Indefinite Damping in Mechanical Systems and Gyroscopic Stabilization

Wolfhard Kliem

Abstract
The paper deals with gyroscopic stabilization of unstable linear mechanical systems with positive definite mass and stiffness matrices, respectively, and an indefinite damping matrix. A stabilization is obtained by adding a suitable skew symmetric gyroscopic matrix to the damping matrix. After investigating several special cases we find an appropriate solution of the Lyapunov matrix equation for the general case. An example shows the deviation of the stability limit found by the Luapunov method from the exact value.

Introduction
Only few papers are dealing with indefinite damping matrices in linear systems of 2nd order differential equations. Indefinite damping matrices can cause instability. In the meagre literature on the subject we can find remarks that modelling of sliding bearings and cutting of metals can lead to negative damping (dry friction) and therefore to instability (self-excited vibrations).
One of the motivations for the present work is the industrial problem of avoiding shrieking of car breaks. Models show negative damping terms in the governing equations induced by a decreasing friction characteristic, see [1].
Consider a linear mechanical system of differential equations of the form

$$M\ddot{x} + D\dot{x} + Kx = 0 \ . \qquad (1)$$

The mass matrix M and the stiffness matrix K are both real symmetric and positive definite ($M^T = M > 0,\ K^T = K > 0$), and the symmetric damping matrix $D^T = D$ is assumed to be *indefinite.* In the following we choose $M = I$ (the identity matrix). The system (1) can be stable or unstable due to the indefinite damping matrix. Let us assume instability, then the question arises how to stabilize the system.

If we stick to linearity, the addition of a gyroscopic force $G\dot{x}$ with a skew-symmetric matrix $G, (G^T = -G)$ on the left hand side of equation (1) might perform a desired *gyroscopic stabilization.*

Special cases

I) In the case of *sufficiently small damping* a simple perturbation approach leads to a condition for system $I\ddot{x} + D\dot{x} + Kx = 0$ to be *unstable* as well as to a condition for the system $I\ddot{x} + (D+G)\dot{x} + Kx = 0$ to be *stable*, see [2].

II) Let the unstable system (1) have a form where all diagonal entries of the indefinite matrix D are *positive* (this can always be achieved by a change of coordinates). Moreover, let K be *diagonal*. Then the gyroscopic matrix G with $g_{ik} = d_{ik}$ for $i < k$, $g_{kk} = 0$, and $g_{ik} = -d_{ik}$ for $i > k$ will stabilize the system.

III) System $I\ddot{x} + (D+G)\dot{x} + cIx = 0,\ c > 0$, is stable if and only if $B = D + G$ is *positive stable,* which means that all eigenvalues of B have positive real parts.

Solution of the Lyapunov matrix equation
We rewrite system

$$I\ddot{x} + (D+G)\dot{x} + Kx = 0 \qquad (2)$$

as a first order system (3)

$$\dot{z} = Lz,\ L = \begin{bmatrix} 0 & I \\ -K & -B \end{bmatrix},\ B = D + G,\ z = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

.

System (3) is stable, if there exist symmetric matrices $P > 0$ and $Q \geq 0$ which satisfy the *Lyapunov matrix equation*.

$$L^T P + PL = -Q \ . \qquad (4)$$

The solutions P and Q to the Lyapunov equation (4) are (5)

$$P=\begin{bmatrix} R & T \\ T^T & S \end{bmatrix}, \quad R=R^T, \quad S=S^T,$$

$$Q=\begin{bmatrix} TK+KT^T & KS-R+TB \\ SK-R^T+B^TT^T & -T-T^T+SB+B^TS \end{bmatrix}$$

We have to determine suitable matrices R, S and T such that the matrices P and Q in (5) are positive definite and positive semi-definite, respectively.

Theorem *Let $B=D+G$ be positive stable and let $K^{-1}B$ and $-B^TK^{-1}$ have no eigenvalues in common (this is e.g. the case if $K^{-1}B$ is positive stable). Then*

1. $$SB+B^TS=H \qquad (6)$$

with an arbitrary symmetric matrix $H>0$ has a unique solution $S=S^T>0$.

2.) $$KVB+B^TVK=SK-KS \qquad (7)$$

has a unique skew-symmetric solution $V=-V^T$.

Sufficient conditions for stability of system (2) are

a.) $H-T-T^T \geq 0$ *with* $T=KV$.

b.) $R-TS^{-1}T^T>0$ *with* $R=KS+TB$.

Proof: After solving equation (6) we want to find matrices T and $R>0$ such that $Q_{11}=TK+KT^T=0$ and $Q_{12}=KS-R+TB=0$. $Q_{11}=0$ implies $T=KV$ with a skew-symmetric matrix V. To satisfy $Q_{12}=0$ we put first the skew-symmetric part of Q_{12} to be zero. This means solving the $n(n-1)/2$ linear equations (7) for the $n(n-1)/2$ unknowns in V. The symmetric part of $Q_{12}=0$ results in $R=KS+TB$. In this way we end with $Q_{11}=Q_{12}=Q_{21}=0$. One of the conditions for stability of system (2) is $Q\geq 0$ which leads to $Q_{22}\geq 0$ and means a.). If this condition is not satisfied, we can not use the assumed matrix G as a stabilizing matrix, and we have to start the procedure again with another choice of G.

If $Q\geq 0$, we still have to investigate whether $P>0$. This can be done using condition b.)

The following example shows the deviation of the stability limit found by this direct method of Lyapunov from the exact value.

Example
Let system (1) have the form

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}+\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (8)$$

with

$$d_1>0,\ d_2<0,\ \mathrm{tr}(D)=d_1+d_2>0,\ K>0,$$
$$\mathrm{tr}(K^{-1}D)>0. \qquad (9)$$

For sufficiently small values of k_{12} the system is unstable. We want to stabilize the system by adding a term $G\dot{x}$ where

$B=D+G=\begin{bmatrix} d_1 & g \\ -g & d_2 \end{bmatrix}$ is positive stable which means $g^2>-d_1d_2$. To this end we choose in (6) $H>0$ as $H=\begin{bmatrix} gd_1 & -d_1d_2 \\ -d_1d_2 & -gd_2 \end{bmatrix}$.

The conditions (9) are sufficient for solving the matrices S, V, T, and R along the guidelines of the theorem. For the sake of simplicity we choose in the following $k_{12}=0$. Then condition a.) of the theorem results in

$$g^2\geq -d_1d_2x^2 \quad \text{with}$$
$$x=1+\frac{(k_{11}-k_{22})^2}{(d_1+d_2)(k_{11}d_2+k_{22}d_1)}\geq 1. \qquad (10)$$

It can be shown that in our example all values of g^2 satisfying (10) automatically satisfy the condition b.) of the theorem such that condition a.) alone represents the stability requirements. It is interesting to compare (10) with the condition for asymptotic stability gained by Routh-Hurwitz

$$g^2\geq -d_1d_2x. \qquad (11)$$

Finally we mention that the formulas (10) and (11) have to be changed slightly if $k_{12}\neq 0$. We can conclude that the conditions $\text{tr}(D) > 0$ and $\text{tr}(K^{-1}D) > 0$ are necessary and sufficient for gyroscopic stability of system (8).

Concluding remarks

Although damping matrices $B = D + G$ need not to be positive stable for the stability of system (2) this assumption is convenient and tempting, since it is successful in special cases like III). Under this condition we solved the Lyapunov matrix equation (4) and received the above theorem. For the shown example condition b.) of the theorem is unnecessary. It is an open question whether condition b.) can be skipped as well in the case of matrix order $n > 2$. In this case the two conditions $\text{tr}(D) > 0$ and $\text{tr}(K^{-1}D) > 0$ are only necessary but in general not sufficient for gyroscopic stabilization with help of the above theorem. For $n > 2$, a numerical procedure is suitable for computation of the matrices S, V, T, and R. It is then convenient to choose $H = I$ in (6). The procedure of the theorem can be extended easily to systems (2) where the stiffness matrix contains an additional skew-symmetric part.

[1] Popp, K., Rudolph, M., Kruger, M., Lindne, M.: Mechanisms to Generate and to Avoid Friction Induced Vibrations, VDI-Berichte 1736, VDI-Verlag Düsseldorf 2002, 1-15

[2] Kliem, W., Müller, P. C.: Gyroscopic Stabilization of Indefinite Damped Systems, Z Angew Math Mech 77 (1997), S1, T163-T164