Abstract: The stability of adaptive control systems has been studied extensively for minimum phase systems, mainly for model reference adaptive systems, but complete stability proof for non-minimum phase systems have not been given. In this paper, the stability of two types of self-tuning controllers for discrete time minimum and non-minimum phase plants is studied, namely: recursive estimation of the implicit self-tuning controller parameters based on generalized minimum variance criterion (REGMVC), and another based on generalized minimum variance criterion - $\beta$ equivalent control approach (REGMVC-$\beta$). Stability of the algorithms are proved by the Lyapunov theory.

Keywords: Self-tuning control, discrete-time systems, generalized minimum variance control, and sliding-mode control, linear control systems.

1. INTRODUCTION

Åström and Wittenmark (Astrom and Wittenmark, 1973) developed and studied the convergence of the implicit self-tuning controller in a stochastic setting. The stability of self-tuning algorithms for Model Reference Adaptive Systems (MRAS) have been studied for the strictly positive real model by Landau (Landau, 1980) (Landau, 1982). Latter on, Johansson (Johansson, 1986) studied the stability of MRAS for a minimum phase system, using Lyapunov theory. Global convergence for a class of adaptive control algorithms applied to discrete-time single-input single-output (SISO) and multi-input multi-output (MIMO) linear systems were studied for the minimum variance criterion in a seminal paper by Goodwin, et al. (Goodwin et al., 1980). However in these approaches the considered system should be minimum phase and the extension to consider the measurement noise may be difficult.

Extending the results of Åström (Astrom and Wittenmark, 1973), Clarke and Gawthrop (Clarke et al., 1975) proposed the Generalized Minimum Variance Control (GMVC) for non-minimum phase systems, using a cost function which incorporates system input and set-point variation. For the case of the unknown system parameters, the unknown parameters are estimated using a re-
cursive least-squares algorithm. Latter, in (Clarke et al., 1979) the convergence of the closed-loop system is analyzed using the positive real condition. In (Gawthrop, 1980) some stability analysis are given based on the notion of dissipative systems and conicity properties. However a complete global stability proof of self-tuning control for non-minimum phase systems have not been yet studied.

Sliding mode control (SMC) based on the variable structure systems (VSS), in the continuous-time case, is not robust when uncertainty excess the bound assumed in the design. Slotine (Slotine and Li, 1964) combined variable structure and adaptive control to solve this problem. Furuta (Furuta, 1993a) presents a discrete-time VSS type method for the case where systems parameters are unknown. The VSS is designed based on minimum variance control (MVC) or generalized minimum variance control (GMVC) using recursive parameter estimation. Extending (Furuta, 1993a), in (Furuta, 1993b) a designed parameter is introduced in the control law while maintaining the use of VSS. The stability of self-tuning control based on the certainty equivalence principle has been studied in (Morse, 1992). This approach is said implicit self-tuning control. However parameters are not identified accurately in the closed loop, and the stability is not assured based on the certainty equivalence principle.

In this paper, the stability of two types of implicit self-tuning controllers for discrete time minimum and non-minimum phase plants, when the system parameters are unknown, is proved. One is the combination of the generalized minimum variance control and identification of control parameter recursively (REGMVC), which has been used in many self-tuning controllers. The stability of the overall adaptive system is proved in this paper, although the parameters are not assured to converge to the true values. The other one (REGMVC-$\beta$) considers delay in control input. Stability of the algorithm is proved by the Lyapunov theory. It is not necessary to use VSS nor any additional condition to ensure closed loop system stability for the algorithms studied in this paper. The stability of the closed-loop system is proved in straight forward way in comparison with Goodwin, et al. (Goodwin et al., 1980), and may be extended to the case including the measurement noise (Patete et al., n.d.). This paper consider the non-minimum phase systems contrary to (Goodwin et al., 1980) paper.

The paper is organized as follows; in section 2, the generalized minimum variance criterion is presented. Section 3 deals with parametric uncertainties using self-tuning control based on generalized minimum variance criterion. A simulate examples is given in section 4. Concluding remarks are in 5.

2. GENERALIZED MINIMUM VARIANCE CRITERION

This paper considers a single-input single-output (SISO) time-invariant system. The representation of the plant with input $u_k$ and output $y_k$ is

$$A(z^{-1})y_k = z^{-d}B(z^{-1})u_k$$ (1)

where $A(z^{-1})$ and $B(z^{-1})$ have no common factor and $z$ denotes the time shift operator $z^{-1}y_k = y_{k-1}$. In the Laplace transformation, $z = e^{sT_0}$ where $T_0$ is the sample period (for simplicity, and without loss of generality, we may assume $T_0 = 1$).

The polynomials $A(z^{-1})$ and $B(z^{-1})$ are assumed to be known, and represented as:

$$A(z^{-1}) = 1 + a_1z^{-1} + a_2z^{-2} + ... + a_nz^{-n}$$

$$B(z^{-1}) = b_0 + b_1z^{-1} + b_2z^{-2} + ... + b_mz^{-m}$$

where $b_0 \neq 0$ and delay step $d$, is also assumed to be known.

The objective of the control is to minimize the variance of the controlled variables $s_{k+d}$, that is defined in the deterministic case as

$$s_{k+d} = C(z^{-1})(y_{k+d} - r_{k+d}) + Q(z^{-1})u_k$$ (2)

The polynomials

$$C(z^{-1}) = 1 + c_1z^{-1} + c_2z^{-2} + ... + c_nz^{-n}$$

$$Q(z^{-1}) = q_0(1 - z^{-1})$$

are to be designed, $r_k$ is the reference signal, and the error signal $e_k$ is defined as $e_k = y_k - r_k$.

The idea is similar to the discrete time sliding mode control, see (Xionghuo and Jian-Xin, 1992) and (Zinobor, 1994). In the case of Goodwin, et al. (Goodwin et al., 1980), $Q(z^{-1})$ is not considered and $C(z^{-1})$ is chosen as $C(z^{-1}) = 1$.

The polynomial $C(z^{-1})$ is Schur, hence the error signal will vanish if (2) is kept to zero. The polynomial $C(z^{-1})$ may be determined by assigning all characteristic roots inside the unit disk of $z$-plane.

Equation (2) is rewritten as:

$$s_{k+d} = G(z^{-1})u_k + F(z^{-1})y_k - C(z^{-1})r_{k+d}$$ (3)

where the polynomial $G(z^{-1})$ is defined as $G(z^{-1}) = E(z^{-1})B(z^{-1}) + Q(z^{-1})$, and polynomials $E(z^{-1})$ and $F(z^{-1})$ satisfy the equality,

$$C(z^{-1}) = A(z^{-1})E(z^{-1}) + z^{-d}F(z^{-1})$$ (4)
Then the minimum generalized variance control input needed to vanish $s_{k+d}$ in (2) is given by:

$$u_k = -G(z^{-1})^{-1}[F(z^{-1})y_k - C(z^{-1})r_{k+d}]$$ (5)

where, the polynomials $C(z^{-1})$ and $Q(z^{-1})$ are chosen to make the close-loop characteristic equation to have all zeros inside the unit disc. If the process is minimum phase, $Q(z^{-1}) = 0$ may be chosen, which is the case of minimum variance control.

3. SELF-TUNING CONTROL BASED ON GENERALIZED MINIMUM VARIANCE CRITERION

In the previous section, the generalized minimum variance criterion was described for the case where all plant parameters are known. Yet, in general, plant parameters are not accurately known and parameter identification should be performed. The assumption here is that while model parameters are unknown, the structure of the model (model order) is known a-priori.

When the parameters of the plant are not accurately known, the controller polynomials $G(z^{-1})$ and $F(z^{-1})$ are estimated, i.e.

$$\hat{F}(z^{-1}) = \hat{f}_0 + \hat{f}_1z^{-1} + \ldots + \hat{f}_{n-1}z^{-n+1}$$
$$\hat{G}(z^{-1}) = \hat{g}_0 + \hat{g}_1z^{-1} + \ldots + \hat{g}_{m+d-1}z^{-(m+d-1)}$$

Let,

$$\theta^T = [y_k, y_{k-1}, \ldots, y_{k-n+1}, u_k, \ldots, u_{k-m-d+1}]$$

be a vector containing measured and control signal data,

$$\theta^T = [f_0, f_1, \ldots, f_{n-1}, g_0, \ldots, g_{m+d-1}]$$

the vector containing the controller parameters, and

$$\hat{\theta}^T = [\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_{n-1}, \hat{g}_0, \ldots, \hat{g}_{m+d-1}]$$

the estimation of $\theta$.

Åström and Wittenmark (Astrom and Wittenmark, 1989) have studied self-tuning estimation by minimizing the least-squares criterion function

$$J = \frac{1}{2} \sum_{j=0}^{k} [\varepsilon_j]^2$$ (6)

by the recursive least-squares algorithm

$$\hat{\theta}_k = \hat{\theta}_{k-1} + \frac{P_{k-1}\theta_{k-d}}{1 + \phi_{k-d}^T P_{k-1}\phi_{k-d}} \varepsilon_k$$ (7)

Then $P_k = P_{k-1} - \frac{P_{k-1}\phi_{k-d}\phi_{k-d}^T P_{k-1}}{1 + \phi_{k-d}^T P_{k-1}\phi_{k-d}}$ (8)

where $\varepsilon_k$ is the prediction error.

In this paper the control law is:

$$u_k = -\hat{G}(z^{-1})^{-1}[\hat{F}(z^{-1})y_k - C(z^{-1})r_{k+d}]$$ (9)

where the estimates of control parameters are given by (Furuta, 1993a),

$$\hat{\theta}_k = \hat{\theta}_{k-1} + \frac{\Gamma_{k-1} \theta_{k-d}}{1 + \phi_{k-d}^T \Gamma_{k-1} \phi_{k-d}}$$ (10)

$$\theta_{k} = \hat{\theta}_{k-1} - \frac{\Gamma_{k-1} \theta_{k-d}}{1 + \phi_{k-d}^T \Gamma_{k-1} \phi_{k-d}}$$ (11)

The stability of the self-tuning control algorithm, recursive estimates of controller parameters based on generalized minimum variance criterion (REG-MVC), is analyzed by using Lyapunov function as follows:

**Theorem 1.** Recursive estimates of controller parameters based on generalized minimum variance criterion (REG-MVC), is analyzed by using Lyapunov function as follows:

$$V_k = \frac{1}{2} s_k^2 + \frac{1}{2} \bar{\theta}_k^T \bar{\theta}_k, (\Gamma_0 > 0)$$ (14)

The time difference of (14) is:

$$\Delta V_k = V_k - V_{k-1}$$ (15)

$$= -\frac{1}{2} s_k^2 + \frac{1}{2} \bar{\theta}_k^T \Gamma_{k-1} \bar{\theta}_k - \frac{1}{2} \bar{\theta}_{k-1}^T \Gamma_{k-1} \bar{\theta}_{k-1}$$ (16)

$$+ \frac{1}{2} \bar{\theta}_k^T \Gamma_{k-1} \bar{\theta}_{k-1}$$ (17)

Consider the candidate Lyapunov function:

$$s_{k+d} = \phi_{k+d}^T \bar{\theta}_{k+d}$$ (13)

Proof. $s_{k+d}$ is written as

$$s_{k+d} = \hat{G}(z^{-1}) u_k + \hat{F}(z^{-1}) y_k - C(z^{-1}) r_{k+d} + \phi_{k+d}^T \bar{\theta}_{k+d}$$

where $\bar{\theta}_k = \theta - \hat{\theta}_k$. Using the control law (9), (12) is rewritten as

Then the estimation of $\theta$, Åström and Wittenmark (Astrom and Wittenmark, 1989) have studied self-tuning estimation by minimizing the least-squares criterion function

$$J = \frac{1}{2} \sum_{j=0}^{k} [\varepsilon_j]^2$$ (6)

by the recursive least-squares algorithm

$$\hat{\theta}_k = \hat{\theta}_{k-1} + \frac{P_{k-1}\theta_{k-d}}{1 + \phi_{k-d}^T P_{k-1}\phi_{k-d}} \varepsilon_k$$ (7)

where $\varepsilon_k$ is the prediction error.
Thus (10) is derived.

If \( \theta_k \) then,

substituting (19) into (18), the following relation is derived:

\[
\Delta V_k = -\frac{1}{2}(\hat{\theta}_k - \hat{\theta}_{k-1})^T \Gamma_{k-1}^{-1}(\hat{\theta}_k - \hat{\theta}_{k-1}) \tag{20}
\]

\[
-\frac{1}{2}s_{k-1}^2 + \frac{1}{2}\hat{\theta}_k^T (\Gamma_{k-1}^{-1} - \phi_{k-d} \phi_k^T \phi_{k-d}) \hat{\theta}_k
\]

\[
+ \frac{1}{2}\hat{\theta}_k \Gamma_{k-1}^{-1}(\hat{\theta}_k - \hat{\theta}_{k-1} + \Gamma_{k-1} \phi_{k-d} \phi_k^T \phi_{k-d} \hat{\theta}_k) = 0
\]

If

\[
\frac{1}{2}\hat{\theta}_k (\Gamma_{k-1}^{-1} - \phi_{k-d} \phi_k^T \phi_{k-d}) \hat{\theta}_k
\]

then, \( \Delta V_k \) is proven to be negative semidefinite, i.e. \( \Delta V_k \leq 0 \).

From the first term on the left of (21) the following is derived:

\[
\Gamma_{k-1}^{-1} - \phi_{k-d} \phi_k^T \phi_{k-d} = 0 \tag{22}
\]

\[
\Gamma_k = (\Gamma_{k-1}^{-1} + \phi_{k-d} \phi_k^T \phi_{k-d})^{-1} \tag{23}
\]

\[
\Gamma_k = \Gamma_{k-1} \tag{24}
\]

\[
\Gamma_{k-1} \phi_{k-d} \phi_k^T \phi_{k-d} (\Gamma_{k-1}^{-1} + \phi_{k-d} \phi_k^T \phi_{k-d})^{-1}
\]

this yields (11) by the matrix inversion lemma.

From the second term of (21) the following is derived

\[
\hat{\theta}_k - \hat{\theta}_{k-1} + \Gamma_{k-1} \phi_{k-d} \phi_k^T \phi_{k-d} \hat{\theta}_k = 0 \tag{25}
\]

\[
\hat{\theta}_k + \Gamma_{k-1} \phi_{k-d} \phi_k^T \phi_{k-d} \hat{\theta}_k = \hat{\theta}_{k-1} \tag{26}
\]

\[
(I + \Gamma_{k-1} \phi_{k-d} \phi_k^T \phi_{k-d}) \hat{\theta}_k = \hat{\theta}_{k-1} \tag{27}
\]

\[
(I + \Gamma_{k-1} \phi_{k-d} \phi_k^T \phi_{k-d}) \hat{\theta}_{k-1} - \Gamma_{k-1} \phi_{k-d} \phi_k^T \phi_{k-d} \hat{\theta}_{k-1} = \hat{\theta}_k \tag{28}
\]

\[
\hat{\theta}_k = \hat{\theta}_{k-1} + \Gamma_{k-1} \phi_{k-d} (1 + \phi_k^T \Gamma_{k-1} \phi_{k-d})^{-1} \phi_k^T \phi_{k-d} (\theta - \hat{\theta}_{k-1})
\]

from (3),

\[
s_k = \phi_k^T \phi_{k-d} \theta - C(z^{-1}) r_k \tag{29}
\]

thus (10) is derived.

Thus \( \Delta V_k \leq 0 \), which implies that as \( k \) goes to infinity \( s_k \) and \( \hat{\theta}_k - \hat{\theta}_{k-1} \) approach to zero then the stability of the overall self-tuning is proved. The stability is proved in straight forward way. The results are just the extension of Goodwin, et al. (Goodwin et al., 1980); but the proof is so straight forward and simple.

(20) gives, \( s_k \) shall vanish when \( \Delta V_k \) is negative semidefinite for all \( k \), and the generalized minimum variance is minimized.

Remark 1. Note that we do not prove, or claim, that \( \hat{\theta}_k \) converges to \( \theta \). Instead each element of \( \hat{\theta}_k \) approaches to constant values.

Now, instead of (9), consider the following control law,

\[
u_k = \hat{G}(z^{-1})^{-1} C(z^{-1}) r_{k+d} - \hat{F}(z^{-1}) y_k + \beta s_k \tag{30}
\]

where \( 0 < \beta < 1 \). Substituting (30) into (3), if the controller parameters are known exactly, the following relation is derived

\[
s_{k+d} = \beta s_k \tag{31}
\]

The idea is similar to the introduction of an additional control parameter to give system robustness. Some results on this topic are found in (Furuta, 1993b). When (30) is used instead of (9), the method will be referred as generalized minimum variance criterion - \( \beta \) equivalent control approach (GMVC-\( \beta \)). The following theorem, recursive estimates of controller parameters based on generalized minimum variance criterion - \( \beta \) equivalent control approach (REGMVC-\( \beta \)), establishes overall system stability when (30) is used.

**Theorem 2.** Recursive estimates of controller parameters based on generalized minimum variance criterion - \( \beta \) equivalent control approach: Given \( \Gamma_i \) and \( \theta_i \), \( i = 0, -1, -2, \ldots, -d \), the estimate \( \hat{\theta}_k \) of the controller (30) satisfies the recursive equations,

\[
\hat{\theta}_k = \hat{\theta}_{k-d} + \frac{\Gamma_{k-d} \phi_{k-d}}{1 + \phi_k^T \Gamma_{k-d} \phi_{k-d}} \tag{32}
\]

\[
\phi_k^T \hat{\theta}_{k-d} + \beta s_{k-d}
\]

\[
\Gamma_k = \Gamma_{k-d} - \frac{\Gamma_{k-d} \phi_{k-d} \phi_k^T \Gamma_{k-d}}{1 + \phi_k^T \Gamma_{k-d} \phi_{k-d}} \tag{33}
\]

then the overall self-tuning controller combining (30), (32) and (33) for system (1) is stable.

Proof. Using (12) and control law (30), \( s_{k+d} \) is

\[
s_{k+d} = \beta s_k + \phi_k^T \hat{\theta}_{k+d} \tag{34}
\]

The time difference of the candidate Lyapunov function is given by:

\[
\Delta V_k = V_k - V_{k-d} \tag{35}
\]
where the candidate Lyapunov function is the same as in (14). Then,

$$\Delta V_k =$$

(36)

$$\frac{1}{2} s_k^2 - \frac{1}{2} s_{k-2}^2 + \frac{1}{2} \theta_k^T \Gamma_k^{-1} \hat{\theta}_k - \frac{1}{2} \hat{\theta}_k^T \Gamma_k^{-1} \hat{\theta}_k -$$

(37)

$$= - \frac{1}{2} s_{k-2}^2 + \frac{1}{2} \theta_k (\theta_k - \hat{\theta}_k) \Gamma_k^{-1} (\theta_k - \hat{\theta}_k)$$

(38)

$$+ \frac{1}{2} \theta_k^T (\Gamma_k^{-1} - \Gamma_k^{-1} \hat{\theta}_k + \hat{\theta}_k^T \Gamma_k^{-1} \hat{\theta}_k - \hat{\theta}_k^T \Gamma_k^{-1} \hat{\theta}_k -$$

$$\frac{1}{2} s_k^2 + s_k^2$$

(39)

Substituting (39) into (38), the following relation is derived

$$\Delta V_k = \frac{1}{2} \theta_k^T (\Gamma_k^{-1} - \Gamma_k^{-1} \hat{\theta}_k + \hat{\theta}_k^T \Gamma_k^{-1} \hat{\theta}_k)$$

(40)

$$- \frac{1}{2} (\theta_k - \hat{\theta}_k) \Gamma_k^{-1} (\theta_k - \hat{\theta}_k) + \frac{1}{2} \theta_k \Gamma_k^{-1}$$

$$+ (\theta_k - \hat{\theta}_k) \Gamma_k^{-1} (\theta_k - \hat{\theta}_k) + \frac{1}{2} \theta_k \Gamma_k^{-1}$$

$$\frac{1}{2} s_k^2 (1 - \beta^2)$$

(41)

If

$$\frac{1}{2} \theta_k^T (\Gamma_k^{-1} - \Gamma_k^{-1} \hat{\theta}_k + \hat{\theta}_k^T \Gamma_k^{-1} \hat{\theta}_k) = 0$$

then the difference of the candidate Lyapunov function (35) is proven to be negative semidefinite, \(\Delta V_k \leq 0\). From the first term on the far left of (41) the following is derived:

$$\Gamma_k^{-1} - \Gamma_k^{-1} \hat{\theta}_k + \hat{\theta}_k^T \Gamma_k^{-1} \hat{\theta}_k = 0$$

(42)

$$\Gamma_k = (\Gamma_k^{-1} + \phi_k \Gamma_k^{-1} \phi_k^T)^{-1}$$

(43)

$$\Gamma_k = \Gamma_k^{-1}$$

(44)

$$\Gamma_k \phi_k \Gamma_k^{-1} \phi_k^T = \Gamma_k^{-1} \phi_k \Gamma_k^{-1} \phi_k^T$$

which yields (33). From the second term of (41),

$$\hat{\theta}_k - \hat{\theta}_k - \Gamma_k \phi_k \phi_k^T \hat{\theta}_k$$

(45)

$$+ \Gamma_k \phi_k \phi_k^T \phi_k \phi_k^T \hat{\theta}_k = 0$$

$$\hat{\theta}_k = \hat{\theta}_k - \Gamma_k \phi_k \phi_k^T (1 + \phi_k \phi_k^T \Gamma_k \phi_k \phi_k^T)^{-1}$$

(46)

$$\Gamma_k \phi_k \phi_k^T \phi_k \phi_k^T \hat{\theta}_k = \Gamma_k \phi_k \phi_k^T \phi_k \phi_k^T \hat{\theta}_k$$

Noting from (3),

$$s_k = \phi_k^T \theta - C(z^{-1})\Gamma_k$$

(47)

(32) is obtained from (46).

Thus \(\Delta V_k \leq 0\), and \(s_k \leq 0\) approach to zero as \(k\) goes to infinity, and the stability of the overall self-tuning has been proved.

Remark 2. Note that for the case \(d = 1\) when \(\beta = 0\), REGMVC\(\beta\) algorithm is same as REGMVC.

4. SIMULATION RESULTS

As an hypothetical example consider the following true non-minimum phase plant with \(d = 2\),

$$\frac{1}{2} s_k^2 + 2 \beta s_k \phi_k \phi_k^T \hat{\theta}_k$$

$$= 1$$

(48)

The nominal plant is

$$1 - 0.7z^{-1} y_k = z^{-d} (1 + 2.8z^{-1}) u_k$$

(49)

For the nominal controller design using the generalized minimum variance criterion, the following polynomials are chosen:

$$C(z^{-1}) = 1 + z^{-1} + 0.3z^{-2}$$

(50)

$$Q(z^{-1}) = 40 (1 - z^{-1})$$

(51)

which lead to the following polynomials for the controller law,

$$F(z^{-1}) = 1.05, \hat{G}(z^{-1}) = 41 - 36z^{-1} + 3.75z^{-2}$$

$$F(z^{-1})$$ and \(G(z^{-1})\) give initial estimates of the controller parameters.

When polynomial \(C(z^{-1})\) and \(Q(z^{-1})\) in (50) and (51) respectively, are used to computed the optimal controller polynomials \(F(z^{-1})\) and \(G(z^{-1})\) for the real plant (48), the following polynomials are obtained,

$$F(z^{-1}) = 1.49, G(z^{-1}) = 41 - 35.5z^{-1} + 4.76z^{-2}$$

In Fig. 1 the output responses of the system (48), using the fixed controller GMVC, and the self-tuning controllers REGMVC and REGMVC\(\beta\) (for selected values of \(\beta\): \(\beta = 0.4\) and \(\beta = 0.8\), are shown. The initial condition for \(\Gamma\) is set to the identity matrix, i.e. \(\Gamma_i = I\), \((i = 0, -1)\).

The reference signal is chosen as a sequence of steps with length of 50 samples. Note that the generalized minimum variance controller (GMVC) is very sensitive to the presence of parametric
uncertainties, and that the self-tuning controller can improve the responses as the parameters are identified. In this particular example, the case REGMVC-β for small values of β, e.g. β ≤ 0.2, the response behave very similar to the case REGMVC.

5. CONCLUSIONS

Two design methods for the self-tuning stabilizing controller based on generalized minimum variance for discrete-time systems were studied: REGMVC and REGMVC-β. Parameter identification has been used in the presence of parametric uncertainties. The overall stability for the two algorithms was proved by the Lyapunov theory. The robustness and performance of the self-tuning algorithms has been shown through a simulated example.

REFERENCES
