Non-holonomic distributions and dynamical systems

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***Abstract*— We study a nonlinear dynamical model defined in terms of a non-holonomic distribution Δ of polynomial vector fields, satisfying the bracket generating condition of control theory. The case when the Lie algebra span(Δ) is nilpotent is discussed. In low dimensional cases, the step-2 models classical particles in homogeneous magnetic fields whereas the step-3 models the case of linear fields.**

I. Introduction

Certain nonholonomic dynamical systems can be described in terms of distributions with a finite number of vector fields, the structure of the spanned Lie algebra, that is, the Lie algebra obtained by Lie bracketing iteratively the vector fields, determines most of the relevant properties of the system.

We study a nonlinear dynamical model defined in terms of distributions of polynomial vector fields satisfying the so-called bracket generating condition, condition that guarantees the existence of trajectories provided the base manifold is connected. Examples of problems which can be expressed in these terms can be found in plasma physics [1] and nonholonomic mechanics [2].

Non-holonomic distributions are in the opposite side of integrable ones, they provide the main object of study of what is known as Carnot-Caratheodory or sub-Riemannian geometries. A distribution Δ of rank k on a smooth n-dimensional manifold M is a smooth rank k sub bundle of the tangent bundle T M. The iteration of the Lie bracket of vector fields in Δ yields the following flag of modules of vector fields

$$\Delta^1 \subset \Delta^2 \subset \cdots \subset \Delta^l \cdots \subset \mathrm{T\ M}$$

here $\Delta^1 = \Delta$ and $\Delta^{i+1} = \Delta^i + [\Delta, \Delta^i]$. The

distribution is said to be non-holonomic or bracket generating, if for each $p \in$ M, there exist a positive integer m for which $\Delta_p^m = \mathrm{T}_p$ M. The first m for which this occur is called the degree of non-holonomy of Δ at p. Let $n_j(p) = \dim(\Delta_p^j)$, the growth vector of Δ at p is defined as $(n_1, \ldots, n_m)$, the distribution is said to be regular if the growth vector is independent of the base point. In this case there is a graded vector space associated to the above flag, defined as follows

$$\mathcal{N}_\Delta = N_1 \oplus N_2 \oplus \cdots \oplus N_m,$$

where $N_l = \Delta_p^l / \Delta_p^{l-1}$. The Lie bracket induces the gradation $[N_i, N_j] \subset N_{i+j}$, and $\mathcal{N}_\Delta$ becomes a graded nilpotent Lie algebra, for details see for instance, [3]. Therefore, it is natural in this context, to begin for analyzing the situation for nilpotent Lie algebras as we do in the present paper.

We consider that a regular non-holonomic distribution Δ is given once for all, and we take $\mathcal{G}$ to be the Lie algebra $\Delta_p^m = \mathrm{T}_p$ M. Through all the paper we shall assume that $\mathcal{G}$ is nilpotent. An inner product in $\mathcal{G}$ determines a natural decomposition $\mathcal{G} = \mathcal{H}_p \oplus \mathcal{V}_p$, in terms of the horizontal vector space $\mathcal{H}_p = \Delta_p$ and the vertical vector space $\mathcal{V}_p = \mathcal{H}_p^\perp$.

An absolutely continuous curve $t \mapsto p(t)$, is said to be horizontal, if $\dot{p}(t) \in \Delta(p(t))$, almost everywhere. Chow-Rashevski's theorem [3], guarantees that any two points can be connected by an horizontal curve, provided M is connected. A sub-Riemannian metric on is defined by a smooth varying inner product $p \mapsto \langle \cdot, \cdot \rangle_p$ in $\Delta(p)$. For horizontal curves $t \mapsto p(t)$, the length and the energy functionals are defined as usual

$$\ell(p) = \int \langle \dot{p}, \dot{p} \rangle^{\frac{1}{2}}, \quad \text{and} \quad \mathcal{E}(p) = \frac{1}{2} \int \langle \dot{p}, \dot{p} \rangle,$$

respectively. For horizontal curves parametrized by arc-length, the variational problems for the functionals ℓ and $\mathcal{E}$ are equivalent.

In this paper we consider nonlinear dynamical systems given by a non-holonomic distribution of real vector fields. We approach the problem as a sub-Riemannian geodesic problem, that is, the one of minimizing functional ℓ (or equivalently functional $\mathcal{E}$), in the class of horizontal curves. We use the Hamiltonian formalism to set the problem as an optimal control problem, we integrate some cases of the Hamiltonian equations and derive some geometric properties of the geodesics.

Apart from this introduction this paper contains four sections, in section II we set the dynamical problem that plays the role of archetype of the theory, and is related with variational problems in nonlinear dynamics, for this case, we depict the Lie algebraic structure and exhibit a gauge transformation under which such structure remains invariant. In section III, we write explicitly a Phillip Hall basis for the Lie algebra. This basis clarifies the hierarchy of non-linear dynamical systems proposed by R.W. Brockett and L. Dai in [4]. In sections

IV and V we apply the Pontryagin Maximum Principle for deriving necessary conditions for minimizing trajectories, and develop also some low dimensional examples in both step-2 and step-3. At the end, in section VI we derive some conclusions and discuss further research perspectives.

II. A MODEL FOR NON-HOLONOMIC DYNAMICAL SYSTEMS

There is an non-holonomic dynamical system that stands for an archetype of our discussion. It consists of the nonlinear dynamical system in R^{n+1} determined by a non-holonomic contact distribution distribution $\Delta = \{X_1, \dots, X_n\}$. By taking coordinates $(x, y) \in R^n \times R$, horizontal curves in general satisfy

$$(\dot{x}, \dot{y}) = \sum_{j=1}^{n} \alpha_j(x, y) X_j(x, y),$$

for certain smooth functions $\alpha_1, \dots, \alpha_n$, and the vector fields in general are written as follows

$$X_i(x, y) = \sum_{j=1}^{n} f^{ij} \partial x_j + \xi^j \partial y$$

for certain smooth functions f^{ij} and $\xi^j, i, j = 1, \dots, n$.

Privileged coordinates can be found in such a way that the distribution is written in the so-called normal form, that is $f^{ij} = \delta_{ij}$ (Kronecker delta), for details on normal forms, see for instance [5]. In conclusion, horizontal curves $t \mapsto q(t) = (x(t), y(t))$ are solutions of the system

$$\dot{q} = \sum_{i=1}^{n} \dot{x}_i \, X_i(q),$$

where $X_i = \partial x_i + \xi^i \partial y$, with $\xi = \xi(x) = (\xi^1(x), \dots, \xi^n(x))$ smooth function that does not depend on y and $\dot{y} = \langle \dot{x}, \xi \rangle$ with the standard inner product of R^n.

The first degree Lie brackets of the distribution are easily calculated as follows

$$X_{ij} := [X_i, X_j] = g^{ij} \partial y, \text{ with } g^{ij} = \xi^i_{x_j} - \xi^j_{x_i}$$

and for all $i, j = 1, \dots, n$. A lengthy but easy calculation shows that Jacobi identity is written as follows

$$g^{jk}_{x_i} + g^{ki}_{x_j} + g^{ij}_{x_k} = 0.$$

Since function ξ does not depend on y, then all fields $\mathrm{ad}_{X_{i_1}} \mathrm{ad}_{X_{i_2}} \cdots \mathrm{ad}_{X_{i_k}}(X_j)$, for $k > 0$ commute among themselves. In consequence the Lie algebra $\mathcal{G}$ spanned by the distribution is solvable, filtered and graded, for details on these definitions see for instance [6].

Furthermore, together with the anti commutativity of the Lie bracket, the Jacobi identity bounds the number of linearly independent vector fields obtained by Lie bracketing the elements of Δ. In fact, we start by defining the fields of depth r with $r \geq 1$ to be the fields

$$X_\kappa = \mathrm{ad}_{X_{\kappa_r}} \mathrm{ad}_{X_{\kappa_{r-1}}} \cdots \mathrm{ad}_{X_{\kappa_2}}(X_{\kappa_1})$$

with multi index $\kappa = (\kappa_r, \kappa_{r-1}, \dots, \kappa_1)$ and $\kappa_j = 1, \dots, n$.

If all fields are distinct and of depth one, the Jacobi identity is a condition relating fields of depth three. For more than one field of depth two, the identity is trivial. Moreover, when two fields, say X_i and X_j, are of depth one, and the third one, say X, is of depth $s \geq 1$ then

$$[X_i, [X_j, X]] = [X_j, [X_i, X]],$$

expression that relates fields of depth $s + 2$, which is just the statement of the commutativity of the partial differentiation. In conclusion, the Jacobi identity states that for $r > 3$ the order of the κ_i for $i > 3$ in X_κ is irrelevant.

Any sufficiently smooth function $x \mapsto \psi(x)$ determines a gauge transformation $\xi \mapsto \xi + \nabla\psi$, under which the fields are mapped to $X_i + \psi_{x_i} \partial y$, and the Lie algebra $\mathcal{G}$ remains invariant.

In terms of differential forms, associated to Δ we have the connection 1-form $\mathcal{A} = dy - \langle \xi, dx \rangle$, and the curvature 2-form

$$\mathcal{K} = 2d\mathcal{A} = \sum_{j,k} g^{jk} \, dx_j \wedge dx_k.$$

In this formulation, $d\mathcal{K} = 0$ corresponds to the homogeneous Maxwell equations (Jacobi identity) for this problem. Further, $\mathcal{A}$ is a higher dimensional analog of the *vector potential* and $d\mathcal{A}$ is the corresponding analog of the *magnetic field*. The above gauge transformation leads to

$$\mathcal{A} \mapsto \mathcal{A} - d\phi \quad \text{and} \quad \mathcal{K} \mapsto \mathcal{K}.$$

In consequence, by means of gauge transformations we can add to our convenience exact differentials leaving both, the equations of motion and the curvature, invariant.

Remark 2.1: The variational problem for the energy functional $\mathcal{E}$ can be formulated by considering the Lagrangian $\mathcal{L} = \lambda_0 \|\dot{x}\|^2 + \lambda(\dot{y} - \langle \xi, \dot{x} \rangle)$. For $\lambda_0 \neq 0$, Euler-Lagrange equations can be easily calculated: $\ddot{x} = \lambda g \dot{x}$ and $\dot{\lambda} = 0$, here $g = (g^{ij})$ is a skew-symmetric matrix. The Lagrange parameter λ is a constant of motion and the Euler-Lagrange equations are gauge-invariant. However we pursue in this paper the Hamiltonian formalism which is more appropriate to tackle the problem as an optimal control one.

III. PHILLIP HALL BASIS FOR NILPOTENT LIE ALGEBRAS

The distribution introduced in the preceding section can be studied by considering the Taylor series expansion around the origin for the function $\xi(x)$. The analytic case has been studied among others by H. Sussmann et al [7], in the context of the so-called motion planning problem for non-holonomic mechanical systems. In a different perspective, R.W. Brockett and L. Dai [4] have suggested a hierarchy of non-linear dynamical systems by considering a cut off in the Taylor series, that is, by taking polynomial vector fields in particular coordinate systems. In such a case, the Lie algebra $\mathcal{G}$ is of finite dimension and nilpotent.

Our treatment is more general, it is coordinate free and leads naturally to the classification problem of isomorphism classes of finite dimensional nilpotent Lie algebras. We consider a rank n distribution Δ, as that given in the preceding section, but for general linearly independent polynomial functions ξ_i of degree m. The corresponding Lie algebra is filtered, graded and nilpotent of step $m+1$. A generic formulation of the problem for nilpotent Lie algebras would be reached once a basis for the commutators is explicitly given. For then the underlying simply connected Lie group can be obtained by means of the exponential map and the BCH formula, privileged coordinates can be used to write a canonical basis of invariant vector fields.

We start by counting the number of linearly independent vector fields of depth $r > 0$, that is, the cardinality of Δ^r. Elementary combinatorics arguments together with a careful use of the Jacobi identity for brackets of different depths imply that such a number is given by

$$D_{n,r} = r \begin{pmatrix} n+r-1 \\ r+1 \end{pmatrix},$$

in consequence the Lie algebra $\mathcal{G}$ has dimension

$$n+1 + D_{n+1,m} - \begin{pmatrix} n+m \\ m \end{pmatrix}.$$

Now we provide a basis for $\mathcal{G}$, that has been introduced in the general theory of free Lie algebras, namely the so-called Phillip Hall basis [8].

Definition 3.1: A Phillip Hall basis of the Lie algebra $\mathcal{G}$ generated by the distribution Δ, is a totally ordered set $\{\mathcal{P}, \prec\}$ that satisfies the following three properties

1) The X_i belong to $\mathcal{P}$
2) If $A, B \in \mathcal{P}$ and length$(A) <$ length(B), then $A \prec B$
3) If C is not in Δ, then $C \in \mathcal{P}$ iff $C = [A, B]$ with $A, B \in \mathcal{P}$, $A \prec B$ and either $B \in \Delta$ or $B = [D, E]$, with $D, E \in \mathcal{P}$, $D \preceq A$ and $D \prec E$.

The following result provides a complete description of a Phillip Hall basis for $\mathcal{G}$ by taking the total order $\prec$ defined by the depth of the brackets.

Theorem 3.1: A Phillip Hall basis for the step m+1 nilpotent Lie algebra $\mathcal{G}$ consists of

1) n elements of Δ and $D_{n,1} = n(n-1)/2$ fields of depth two: $X_{i_1 i_2} = [X_{i_1}, X_{i_2}]$, for $i_1 < i_2$,
2) $D_{n,2} = n(n^2-1)/3$ linearly independent fields of depth three:
$$\begin{aligned} X_{i_3\, i_1\, i_2} &= \mathrm{ad}_{X_{i_3}} \mathrm{ad}_{X_{i_1}} X_{i_2}, \text{ for } i_1 < i_2 \leq i_3 \text{ and} \\ X_{i_2\, i_1\, i_3} &= \mathrm{ad}_{X_{i_2}} \mathrm{ad}_{X_{i_1}} X_{i_3}, \text{ for } i_1 \leq i_2 < i_3, \end{aligned}$$
3) $D_{n,r+2}$ fields of depth $r+3$ for $r = 1, \ldots, m-3$:
$$\begin{aligned} X_{ji} &= \mathrm{ad}_{X_{j_r}} \cdots \mathrm{ad}_{X_{j_1}} \mathrm{ad}_{X_{i_3}} \mathrm{ad}_{X_{i_1}} X_{i_2} \text{ and} \\ X_{ji'} &= \mathrm{ad}_{X_{j_r}} \cdots \mathrm{ad}_{X_{j_1}} \mathrm{ad}_{X_{i_2}} \mathrm{ad}_{X_{i_1}} X_{i_3}, \end{aligned}$$
with $j_1 \leq j_2 \leq \cdots \leq j_r$.

The proof of this result is based on a careful use of the Jacobi identity together with elementary but lengthy counting arguments, it shall be given elsewhere, we shall restrict ourselves here to the following examples

Example 3.1: For the step-2 algebra in n variables, up to isomorphisms, the basis is given by

$$[X_i, X_j] = X_{ij}, \quad i < j \quad i, j = 1, \ldots, n$$

the remaining non-trivial elements of the algebra are the following $X_{i_2,i_1} = -X_{i_1,i_2}$.

Example 3.2: For step-3 the basis is given by

$$\begin{aligned} [X_{i_1}, X_{i_2}] &= X_{i_1 i_2}, \quad i_1 < i_2 \\ [X_{i_3}, X_{i_1 i_2}] &= X_{i_3, i_1 i_2}, \quad i_1 < i_2 \leq i_3 \\ [X_{i_2}, X_{i_1 i_3}] &= X_{i_2, i_1 i_3}, \quad i_1 \leq i_2 < i_3. \end{aligned}$$

The remaining non-trivial elements of the algebra are again $X_{i_2,i_1} = -X_{i_1,i_2}$ together with

$$[X_{i_1}, X_{i_2 i_3}] = X_{i_3, i_1 i_2} - X_{i_2, i_1 i_3}, \quad i_1 < i_2 < i_3,$$

with $i_1, i_2, i_3 = 1, \ldots, n$.

IV. Optimal control and extremal curves

We consider now G to be the simply connected Lie group that corresponds to the nilpotent Lie algebra $\mathcal{G}$ with order of nilpotency $m+1$, generated by the rank n nonholonomic distribution Δ. A smooth varying inner product $g \mapsto \langle \cdot, \cdot \rangle_g$ is defined in the planes span$\Delta(g)$ by declaring the set $\{X_1(g), \ldots, X_n(g)\}$ orthonormal. The variational problem associated to the kinetic energy $\mathcal{E}$, is equivalent to the sub-Riemannian geodesic problem on G, which in turn, can be formulated as the following left invariant optimal control problem on the group G

$$\dot{g} = u_1 X_1(g) + \cdots + u_n X_n(g), \qquad \mathcal{E}(g) \to \min. \qquad (1)$$

Here the admissible control laws $u = (u_1, \ldots, u_n)$ are taken to be measurable and bounded. The family of such control laws shall be denoted as $\mathcal{U}$.

We profit on the symplectic structure of the cotangent bundle T^*G to derive necessary conditions for the optimal controls. It is known that $T^*G \simeq G \times \mathcal{G}^*$, then we take coordinates $(g, p) \in T^*G$. Each left invariant vector field X determines a Hamiltonian $H_X(g, p) = p(X(e))$, where $e \in G$ is the group identity. The Poisson bracket for Hamiltonians determines then a dual Lie algebra structure in the sense that $\{H_X, H_Y\} = H_{[X,Y]}$, for any pair of left invariant vector fields X and Y, for details see for instance [9].

Let H_i denote the Hamiltonian H_{X_i} for each $i = 1, \ldots, n$. We consider the co vector $h = (H_1, \ldots, H_n)$ As usual in optimal control theory, one incorporates the energy to the dynamics to define the following control-parametrized Hamiltonian

$$\mathcal{H}_{\lambda,u} = -\frac{\lambda}{2}\|u\|^2 + \langle u, h\rangle.$$

In this paper we consider only the normal case that corresponds to $\lambda = 1$, although the abnormal case ,$\lambda = 0$, is relevant and must be studied.

Pontryagin Maximum Principle [10], tells us that a solution $t \mapsto (g, u)$ of the problem (1) is optimal, if it is the projection of a solution $(g, p) \in G \times T^*G$ of the Hamilton equations corresponding to $\mathcal{H}_u$, along which the inequality

$$\mathcal{H}_u \geq \mathcal{H}_\nu \tag{2}$$

holds a.e., for all $\nu \in \mathcal{U}$.

Solutions of the Hamiltonian system corresponding to the Hamiltonian $\mathcal{H}_u$ are customarily called extremal curves.

For the Phillip Hall basis for $\mathcal{G}$ depicted in the preceding section, we have the corresponding dual basis for $\mathcal{G}^*$ for which we shall adopt the following notation

$$\{\underbrace{H_1, \ldots, H_n}_{h}, \overbrace{H_{i_1,i_2}}^{H}, \underbrace{H_{i_3 i_1 i_2}, H_{i_2 i_1 i_3}, H_{ji}, H_{ji'}}_{\mathsf{H}}.\}$$

In consequence the co vector can be written as follows

$$p = (h, H, \mathsf{H}) \in R^n \times so_n^* \times R^J,$$

where $J = D_{n,2} + D_{n,3} + \cdots + D_{n,m-1}$.

The optimality condition of the Maximum Principle given by (2), implies that $u_i = H_i$, for all $i = 1, \ldots, n$, therefore the system Hamiltonian is quadratic

$$\mathcal{H} = \frac{1}{2}\left(H_1^2 + \cdots + H_n^2\right),$$

and the first Hamiltonian equation writes as follows

$$\frac{dg}{dt} = H_1 X_1(g) + \ldots + H_n X_n(g).$$

The second Hamiltonian equation is by far more complicated, it is obtained by differentiating along the extremal, that is, by Poisson bracketing the entries of the co vector with the Hamiltonian $\mathcal{H}$, for instance, $\dot{H}_i = \{H_i, \mathcal{H}\}$ readily implies

$$\dot{h}^T = H\ h^T,$$

the skew-symmetry of H yields the energy conservation law

Proposition 4.1: The system Hamiltonian $\mathcal{H}$ is constant along the extremal.

Proof: A direct differentiation along the extremal yields

$$\dot{\mathcal{H}} = H_1\{H_1, \mathcal{H}\} + \cdots + H_n\{H_n, \mathcal{H}\} = \langle h^T, Hh\rangle = 0.$$

■

The step-2 has been studied by R.Brockett [11], and A.M.Vershik and V. Gershkovich [2], a coordinate-free presentation together with a complete integration has been presented in our previous paper [12]. The general case with nilpotency $m+1$, shall be presented elsewhere, here we summarize the general step-2 and present some low dimensional particular cases.

For step-2 the co vector writes as $p = (h, H)$ and $\dot{H} = 0$, therefore a complete set of of integrals of motion allows the complete integration. the extremal curves are given by

$$t \mapsto (h(t)^T, H(t)) = (\exp(tH)h_0^T, H(0)).$$

If $\sigma \subset i\mathcal{R}$ is the spectrum of H, and π its characteristic polynomial, then $\mu \in \sigma$ determines its spectral projector

$$\pi_\mu = \frac{1}{\pi'(\mu)} \prod_{\nu \in (\sigma - \{\mu\})} (H - I\nu),$$

and the Lagrange-Sylvester formula yields

$$\exp(tH) = \sum_{\mu \in \sigma} e^{t\mu}\pi_\mu,$$

from these expressions we can pursue the geometric analysis of geodesics.

V. Geometric properties of geodesics for step-3

As mentioned above the general step-2 case is already complete, in previous papers [12] we present the low dimensional cases $n = 2$ and $n = 3$, that model classical particles in the plane and three-dimensional space respectively. In the first case the field points orthogonally to the plane, whereas in the second it has an arbitrary direction. Here we develop the case of step-3 that models a classical particle and linear magnetic fields with nontrivial slope, surprisingly this corresponds to the formulation of the famous Cartan five dimensional Pfaffian system [13].

We consider the rank-two distribution $\mathcal{D}$ in R^5 with coordinates (x, y, z, u, v) given by the vector fields

$$\begin{aligned} \tilde{X}_1 &= \partial x + z\partial u + y\partial z \\ \tilde{X}_2 &= \partial y + \frac{z}{2}\partial v, \end{aligned}$$

the non-trivial Lie brackets are as follows

$$\begin{aligned} \left[\tilde{X}_1, \tilde{X}_2\right] &= \frac{y}{2}\partial v - \partial z =: \tilde{X}_3 \\ \left[\tilde{X}_1, \tilde{X}_3\right] &= \partial u =: \tilde{X}_4 \\ \left[\tilde{X}_2, \tilde{X}_3\right] &= \partial v =: \tilde{X}_5. \end{aligned}$$

The growth vector of $\mathcal{D}$ is $(2, 3, 5)$. This system is usually called a system of Cartan type, it has been largely studied beginning with E. Cartan [13], who showed their relation with the exceptional Lie algebra $\mathcal{G}_2$. A modern treatment can be found for instance in [14].

Let us denote as N_5 Lie group corresponding to this Lie algebra. This group can be represented by R^5 endowed with the following group law

$$\begin{aligned} & (\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)(\beta_1,\beta_2,\beta_3,\beta_4,\beta_5) \\ = \quad & (\alpha_1+\beta_1,\alpha_2+\beta_2,\alpha_3',\alpha_4',\alpha_5'), \end{aligned}$$

where

$$\begin{aligned} \alpha_3' &= \alpha_3+\beta_3+\frac{1}{2}(\alpha_1\beta_2-\alpha_2\beta_1) \\ \alpha_4' &= \alpha_4+\beta_4+\frac{1}{2}(\alpha_1\beta_3-\alpha_3\beta_1) \\ &- \frac{1}{12}(\beta_1-\alpha_1)(\alpha_1\beta_2-\alpha_2\beta_1) \\ \alpha_5' &= \alpha_5+\beta_5+\frac{1}{2}(\alpha_2\beta_3-\alpha_3\beta_2) \\ &- \frac{1}{12}(\beta_2-\alpha_2)(\alpha_1\beta_2-\alpha_2\beta_1). \end{aligned}$$

The group N_5 acts on itself by left multiplication leaving the following two vector fields invariant

$$\begin{aligned} X_1 &= \partial x-\frac{y}{2}\partial z-\left(\frac{xy}{12}+\frac{z}{2}\right)\partial u-\frac{y^2}{12}\partial v, \\ X_2 &= \partial y+\frac{x}{2}\partial z+\frac{x^2}{12}\partial u+(\frac{xy}{12}-\frac{z}{2})\partial v. \end{aligned}$$

A direct calculation shows that

$$[X_1,X_2]=\partial z+\frac{x}{2}\partial u+\frac{y}{2}\partial v:=X_3.$$

There is a rotational action on N_5, which is encoded in the left invariant vector field

$$W=x\partial y-y\partial x+u\partial v-v\partial u,$$

which satisfies the following commuting relations

$$\begin{aligned} [X_1,W] &= \quad X_2, \\ [X_2,W] &= \; -X_1, \\ [X_4,W] &= \quad X_5, \\ [X_5,W] &= \; -X_4, \end{aligned}$$

these brackets allows us to consider the semi-direct sum $\mathcal{N}_5\oplus so_2$ from which the action of $N_5\propto \mathrm{SO}_2$ becomes transparent.

VI. Conclusion and perspectives

We have presented a model for non-holonomic system that leads naturally to the consideration of nilpotent Lie algebras of arbitrary nilpotency degree. We depict a Phillip Hall basis for the Lie algebra generated by means of a non-holonomic distribution of vector fields, that determine the problem. By using the Hamiltonian formalism we set the problem as a geodesic sub-Riemannian problem or equivalently, as an optimal control one. Necessary condition of the Pontryagin Maximum Principle provides tools for analyzing geometric properties of the geodesics, The case of step-3 in five variables is analyzed in this formalism. Further investigation on levels of higher degrees of nilpotency in arbitrary dimensions have to be pursued, as well as examples of this theoretical framework in particular non-holonomic dynamical systems.

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