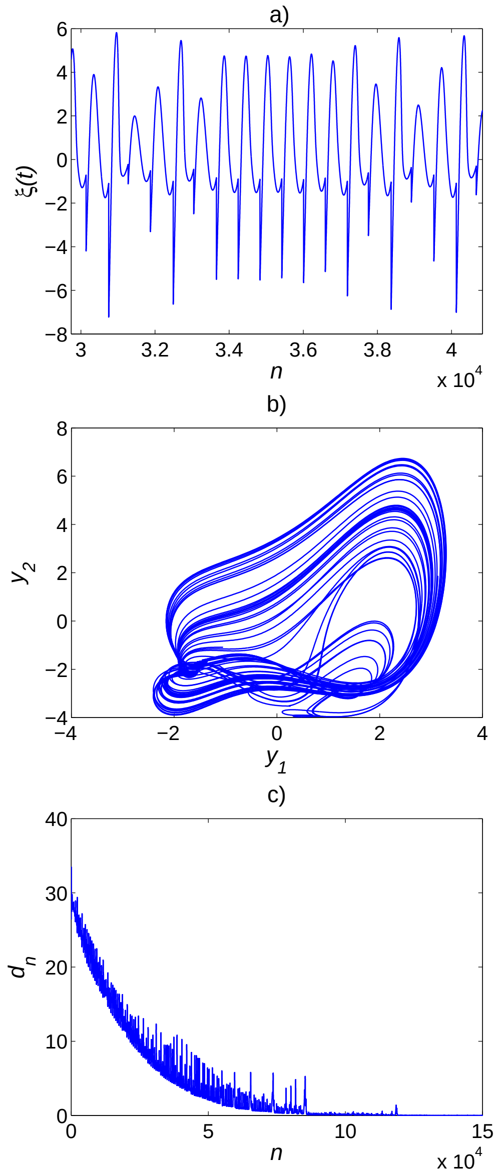

Figure 3. a) Coupling signal $\xi(t)$ from Eq. 2 generated from the monitored Rössler system. b) Projection onto the (y_1, y_2) plane of the forced system. c) Euclidean distance $d_n(\mathbf{y}(i), \mathbf{z}(i))$.

4.1 Rössler as Master System

First we use the chaotic Rössler system given by Eq. (3) as the monitored master system. The sets of initial conditions for the systems are $\mathbf{x}_0 = [-1.1,\ 1.15,\ 0.6116]$, $\mathbf{y}_0 = [-0.5,\ 0.1,\ 0.6116]$ and

$\mathbf{z}_0 = [-7,\ -8,\ 24]$. The resulting coupling signal $\xi(t)$ with this configuration is shown in Figure 3 a). Figure 3 b) depicts the projection onto the (y_1, y_2) plane of the forced system after the coupling, and in Figure 3 c) it can be seen the Euclidian distance d_n between the forced system and the auxiliary system. The systems initialized with different initial conditions loose their transient states after $n = 120,000$ iteration. Note that $d_n \leftarrow 0$ after this value showing their convergence. With this and with the maximum conditional Lyapunov exponent which is -0.0327, we can assure that the control signal forces the systems $\mathbf{y}'$ and $\mathbf{z}'$ to oscillate equally meaning both systems are synchronized with the master system.

4.2 Lorenz as Master System

Changing now the monitored system as the Lorenz system given by Eq. (4) results in the coupling signal depicted in Figure 4 a). Using the same sets of initial conditions as above outcome in the attractor depicted in Figure 4 b), where a projection onto the (y_1, y_2) plane is shown. From Figure 4 c) it can be observed that distance between the attractors d_n also converge to zero after a transient state, for this case $150,000$ iterations. The maximum conditional Lyapunov exponent for this system is -0.0236 meaning that also the master and forced systems are synchronized.

4.3 β cell as Master System

Based on the results shown above we decided to implement one last experiment, considering the β cell given by Eq. (6) as a monitored system $\mathbf{x}'$. By locating the same Σ_L Poincaré plane with $\alpha_1 = -0.179, \alpha_2 = -0.1739, \alpha_3 = 0, \alpha_4 = 0.0246$ we obtained the following results. Figure 5 a) shows the coupling signal $\xi(t)$ generated by the crossing events of this monitored system with Σ_L. Figure 5 b) depicts the projection onto the (y_1, y_2) plane of the resulting forced system, and the Euclidean distance d_n between $\mathbf{y}'$ and $\mathbf{z}'$ can be appreciated in Figure 5 c). The forced systems converge after $200,000$ iterations, and the maximum conditional Lyapunov exponent is -0.0704 therefore the master and forced systems are also synchronized.

5 Conclusion

We demonstrated in this work that with the Poincaré coupling and a unidirectional coupling, one is able to generate a coupling signal sent from the master to the slave system, each time that the master system crosses the previously defined plane. This coupling is capable of synchronize a biological system described by mathematical models. The coupling signal was applied in chaotic intervals of time due to the 2 chaotic monitored systems, and periodically when the β cell was monitored. The stability of the systems was determined through the Lyapunov exponents.

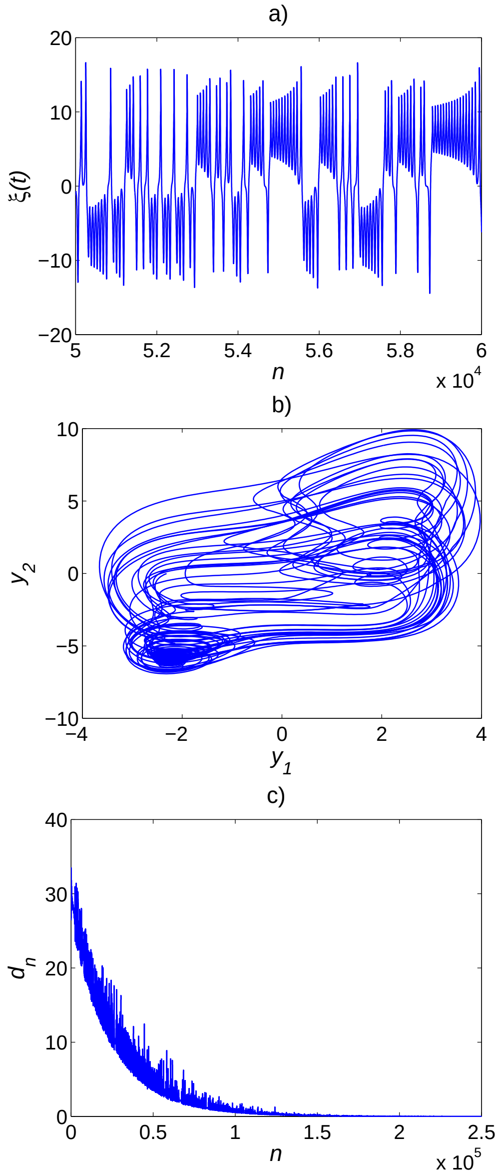

Figure 4. Coupling signal $\xi(t)$ from Eq. 2 generated from the monitored Lorenz system. b) Projection onto the (y_1, y_2) plane of the forced system. c) Euclidean distance $d_n(\mathbf{y}(i), \mathbf{z}(i))$.

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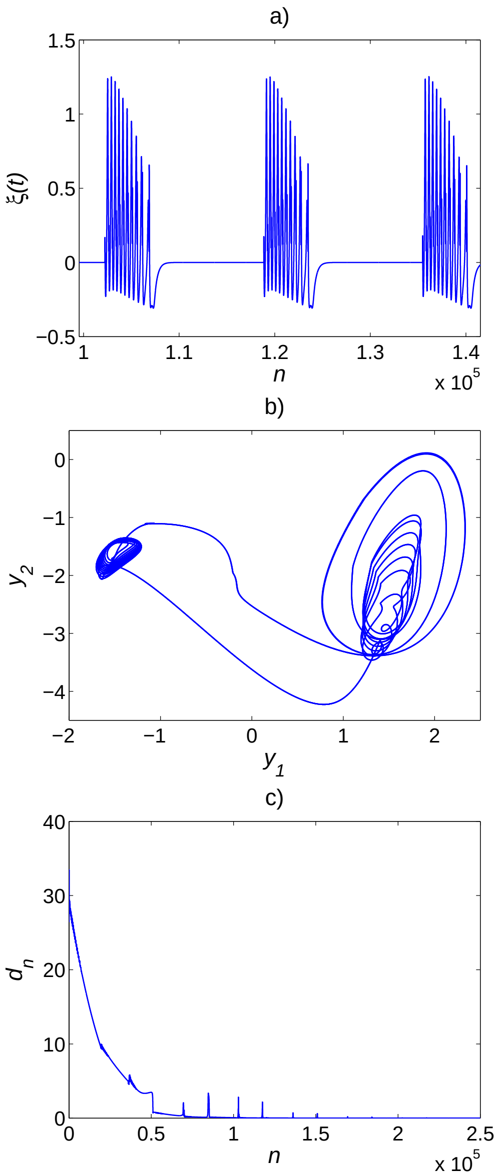

Figure 5. Control signal $\xi(t)$ from Eq. 2 generated from the monitored β cell system. b) Projection onto the (y_1, y_2) plane of the forced system. c) Euclidean distance $d_n(\mathbf{y}(i), \mathbf{z}(i))$.

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