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ON AVERAGE VALUES OF NONLINEAR FUNCTIONS OF OSCILLATING QUANTITIES AND THEIR APPLICATIONS

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Abstract

Oscillations of arguments in non-linear dependencies change the average values of respective functions. As the simplest example, the harmonic oscillations of the ball radius increase the average volume and surface area, while the average radius remains unchanged. Despite their elementary nature, such considerations are often ignored, which may lead to inaccuracies and errors. This paper presents a study of such effects in algebraic, geometric, and trigonometric relations, as well as in certain basic formulas of mathematical analysis. A number of applications in solving technical problems are considered; in particular, the influence of parameter oscillations on the efficiency of industrial operations. The results of the study may be of interest for the theory of vibrational processes and devices, and the theory of accuracy, as well as for the theory of control and optimal processes.

Key words

Parameter oscillations, average values, non-linear relations, paradoxes, oscillation mathematics, applications.

1 Introduction

Oscillations of parameters in basic mathematical formulas may change the average values of the functions for these parameters quite significantly. For example, under harmonic oscillations of the radius of a compressible spherical particle, such as a bubble, its average (effective) surface area and volume increase due to the non-linearity of their dependence on the radius. Such effects are typical for all non-linear mathematical relations and are therefore essential for all areas of engineering. This raises the following two main questions: 1) what are these effects, and how significant are they when solving a specific technical problem; and 2) can they be used to achieve useful goals?

This paper provides a general overview of the corresponding effects, establishes the main regularities, and illustrates them in application to certain basic formulas of algebra, geometry, trigonometry and mathematical analysis. The potential applications of these effects in the theory of vibrational processes and devices, the theory of accuracy, the theory of control and optimal processes are then covered.

With regard to the first group of applications mentioned, it should be noted that the problems of oscillatory effects on dynamic systems described by differential equations have currently been studied in detail [Blekhman, 2000; Blekhman, 2012; Kremer, 2016; Blekhman and Sorokin, 2016; Blekhman, 2018]. The study presented here covers the simplest case of parameter oscillation effects on final relations (formulas). It is also to these relations that the explanation of the above-mentioned more complex effects may be reduced to. The connection between the above and the concept of hidden parameters ([Blekhman, 2018], pp. 53–68) should also be noted: oscillations of quantities may be regarded as caused by hidden parameters, leading to unexpected phenomena.

2 Oscillating argument function averaging

2.1 General case

Let us assume a sufficiently smooth function of several variables

$$y = f(x_1, x_2, ..., x_k)$$

with the variables x_s subject to minor oscillations Δx_s with respect to certain fixed values $x_{10}, ..., x_{k0}$, i.e.

$$x_s = x_{s0} + \Delta x_s. \qquad (1)$$

The functions of $\Delta x_s = \Delta x_s\,(x_{s0}, t)$ shall be regarded as periodic functions of time t (or another parameter) with a period of $T = 2\pi/\omega$, where ω is the frequency.

Expansion of the function f into a Taylor series near the values of $x_s = x_{s0}$ renders

$$f\,(x) = f\,(x_{10}, ..., x_{k0}) + \sum_{s=1}^{k} \frac{\partial f}{\partial x_s}\Delta x_s + \frac{1}{2}\sum_{s=1}^{k}\sum_{r=1}^{k}\left(\frac{\partial^2 f}{\partial x_s \partial x_r}\right)\Delta x_s \Delta x_r + ..., \tag{2}$$

where the derivatives of the function f are calculated at the point of $x_{10}, ..., x_{k0}$, and the implicit terms are of more than the second order of smallness as compared to Δx_s. While assuming the small deviations of Δx_s, the calculation will be reduced to the explicit terms only. Besides, after the following averaging summands with odd numbers Δx_s becomes zero and only terms of fourth and higher orders relative to small deviations of Δx_s would be included in Eq. (2). With the average values of the function over a period of T in angle brackets, let us assume that

$$\langle \Delta x_s \rangle = \frac{1}{T}\int_0^T \Delta x_s dt = 0,$$

i.e. that the deviations of Δx_s average to zero.

Then, the averaging of expression (2) renders

$$\langle f\,(x)\rangle = F\,(x_{10}, ..., x_{k0}) = f\,(x_{10}, ..., x_{k0}) + \frac{1}{2}\sum_{s=1}^{k}\sum_{r=1}^{k}\left(\frac{\partial^2 f}{\partial x_s \partial x_r}\right)\langle \Delta x_s \Delta x_r \rangle\,. \tag{3}$$

Here and below, all expressions for average values, within the accepted accuracy, will use the equal sign instead of $\approx$.

Assuming that the values of Δx_s change harmonically with the same frequency of $\omega = 2\pi/T$, but with different amplitudes a_s and phases β_s, i.e.

$$\Delta x_s = a_s \sin\left(\omega t + \beta_s\right), \tag{4}$$

we obtain

$$\langle \Delta x_s \Delta x_r \rangle = \frac{1}{2} a_s a_r \cos\left(\beta_s - \beta_r\right),$$

and then

$$F\,(x_{10}, ..., x_{k0}) = f\,(x_{10}, ..., x_{k0}) + \frac{1}{4}\sum_{s=1}^{k}\sum_{r=1}^{k}\left(\frac{\partial^2 f}{\partial x_s \partial x_r}\right) a_s a_r \cos\left(\beta_s - \beta_r\right). \tag{5}$$

Note that the amplitudes a_s and phases β_s may be functions x_{s0}. This is specifically indicated where necessary. It should also be noted that the additional terms in Eqs. (3), (4), and (5) are of the second order of smallness with respect to the deviations Δx_s.

2.2 Single variable case

Let us consider the simplest special case with one argument $x_1 = x$ that may be used to establish a number of general patterns. In this case,

$$F\,(x_0) = \langle f\,(x_0 + a\sin\omega t)\rangle = f\,(x_0) + \frac{1}{4} f''\,(x_0)\,a^2. \tag{6}$$

If $f''\,(x_0) \neq 0$, which is expected, then, due to the oscillations of the argument, the value of Δx added to the function f is positive for a concave curve $y = f(x)$ at the point of x_0 and is negative for a convex curve (see Figure 1).

Similarly to [Blekhman and Sorokin, 2016], the $y = F(x_0)$ curve may be referred to as a vibration-transformed curve with respect to the initial $y = f(x_0)$ curve. The vibration-transformed curve is smoother than the initial curve, with the maximum points going down and the minimum points going up (see Figure 2). The same may be deduced from Figure 1.

Figure 2 also shows that argument oscillations shift and may even eliminate the real roots of the equation $f(x_0) = 0$. Note that the latter is observed, for example, for oscillatory effects in the Zeldovich–Frank-Kamenetsky flame propagation model and the nerve impulse propagation model [Blekhman and Sorokin, 2016].

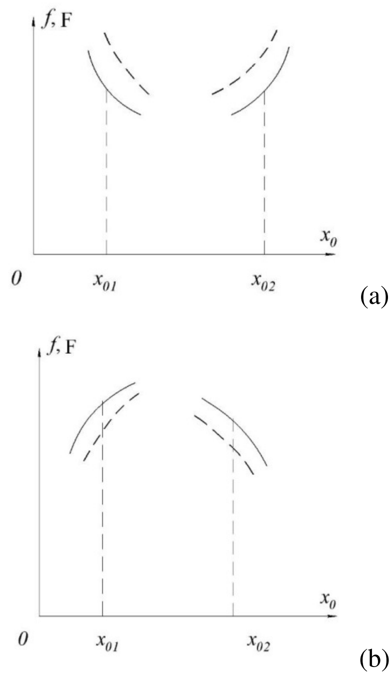

Figure 1. Fragments of curves, solid lines for $f(x_0)$ and dotted lines for $F(x_0)$: (a) showing $f''\,(x_0) > 0$, (b) showing $f''\,(x_0) < 0$.

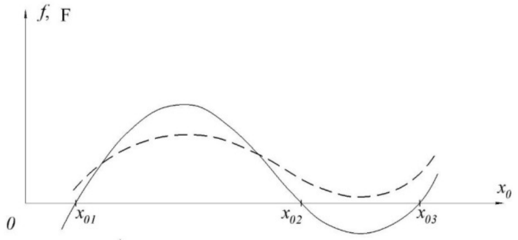

Figure 2. Initial curve $f(x_0)$ (solid line) and vibration-transformed curve $F(x_0)$ (dotted line).

2.3 Two variable case

In the case of two variables (x_1 and x_2), Eq. (5) may be used to find

$$F\left(x_{10}, x_{20}\right) = f\left(x_{10}, x_{20}\right) + \frac{1}{4}\left[a_1^2 \frac{\partial^2 f}{\partial x_1^2} + 2a_1 a_2 \cos\left(\beta_1 - \beta_2\right) \frac{\partial^2 f}{\partial x_1 \partial x_2} + a_2^2 \frac{\partial^2 f}{\partial x_2^2}\right], \quad (7)$$

where the derivatives are calculated at the point of (x_{10}, x_{20}).

Similarly to the simplest case considered above, smoothing effects take place here. At the same time, the vibration-transformed function depends on the phase difference $\beta_1 - \beta_2$. This is essential for the possibility of control; in particular, for vibrational process machines [Fradkov, 2007; Blekhman, 2012; Andrievskii et al., 2016; Fradkov et al., 2016; Tomchina et al., 2018; Tomchina et al., 2021; Tomchina, 2022] , when the function F represents a characteristic of the process (see Sec. 4).

2.4 Feasible generalizations

The results obtained may be easily generalized for the cases of any periodic and random effects. The case of small deviations Δx was covered above. It has been shown that their contribution to the average value of a function is of the order of a squared deviation amplitude. In many cases, and in the case of finite deviations, it is quite easy to find the vibration-transformed function F. Therefore, for $f(x) = x^2 + bx + c$, $F(x_0) = x_0^2 + bx_0 + c + \frac{1}{2}x_1^2$ is true at $x = x_0 + x_1 \sin\omega t$ and $F(x_0) = x_0^3 + ax_0^2 + (b + \frac{3}{2}a^2)x_0 + c + \frac{1}{2}ax_1^2$ is true at $f(x) = x^3 + ax^2 + bx + c$, which is consistent with Eq. (6). Correspondingly, the roots of the equation $f(x) = 0$ also change and may be eliminated.

3 Certain examples of average values

All formulas given below are approximate and obtained by using the two first terms of decompositions, given in Eq. (5) and its special cases (6) and (7).

3.1 Algebra

With x_s and Δx_s obtained using Eqs. (1) and (4), i.e.

$$a = a_0 + a_1 \sin(\omega t + \beta_1), \; b = b_0 + b_1 \sin(\omega t + \beta_2), \quad (8)$$

the following is true:

$$\langle ab \rangle = a_0 b_0 + \frac{1}{2} a_1 b_1 \cos(\beta_1 - \beta_2);$$

$$\left\langle \frac{a}{b} \right\rangle = \frac{a_0}{b_0} + \frac{1}{2}\frac{b_1^2}{b_0^2}\left[\frac{a_0}{b_0} - \frac{a_1}{b_1}\cos(\beta_1 - \beta_2)\right];$$

$$\left\langle a^2 \right\rangle = a_0^2 + \frac{1}{2}a_1^2; \quad \left\langle a^3 \right\rangle = a_0^3 + \frac{3}{2}a_0 a_1^2;$$

$$\left\langle (a+b)^2 \right\rangle = (a_0 + b_0)^2 + \frac{1}{2}\left[a_1^2 + b_1^2 + 2a_1 b_1 \cos(\beta_1 - \beta_2)\right].$$

Scalar product of vectors

1) a = varia, b = varia, α = const. Here a and b are defined by Eq. (8), α is angle between the oscillating vectors $\vec{a}$ and $\vec{b}$.

$$\left\langle \vec{a} \cdot \vec{b} \right\rangle = a_0 b_0 \cos\alpha + \frac{1}{2} a_1 b_1 \cos\alpha \cos(\beta_1 - \beta_2).$$

Note that many physical quantities are expressed in terms of scalar or vector products.

2) a = const, b = const, α = varia ($\alpha = \alpha_0 + \alpha_1 \sin\omega t$).

$$\left\langle \vec{a} \cdot \vec{b} \right\rangle = ab \cos\alpha_0 (1 - \frac{1}{4}\alpha_1^2).$$

3) a = varia, b = varia, α = varia (general case, three variables).

$$\left\langle \vec{a} \cdot \vec{b} \right\rangle = a_0 b_0 \cos\alpha + \frac{1}{4}\left[\cos\alpha_0\left(2a_1 b_1 \cos(\beta_1 - \beta_2) - \alpha_1^2 a_0 b_0\right) - 2\alpha_1 \sin\alpha_0\left(a_1 b_0 \cos(\beta_1 - \beta_3) + b_1 a_0 \cos(\beta_2 - \beta_3)\right)\right].$$

Module of vector product

The formulas are similar to the expressions for the scalar product when $\cos\alpha$ is replaced with $\sin\alpha$ and vice versa. In cases 1) and 2), the following is true:

1) $\left\langle \left|\vec{a} \times \vec{b}\right| \right\rangle = a_0 b_0 \sin\alpha + \frac{1}{2} a_1 b_1 \sin\alpha \cos(\beta_1 - \beta_2)$;

2) $\left\langle \left|\vec{a} \times \vec{b}\right| \right\rangle = ab \sin\alpha_0 (1 - \frac{1}{4}\alpha_1^2)$.

The respective behaviour of the roots of the equation $f(x) = 0$ is covered in Sec. 2.2.

3.2 Geometry

Area of a circle

$$S = \left\langle \pi r^2 \right\rangle = \left\langle \pi\left(r_0 + r_1 \sin(\omega t + \beta_1)\right)^2 \right\rangle = \pi r_0^2 + \frac{\pi}{2} r_1^2.$$

The average area of a circle increases even with the average radius and circumference remaining unchanged. This may be explained by a transition to a certain surface as a result of averaging, that is, by a change in the geometry. The works by Mark Levi (namely, [Levi, 1999]) on the relationship between geometry and physics using the example of the Kapitza pendulum and the Paul trap may be mentioned in this regard.

Ball volume

$$V = \left\langle \frac{4}{3}\pi r^3 \right\rangle = \frac{4}{3}\pi \left\langle [(r_0 + r_1 \sin(\omega t + \beta_1)]^3 \right\rangle = \frac{4}{3}\pi r_0^3 + 2\pi r_0 r_1^2.$$

Rectangle area

$$S = \langle ab \rangle = a_0 b_0 + \frac{1}{2} a_1 b_1 \cos(\beta_1 - \beta_2).$$

Volume of a cuboid

$$V = \langle abc \rangle = a_0 b_0 c_0 + \frac{1}{2} a_0 b_0 c_0 \left[\frac{a_1 a_2}{a_0 b_0} \cos(\beta_1 - \beta_2) + \frac{a_1 a_3}{a_0 c_0} \cos(\beta_1 - \beta_3) + \frac{a_2 a_3}{b_0 c_0} \cos(\beta_2 - \beta_3) \right].$$

Volume of a cube

$$V_k = \left\langle a^3 \right\rangle = a_0^3 + \frac{3}{2} a_0 a_1^2.$$

Distance between two points on a plane, the Pythagorean theorem.
When the sides of a right triangle are expressed as

$$a = a_0 + a_1 \sin(\omega t + \beta_1), \quad b = b_0 + b_1 \sin(\omega t + \beta_2),$$

the average value of the square of the hypotenuse $\left\langle c^2 \right\rangle$ will be:

$$\left\langle c^2 \right\rangle = a_0^2 + b_0^2 + \frac{1}{2}(a_1^2 + b_1^2).$$

Similarly to a circle, $\left\langle c^2 \right\rangle$ differs from the sum of the squares of the sides, despite the fact that the average values for the sides have not changed. It should be noted that $\left\langle c^2 \right\rangle \neq (\langle c \rangle)^2$ (the average value of a function is not equal to a function of the average value) and, unlike $(\langle c \rangle)^2$, $\left\langle c^2 \right\rangle$ does not depend on the phase difference $\beta_1 - \beta_2$. With that, both of these quantities are greater than the sum of the squares of the sides. A more general formula for the distance between two points on a plane that oscillate in arbitrary directions may thus be easily obtained.

3.3 Trigonometry

Let us consider certain relations, assuming, as before, that $\alpha = \alpha_0 + \alpha_1 \sin(\omega t + \beta_1)$, $\gamma = \gamma_0 + \gamma_1 \sin(\omega t + \beta_2)$:

$$\begin{aligned}
\langle \sin \alpha \rangle &= (\sin \alpha_0)\left(1 - \alpha_1^2/4\right); \\
\langle \cos \alpha \rangle &= (\cos \alpha_0)\left(1 - \alpha_1^2/4\right); \\
\langle \tan \alpha \rangle &= (\tan \alpha_0) \left[1 + \frac{1}{2} \left(\tan^2 \alpha_0 + 1 \right) \alpha_1^2 \right]; \\
\langle \sin 2\alpha \rangle &= (\sin 2\alpha_0) \left(1 - \alpha_1^2\right); \\
\langle \cos 2\alpha \rangle &= (\cos 2\alpha_0) \left(1 - \alpha_1^2\right);
\end{aligned}$$

$$\begin{aligned}
\langle \sin(\alpha + \gamma) \rangle &= \sin(\alpha_0 + \gamma_0) \times \left[1 - \frac{1}{4} \left(\alpha_1^2 + 2\alpha_1 \gamma_1 \cos(\beta_1 - \beta_2) + \gamma_1^2 \right) \right]; \\
\langle \cos(\alpha + \gamma) \rangle &= \cos(\alpha_0 + \gamma_0) \times \left[1 - \frac{1}{4} \left(\alpha_1^2 + 2\alpha_1 \gamma_1 \cos(\beta_1 - \beta_2) + \gamma_1^2 \right) \right]; \\
\langle \tan(\alpha + \gamma) \rangle &= \tan(\alpha_0 + \gamma_0) \times \left[1 + \frac{1}{2} \left(1 + \tan^2(\alpha_0 + \gamma_0) \right) \times \left(\alpha_1^2 + 2\alpha_1 \gamma_1 \cos(\beta_1 - \beta_2) + \gamma_1^2 \right) \right].
\end{aligned}$$

3.4 Elements of mathematical analysis

Derivative

$$f'(x) = f'(x_0 + x_1 \sin \omega t) = f'(x_0) + f''(x_0) x_1 \sin \omega t + \frac{1}{2} f'''(x_0) (x_1 \sin \omega t)^2 + \ldots,$$

$$\langle f'(x) \rangle \approx f'(x_0) + \frac{1}{4} f'''(x_0) x_1^2.$$

For a definite integral $\int_a^b f(x)dx = \Phi(b) - \Phi(a)$, where $\Phi'(x) = f(x)$, $\Phi(x)$ is the primitive, and a and b are the variables established by Eq. (8), we obtain

$$\begin{aligned}
\left\langle \int_a^b f(x)dx \right\rangle &= \\
&\langle \Phi[b_0 + b_1 \sin(\omega t + \beta_2)] - \Phi[a_0 + a_1 \sin(\omega t + \beta_1)] \rangle \\
&= \Phi(b_0) - \Phi(a_0) + \frac{1}{4} \left[\Phi''(b_0) b_1^2 - \Phi''(a_0) a_1^2 \right] \\
&= \int_{a_0}^{b_0} f(x)dx + \frac{1}{4} \left[f'(b_0) b_1^2 - f'(a_0) a_1^2 \right].
\end{aligned}$$

For an indefinite integral $\int f(x)dx$, assuming that $x = x_0 + a \sin \omega t$, according to Eq. (6), we get

$$\left\langle \int f(x)dx \right\rangle = \Phi(x_0) + \frac{1}{4} f'(x_0) a^2 + C,$$

where C is an arbitrary constant.

3.5 On the averaging result

The above examples of expressions for the average values of functions provide a certain specification for the multiplication table, the formula for the area of a circle, the Pythagorean theorem, etc. In particular, the average value of a product of two oscillating quantities, perceived as numbers 2 and 2, may be both greater than 4, less than 4, and also equal to 4. When considering functions xy, x^2, $\sqrt{x}$, and a^x with oscillating arguments x and y, the following is true for the average values: $2 \cdot 2 \neq 2^2 \neq \sqrt{16} \neq 4$ and $0 \cdot 0 \neq 0^2 \neq 0$.

In a general case, deviations from the "unperturbed" values may always either be ≥ 0 or ≤ 0, or may have any sign and be equal to zero, as in the case of a product of two quantities.

Let us also note that parameter oscillations may not violate the rule that the arithmetic mean of numbers is never less than their geometric mean (the Cauchy inequality).

4 Industrial processes with parameter oscillations

4.1 Cases of one or two parameters

When analysing the effects of parameter oscillations in industrial processes, it is assumed that the function of $y = f(x_1, x_2, ..., x_k)$ is a measure of process efficiency. In this case, for a single parameter of $x = x_1$, the average value of the indicator is determined by formula (6). It follows from this formula that, if $f''(x_0) > 0$ and the curve is concave, oscillations of the parameter would improve the process efficiency and, if $f''(x_0) < 0$ (the curve is convex), its efficiency would be lower (see Figure 3). The most significant process improvement is therefore observed near the minimum points and the lowest process efficiency is near the maximum points. Note that this case is mentioned in the book [Blekhman, 2018] (pp. 672–673).

For a case of two parameters, the average value of the indicator is expressed by Eq. (7). It is essential that, in this case, process efficiency may be affected both through the selection of oscillation amplitudes a_1 and a_2 and using the phase difference of the oscillations.

4.2 Examples for the single variable case

4.2.1 Effects of oscillations in the particle size distribution of a material on its sizing performance The sizing quality for a material is usually assessed by the recovery of the negative size classes $(-\delta)$ (finer than a certain size δ) into the fine product $\varepsilon_-(\delta)$ or by the Hancock classification efficiency $E_-(\delta)$. Indicators associated with the narrow class recovery curve are also used (the selection of any calculated class is not covered in this paper). The Hancock criterion has been borrowed from the field of ore processing efficiency evaluation, where it is used to assess the recovery of a metal or mineral into the concentrate. The sizing process notably implies no clear differentiation between valuable and non-valuable components (valuable metal and waste rock). It is believed that particles smaller than δ in the fine product represent the valuable component (an equivalent of metal), while larger particles δ represent waste rock. The recovery curve ε_- usually has a characteristic S-shape with a kink resembling a Gaussian distribution function, i.e. has a form similar to the part of the curve in Figure 3 to the right of the maximum point. The efficiency curve E_- in the area of practical interest is concave with a maximum point, similar to the central part of the curve in Figure 3. Numerous attempts are known to approximate these indicators by various dependencies (for example, see [Povarov, 1978] for hydrocyclones).

A qualitative assessment of the effects of oscillating particle size distributions, for example, fluctuations in the amount of negative classes in the initial material, may therefore be made using the shapes of the curves in Figure 3. In accordance with Eq. (6) for the case of a single parameter and the curve shape visualization rule, the values will be decreasing on the section of the recovery curve ε_- from the maximum point $x_{\max}$ to the point of inflection x_2^* and will be increasing at $x > x_2^*$. The curve will become flatter and further away from its ideal stepped shape. The values of E_- will decrease accordingly. Therefore, parameter oscillations will reduce process efficiency.

Knowing the dependencies of E_- and ε_- on the parameter, the respective quantitative change may be easily estimated using Eq. (6). When applied to separation in hydrocyclones, such an assessment may be conveniently performed using the dependences of efficiency E_- and recovery ε_- on the amount $\alpha(\delta)$ of particles smaller than δ in the initial material, as proposed in [Povarov and Blekhman, 1978] in order to predict the separation performance.

4.2.2 Effects of oscillations in metal grades in the ore on its recovery into the concentrate The importance of this dependence is due, in particular, to the fact

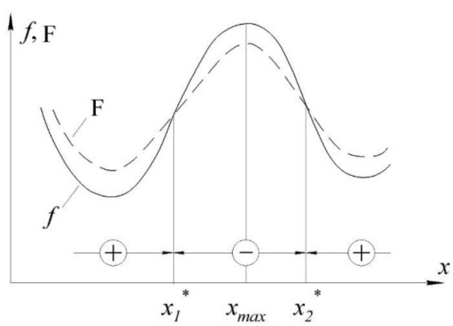

Figure 3. Efficiency indicators with one parameter: F is the indicator or with parameter oscillations, f is the initial indicator; $\oplus$ are indicator upward increments, $\ominus$ are indicator downward increments; and x_1^*, x_2^* are the inflection points of the curve $y = f(x)$

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that it may be used to assess the efficiency of ore blending upstream of ore processing or metallurgical processes.

Metal recovery into the concentrate ε is by definition expressed as

$$\varepsilon = \gamma \frac{\beta}{\alpha} = \frac{\alpha - \upsilon}{\beta - \upsilon} \frac{\beta}{\alpha},$$

where α, β and υ are the metal grades in ore, concentrate, and tailings, respectively, and γ is the concentrate yield.

Considering, as is often believed [Bastan and Klyuchkin, 1976; Mashevsky et al., 1977; Pokhodzey, 1979; Bastan et al., 1979; Boloshin and Gindin, 1981], that the quantities of β and υ may be deemed virtually constant (regulated and largely sought to be stabilized during the process), we obtain

$$\varepsilon(\alpha) = \frac{\beta}{\beta - \upsilon}\Big(1 - \frac{\upsilon}{\alpha}\Big). \quad (9)$$

In [Bastan and Klyuchkin, 1976; Mashevsky et al., 1977; Pokhodzey, 1979; Bastan et al., 1979; Boloshin and Gindin, 1981], this dependence is sometimes referred to as the estimated separation performance curve and is analysed using the theory of probability, assuming that α is a random variable. Here, we will assume that $\alpha = \alpha_0 + \alpha_1 \sin \omega t$ and that the oscillation period is much longer than the process stabilization time. Then, according to (6) and (9), we obtain

$$\langle \varepsilon \rangle = \varepsilon(\alpha_0) - \frac{1}{2} \frac{\beta \upsilon}{\beta - \upsilon} \frac{\alpha_1^2}{\alpha_0^3}. \quad (10)$$

Thus, the wider spread of values for α always reduces the recovery of metal into the concentrate, and Eq. (10) allows us to estimate this reduction. Qualitatively, this result follows from the fact that the expected separation performance curve (9), as well as the real curve, are both convex.

Note that Eq. (10), coincides (up to the numerical coefficient in the second term) with the equation obtained in [Bastan and Klyuchkin, 1976] (further detailed in ([Mashevsky et al., 1977] and subsequently in [Pokhodzey, 1979]) for the changes of ε with α parameter oscillations within $\bar{\alpha} \pm \sigma$ (where σ is the standard deviation of α from the average grades). According to [Bastan and Klyuchkin, 1976; Mashevsky et al., 1977; Pokhodzey, 1979], the change formula for ε takes the following form:

$$M_\varepsilon = -k \frac{\beta \upsilon}{\bar{\alpha}(\beta - \upsilon)} V_\alpha^2,$$

where $k > 0$ is a numerical coefficient.

In Eq. (10), the average value of $\bar{\alpha}$ is equivalent to α_0, σ_α is equivalent to α_1, and the coefficient of variation $V_\alpha = \sigma / \bar{\alpha}$ is equivalent to parameter α_1/α_0.

5 Examples of other applications

Similar to Sec. 4, the following two examples refer to the simplest case of a single parameter.

5.1 Effects of oscillating gas content on the speed of sound in a liquid

The speed of sound in a liquid depends on the availability of air bubbles in it and plays a decisive role in the hydrodynamic effects. In particular, in a vibrating open vessel filled with liquid, low values of the speed of sound in the presence of bubbles are essential for the occurrence of the effects causing air bubble immersion deeper into the vessel and the flotation of heavy particles [Blekhman et al., 2008; Blekhman et al., 2011; Blekhman et al., 2012].

The dependence of the speed of sound c on the volumetric gas content α may be established with acceptable accuracy and in a wide range of values of α using the Batchelor formula [Batchelor, 1968; Loitsyansky, 1970]:

$$c = \frac{10}{\sqrt{\alpha(1-\alpha)}}. \quad (11)$$

The speed c is paradoxically small here and amounts to 20±5 m/s at $\alpha = 0.2 \div 0.8$. At $\alpha \to 0$ and $\alpha \to 1$, Eq. (11) renders infinite values. A more accurate formula rendering the speed of sound in water and in air at α=0 and α=1, respectively, was obtained in [Blekhman et al., 2009]. Here, however, we use the simpler Eq. (11). With oscillating gas content $\alpha = \alpha_0 + \alpha_1 \sin \omega t$, the average value of $\langle c \rangle$, according to Eqs. (6) and (11), will be

$$\langle c \rangle = \frac{10}{\sqrt{\alpha_0(1-\alpha_0)}} \left[1 + \frac{8(\alpha_0 - 1/2)^2 + 1}{16\alpha_0^2(1-\alpha_0)^2} \alpha_1^2\right].$$

Therefore, oscillations of α increase the average speed of sound, with more significant increases at low and high values of α and smaller increases near the minimum point of α=0.5. In particular, the following is true: c=20 m/s and $\langle c \rangle = 20(1 + \alpha_1^2)$ at α=0.5, and $\langle c \rangle$=20.2 m/s at α_1=0.1; c=25 m/s and $\langle c \rangle = 25(1 + 4.20\alpha_1^2)$ at α=0.2 and 0.8, and $\langle c \rangle$=26.05 m/s at α_1=0.1. The increase in $\langle c \rangle$ is also evident from the fact that the dependency graph for $c(\alpha)$ is a concave curve (at α values sufficiently away from the limit values).

5.2 Effects of body velocity oscillations on the theory of relativity relations

The relativistic change in parameter values is described by simple relations. For example, the reduction in the body size l and the increase in body mass m with higher $v_c = v/c$ ratios between body velocity v and the speed of light A are expressed by the following equation:

$$l = l_0\sqrt{1 - v_c^2}, \quad m = \frac{m_0}{\sqrt{1 - v_c^2}}, \quad (12)$$

where l_0 and m_0 are the length and mass of the body in the reference frame relative to which the body is at rest, respectively.

The length change graph is a decreasing convex curve, and the mass change graph is an increasing concave curve. According to the law of $v = v_0 + v_1 \sin \omega t$, i.e. $v_c = v_{c0} + v_{c1} \sin \omega t$, where $v_{c0} = v_0/c$, $v_{c1} = v_1/c$, body velocity changes will therefore additionally reduce the body length and increase its mass. Namely, according to (6) and (12), the following will be true:

$$\langle l \rangle = l_0 \sqrt{1 - v_{c0}^2} \left[1 - \frac{1}{4\left(1 - v_{c0}^2\right)^2} v_{c1}^2 \right],$$

$$\langle m \rangle = \frac{m_0}{\sqrt{1 - v_{c0}^2}} \left[1 + \frac{1 + 2v_{c0}^2}{4\left(1 - v_{c0}^2\right)^2} v_{c1}^2 \right].$$

6 Similarities with other approaches

The approach proposed allows us to obtain mathematical relations for the average values of nonlinear functions of oscillating quantities and is not initially associated with any specific problems. One can, however, draw an analogy with the approach of vibrational mechanics, an analytical research method for studying the effects of vibration on nonlinear dynamic systems [Blekhman, 2000; Blekhman, 2012; Kremer, 2016; Blekhman and Sorokin, 2016; Blekhman, 2018].

Similar to how vibrational mechanics represents mechanics in the perspective of an observer who ignores rapid motion, the approach covered herein, which may rather conditionally be referred to as oscillatory mathematics, represents mathematics in the perspective of an observer who perceives only average values of quantities. The mathematical averaging results are accurate in this case (with due account of the discarded terms of a higher order), while, in vibrational mechanics, they may depend on the accuracy of the solution for the rapid motion equation.

The method for establishing the average values of abstract quantities may be selected arbitrarily; however, the variation of parameters in the physical ratios and laws must not be in conflict with the physical principles these ratios and laws are based on.

As mentioned previously, the above considerations may also be associated with the concept of hidden parameters (the mechanics of systems with hidden parameters), if the oscillations are regarded as a result of the effects of these parameters.

7 Conclusion

This paper demonstrates that an analysis of average values of oscillating quantities in non-linear relations renders a paradoxical result: ordinary, including elementary, dependencies become inaccurate or inapplicable. This result is more than just a mathematical curiosity as it may be tied to a number of complex nonlinear oscillatory phenomena of both fundamental and applied significance. The above research complements the study on the effects of vibration on dynamic systems described by differential equations, while drawing attention to one of their mathematical foundations.

Some applications are outlined for the approach, including those aimed at improving industrial processes by taking into account the perturbing effects and through the use of vibration.

It appears that the data obtained may be used in the theory of vibrational processes and devices, the theory of measurement accuracy, the theory of control and optimal processes, possibly in the theory of fuzzy sets, and in applied problems (such as the theory behind ore blending). In this context, a number of generalizations are required, the case of several parameters has to be considered in more detail and ways have to be developed to implement controlled parameter changes. Respective research is expected to be done in the future.

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References

Andrievskii, B. R., Blekhman, I. I., Blekhman, L. I., Boikov, V. I., Vasil'kov, V. B., and Fradkov, A. L. (2016). Education and research mechatronic complex for studying vibration devices and processes. In *J. Mach. Manuf. Reliab.*, **45**, pp. 369–374.

Bastan, P. P., Klyuchkin, E. I. (1976). On the effect of ore grades on the recovery of metal into concentrate. *Obogashchenie Rud (Mineral Processing Journal)*, No. 1, pp. 24–25.

Bastan, P. P., Azbel, E. I. Klyuchkin, E. I. (1979). *Theory and practice of ore blending*. Nedra, Moscow.

Batchelor, G. K. (1968). Compression waves in a suspension of gas bubbles in a liquid. *Mechanics: Collection of Translations of Foreign Articles*, 4, pp. 65–84.

Blekhman, I. I. (2000). *Vibrational Mechanics. Nonlinear Dynamic Effects, General Approach, Applications*. World Scientific, Singapore.

Blekhman, I. I., Blekhman, L. I., Vaisberg, L. A., Vasil'kov, V. B., Yakimova, K. S. (2008). "Anomalous" phenomena in fluid under the action of vibration. *Doklady Physics.* **53**, pp. 520–524. https://doi.org/10.1134/S1028335808100054

Blekhman, I. I., Vakulenko, S. A., Indeitsev, D. A., Mochalova, Yu. A. (2009). Generation and motion of a gas-liquid suspension in a vibrating vessel containing a liquid with a free surface. Collection Proceedings of the XVI Symposium "Dynamics of Vibro-Impact (Strongly Nonlinear) Systems". Moscow. pp. 61–71.

Blekhman, I. I., Blekhman, L. I., Vasil'kov, V. B., Sorokin, V. S., Yakimova, K. S. (2011). Motion of a gas bubble in an oscillating gas-saturated liquid. *Obogashchenie Rud (Mineral Processing Journal)*, No. 5, pp. 30–37.

Blekhman, I. I., Blekhman, L. I., Sorokin, V. S., Vasilkov, V. B., Yakimova, K. S. (2012). Surface and volumetric effects in a fluid subjected to high-frequency vibration**.** *Proc. of the Institution of Mechanical Engineers, Part C, Journal of Mechanical Engineering Science. London*, **226**, pp. 2028–2043.Doi: 10.1177/0954406211433260

Blekhman, I. I. (2012). Oscillatory strobodynamics – A new area in nonlinear oscillations theory, nonlinear dynamics and cybernetical physics. *Cybernetics and Physics*, **1**, pp. 5–10.

Blekhman, I. I., Sorokin, V. S. (2016). Effects produced by oscillations applied to nonlinear dynamic systems: a general approach and examples. *Nonlinear Dynamics.*, **83**, pp. 2125–2141. Doi: 10.1007/s11071-015-2470-x

Blekhman, I.I. (2018). *Vibratsionnaya mekhanika i vibratsionnaya reologiya (teoriya i prilozheniya) (Vibration Mechanics and Vibration Rheology: Theory and Applications)*. Fizmatlit, Moscow.

Boloshin, N. N., Gindin, M. A. (1981). Method for estimating the expected blending process efficiency with account of the dynamic characteristics of the concentration process. *Obogashchenie Rud (Mineral Processing Journal)*, No. 4, pp. 3–6.

Fradkov, A. L. (2007). *Cybernetical physics: from control of chaos to quantum control*. Springer-Verlag, Heidelberg-New York. 242 p.

Fradkov, A., Tomchin, D., Gorlatov, D., and Tomchina O., (2016). Control of oscillations in vibration machines: Start up and passage through resonance. *Chaos*. **26**, 116310.

Kremer, E. (2016). Slow motions in systems with fast modulated excitation. *Journal of Sound and Vibration*, **383**(5.2), pp. 295–308. http://dx.doi.org/10.1016/j.jsv.2016.07.006

Levi, M. (1999). Geometry and physics of averaging with applications. *Physica D: Nonlinear Phenomena*. **132**(1–2), pp. 150–164.

Loitsyansky, L. G. (1970). *Fluid and gas mechanics*. Nauka, Moscow.

Mashevsky, G. N., Pavlov, A. I., Pokhodzey, B. B. (1977). On the issue concerning the effects of ore grade oscillations on the recovery of metal into concentrate // *Obogashchenie Rud (Mineral Processing Journal)*, No. 1, pp. 13–15.

Povarov, A. I. (1978). *Hydrocyclones at mineral processing plants*. Nedra, Moscow.

Povarov, A. I., Blekhman, L. I. (1978). Method for calculating the size characteristics of classification products obtained in hydrocyclones. *Obogashchenie Rud (Mineral Processing Journal)*, No. 5, pp. 45–47.

Pokhodzey, B. B. (1979). Analysis of the effects of ore grade oscillations on the recovery of metal into concentrate. *Obogashchenie Rud (Mineral Processing Journal)*, No. 4, pp. 17–21.

Tomchina, O. (2018). Control of vibrational field in a cyber-physical vibration unit. *Cybernetics and Physics*, **7**(2), pp. 144–151.

Tomchina, O., Gorlatov, D., Tomchin, D., Epishkin, A. (2021). Control of passage through resonance zone for 1-rotor vibration unit with time-varying load. *Cybernetics and Physics*, **10**(1), pp. 97–105.

Tomchina, O. (2022). Vibration field control of a two-rotor vibratory unit in the double synchronization mode. *Cybernetics and Physics*, **11**(2.1), pp. 246–252.