Structure at Infinity and the Structure of Interactor of LinearMultivariable Systems

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***Abstract*— The purpose of this paper is to discuss the relationship between the structure at infinity of a linear multivariable system and its interactor matrix. It will be shown that the structure at infinity of a given system coincides with row degrees of it's interactor if and only if the interactor is row proper.**

I. Introduction

An interactor was originally defined by Wolovich and Falb [1] as the system invariant under dynamic compensation, which is a lower left triangular polynomial matrix. Since an interactor is the generalization of the relative degree of a scalar transfer function for linear multivariable case, it has been used for many of linear multivariable control design problems, e.g., the model matching control, the decoupling control, the disturbance decoupling control, the adaptive control, etc. [2]-[6]. In these control design problems, especially in the adaptive control problem, the triangular structure of the interactor plays an important role, but, from the algebraic point of view, the triangular structure is not necessarily an inherent property of the interactor. In fact, the spectral interactor matrix discussed in [7] is not a triangular matrix any more. In general, an interactor can be defined by any polynomial matrix which cancels all zeros at infinity of a given plant transfer matrix by premultiplying. This implies that the interactor can be regarded as another expression of the structure at infinity of the plant, and hence, there must be a direct relationship between the structure at infinity of the plant and the structure of degrees of it 's interactor. However, although the structure at infinity is determined uniquely, the interactor and even its structure of degrees are not determined uniquely. In this note, to establish this explicit relationship, the regular interactor in defined as the interactor whose row degrees coincide with the structure at infinity of the plant. It will be shown that the interactor is regular if and only if the interactor is row proper.

II. Preliminaries

Let $T(s)$ be a $m \times m$ nonsingular and strictly proper transfer matrix. Then, there will exist bicausal rational matrices $U(s)$ and $V(s)$ such that

$$U(s)T(s)V(s) = \begin{bmatrix} \frac{1}{s^{f_1}} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{s^{f_m}} \end{bmatrix} \quad (1)$$

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where $f_1 \le f_2 \le \cdots \le f_m$ are positive integers which are determined uniquely.

Definition 1: The set $\{f_1, f_2, \cdots, f_m\}$ is called the structure at infinity of $T(s)$, and f_i is called the order of i-th zero at infinity of $T(s)$.

It is well known that f_i can be determined by the following relations.

$$\begin{aligned} f_1 &= \delta_1 \\ f_i &= \delta_i - \delta_{i-1} \quad (i = 2, \cdots, m) \end{aligned} \quad (2)$$

where δ_i is the minimum relative degree among those of non-zero i-th ordered minors of $T(s)$. On the other hand, an interactor of $T(s)$ is defined by the following.

Definition 2: A $m \times m$ polynomial matrix $L(s)$ satisfying the following equation is called an interactor of $T(s)$.

$$\lim_{s \to \infty} L(s)T(s) = \Lambda, \qquad \text{nonsingular} \quad (3)$$

It should be noted that this definition does not restrict the interactor to a lower left triangular polynomial matrix. It defines the interactor only by the most essential property (eq.(3)) that all interactors must have. Since eq.(3) implies that $L(s)T(s)$ does not have any zeros at infinity, the interactor can be also defined by any polynomial matrix, $L(s)$, which cancels all zeros at infinity of $T(s)$ by premultiplying. From this point of view, the interactor is an alternative expression of the structure at infitnity of $T(s)$, and hence, there must be some direct relationship between the structure at infinity of $T(s)$ and the structure of degrees of its interactor $L(s)$.

Let $T(s)$ and $L(s)$ be represented by

$$\begin{aligned} T(s) &= T_1 s^{-1} + T_2 s^{-2} + \cdots \\ L(s) &= L_w s^w + L_{w-1} s^{w-1} + \cdots + L_0 \end{aligned} \quad (4)$$

where $T_i \in R^{m \times m}$ is the Markov parameter of $T(s)$ and $L_i \in R^{m \times m}$ is the coefficient matrix of an interactor. Then, it is known that $L(s)$ is an interactor of $T(s)$ if and only if T_i and L_i satisfy the following equation [5].

$$\begin{aligned} \Gamma_w \begin{bmatrix} L_w^T \\ \vdots \\ L_1^T \end{bmatrix} &= \left[\begin{array}{ccc|c} & & & 0 \\ & \Gamma_{w-1} & & \vdots \\ & & & 0 \\ \hline T_w^T & \cdots & T_2^T & T_1^T \end{array} \right] \begin{bmatrix} L_w^T \\ \vdots \\ L_1^T \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \Lambda^T \end{bmatrix} \end{aligned} \quad (5)$$

where Γ_i is the i-th truncated block Toeplitz matrix of $T(s)$ defined by the following.

$$\Gamma_i = \begin{bmatrix} T_1^T & 0 & \cdots & 0 \\ T_2^T & T_1^T & & \vdots \\ \vdots & & \ddots & 0 \\ T_i^T & \cdots & \cdots & T_1^T \end{bmatrix} \quad i = 1, 2, \cdots \tag{6}$$

Since, L_0 does not appear in eq.(5), L_0 can be chosen arbitrarily. It was also shown in [5] that the degree of $L(s)$ is the maximum order of zero at infinity of $T(s)$, and the degree of $detL(s)$ equals the relative degree of $detT(s)$, i.e.,

$$w = f_m \tag{7}$$

$$deg\ detL(s) = \delta_m = \sum_{i=1}^{m} f_i. \tag{8}$$

Eqs.(7) and (8) give a relationship between the structure at infinity of $T(s)$ and its interactor to a certain extent. However, under the condition of eqs.(7) and (8), the interactor and even the structure of degrees of the interactor are not still determined uniquely, then further investigation is needed to establish more explicit relationship between the structure at infinity of $T(s)$ and the structure of degrees of its interactor.

To show a property of Γ_i more precisely, f_i's are assumed to satisfy the following relation.

$$\begin{aligned} &0 < f_1 = f_2 = \cdots = f_{t_1} < f_{t_1+1} = \\ &\cdots = f_{t_1+t_2} < f_{t_1+t_2+1} = \cdots \\ &\cdots < f_{t_1+\cdots+t_{d-1}+1} = \cdots = f_{t_1+\cdots+t_d} \end{aligned} \tag{9}$$

where t_i is a number of zeros at infinity of $T(s)$ whose order is all $f_{t_0+t_1+\cdots+t_{i-1}+1}$ ($t_0 = 0$). In other words, t_i is multiplicity of zeros at infinity, and hence,

$$\sum_{i=1}^{d} t_i = m \tag{10}$$

Note that if f_i's are distinct, then $t_i = 1$ ($i = 1, \cdots, m,\ d = m$). It is shown in [8][5] that the rank of Γ_i satisfies

$$rank\Gamma_k = rank\Gamma_{k-1} + \sigma_k, \qquad (k = 1, 2, \cdots, \quad \Gamma_0 = 0) \tag{11}$$

where σ_k is defined as follows.

$$\begin{cases} \sigma_k = \sigma_{k-1} + t_i & if \quad k = f_{t_0+t_1+\cdots+t_{i-1}+1}, \\ & (i = 1, \cdots, d, \quad t_0 = 0) \\ \sigma_k = \sigma_{k-1} & otherwise \end{cases} \tag{12}$$

Here,

$$\begin{cases} \sigma_0 = 0 \\ \sigma_{f_{t_1+\cdots+t_{d-1}+1}} = \sum_{i=1}^{d} t_i = m \end{cases}$$

This property can be derived using eq.(1). Eqs.(11) and (12) will be used in the proof of Theorem 1 in the next section.

III. Regular Interactor

In this section, the relationship between the structure at infinity of $T(s)$ and the structure of degrees of its interactor $L(s)$ will be considered. Before stating the main result, we define a regular interactor as follows.

Definition 3: An interactor of $T(s)$ is called a regular interactor if its row degrees coincide with the structure at infinity.

Note that the lower left triangular interactor defined in [5] is the special case of this definition. The main result of this paper is stated as the following Theorem.

Theorem 1: Let $L(s)$ be an interactor of $T(s)$. Then, $L(s)$ is regular if and only if $L(s)$ is row proper.

Proof: (Necessity) $L(s)$ is assumed to be a regular interactor of $T(s)$, i.e., its row degrees are $\{f_1, f_2, \cdots, f_m\}$. Then, eq.(8) implies that $L(s)$ is row proper because the degree of $detL(s)$ is equal to the sum of the row degrees of $L(s)$ [9].

(Sufficiency) $L(s)$ is assumed to be a row proper interactor of $T(s)$. Let r_i ($i = 1, \cdots, m$) be the i-th row degree of $L(s)$. Then, $L(s)$ can be expanded as

$$\begin{aligned} L(s) \;=\; & \begin{bmatrix} s^{r_1} & & 0 \\ & \ddots & \\ 0 & & s^{r_m} \end{bmatrix} R_q \\ & + \begin{bmatrix} s^{r_1-1} & & 0 \\ & \ddots & \\ 0 & & s^{r_m-1} \end{bmatrix} R_{q-1} + \cdots \\ & + \begin{bmatrix} s^{r_1-q} & & 0 \\ & \ddots & \\ 0 & & s^{r_m-q} \end{bmatrix} R_0 \end{aligned} \tag{13}$$

where $R_i \in R^{m\times m}$ ($i = 0, \cdots, q$) and $q = \max[r_1, r_2, \cdots, r_m]$. Furthermore, in eq.(13), s^j is set to 0 for negative j and the corresponding row vector of a coefficient matrix is also set to 0. Since $L(s)$ is row proper, the leading coefficient matrix R_q is nonsingular. Without loss of generality, it is assumed that

$$\begin{aligned} &0 < r_1 = \cdots = r_{b_1} < r_{b_1+1} = \cdots \\ &\cdots = r_{b_1+b_2} < r_{b_1+b_2+1} = \cdots \\ &< r_{b_1+\cdots+b_{e-1}+1} = \cdots = r_{b_1+\cdots+b_e} \end{aligned} \tag{14}$$

where b_i is the multiplicity of row degrees of $L(s)$, and hence

$$\sum_{i=1}^{e} b_i = m \tag{15}$$

Note that if r_i $(i = 1, \cdots, m)$ are distinct, then $b_i = 1$ $(i = 1, \cdots, e, \quad e = m)$.

Since $L(s)$ is an interactor of $T(s)$, from the definition of the interactor, the expanded form of $L(s)T(s)$ satisfies the following equation.

$$L(s)T(s) = \begin{bmatrix} s^{r_1-1} & & 0 \\ & \ddots & \\ 0 & & s^{r_m-1} \end{bmatrix} G_1 + \begin{bmatrix} s^{r_1-2} & & 0 \\ & \ddots & \\ 0 & & s^{r_m-2} \end{bmatrix} G_2 + \cdots$$
$$= \Lambda + \; lower\ degree\ terms, \quad (\Lambda : nonsingular) \tag{16}$$

where, from eqs.(4) and (13), the first q coefficient matrices $G_1, \cdots, G_q$ are calculated by the following.

$$\begin{bmatrix} T_1^T & 0 & \cdots & 0 \\ T_2^T & T_1^T & & \vdots \\ \vdots & & \ddots & 0 \\ T_i^T & \cdots & \cdots & T_1^T \end{bmatrix} \begin{bmatrix} R_q^T \\ R_{q-1}^T \\ \vdots \\ R_1^T \end{bmatrix} = \begin{bmatrix} G_1^T \\ G_2^T \\ \vdots \\ G_q^T \end{bmatrix} \tag{17}$$

Eq.(16) implies that $G_1, \cdots, G_{r_{b_1+\cdots+b_{e-1}+1}}$ have the following forms.

$$G_1 = \cdots = G_{r_1-1} = 0$$

$$G_{r_1} = \left[\begin{array}{c} X_1 \\ \hline 0 \end{array}\right] \}b_1 \quad , \; X_1 \in R^{b_1 \times m}, \; rankX_1 = b_1$$

$$G_{r_1+1}, \cdots, G_{r_{b_1+1}-1} = \left[\begin{array}{c} * \\ \hline 0 \end{array}\right] \}b_1$$

$$G_{r_{b_1+1}} = \left[\begin{array}{c} * \\ \hline X_2 \\ \hline 0 \end{array}\right] \begin{array}{l} \}b_1 \\ \}b_2 \\ \\ \end{array}$$
$$X_2 \in R^{b_2 \times m}, \quad rankX_2 = b_2$$

$$G_{r_{b_1+1}+1}, \cdots, G_{r_{b_1+b_2+1}-1} = \left[\begin{array}{c} * \\ \hline 0 \end{array}\right] \}b_1 + b_2$$

$$G_{r_{b_1+b_2+1}} = \left[\begin{array}{c} * \\ \hline X_3 \\ \hline 0 \end{array}\right] \begin{array}{l} \}b_1 + b_2 \\ \}b_3 \\ \\ \end{array}$$
$$X_3 \in R^{b_3 \times m}, \; rankX_3 = b_3$$

$$G_{r_{b_1+b_2+1}+1}, \cdots, G_{r_{b_1+b_2+b_3+1}-1} = \left[\begin{array}{c} * \\ \hline 0 \end{array}\right] \}b_1 + b_2 + b_3$$

$$\vdots$$

$$G_{r_{b_1+\cdots+b_{e-1}+1}} = \left[\begin{array}{c} * \\ \hline X_e \end{array}\right] \begin{array}{l} \}b_1 + \cdots + b_{e-1} \\ \}b_e \end{array} ,$$
$$X_e \in R^{b_e \times m}, \quad rankX_e = b_e \tag{18}$$

and

$$\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_e \end{bmatrix} = \Lambda \qquad nonsingular \tag{19}$$

In eq.(18), '*' denotes some matrix with an appropriate size. Since R_q is nonsingular, substituting eqs.(18) and (19) into eq.(17) gives the following relations of the rank of Γ_i in terms of the row degrees of $L(s)$.

$$rank\Gamma_k = rank\Gamma_{k-1} + \xi_k, \quad (\Gamma_0 = 0, \quad k = 1, 2, \cdots) \tag{20}$$

where ξ_k is defined as follows.

$$\begin{cases} \xi_k = \xi_{k-1} + b_i & if \quad k = r_{b_0+b_1+\cdots+b_{i-1}+1}, \\ & (i = 1, \cdots, e, \quad b_0 = 0) \\ \xi_k = \xi_{k-1} & otherwise \end{cases} \tag{21}$$

Here,

$$\begin{cases} \xi_0 = 0 \\ \xi_{r_{b_1+\cdots+b_{d-1}+1}} = \sum_{i=1}^{e} b_i = m \end{cases}$$

By comparing eqs.(11),(12) and eqs.(20),(21), we have

$$e = d, \quad b_i = t_i, \quad r_i = f_i \quad (i = 1, \cdots, m) \tag{22}$$

which means that the row degrees of $L(s)$ coincide with the structure at infinity, and hence, $L(s)$ is a regular interactor of $T(s)$. ■

It was shown in [5] that there always exists an appropriate row permutation matrix $W \in R^{m \times m}$ such that $WT(s)$ has a lower left triangular regular interactor $L_W(s)$. This guarantees the existence of a regular interactor for any nonsingular $T(s)$ because $L_W(s)W$ is one of the regular interactors of $T(s)$. However, it should be noted that there does not necessarily exist a lower left triangular regular interactor for any $T(s)$ unless rows of $T(s)$ are properly reordered.

Example 1: Consider the following transfer matrix.

$$G(s) = \begin{bmatrix} \frac{s^2+1}{s^3} & \frac{1}{s} & \frac{1}{s} \\ \frac{s+1}{s^2} & \frac{s^2+2s+2}{s^3} & \frac{s+3}{s^2} \\ \frac{1}{s} & \frac{2}{s} & \frac{3s^2+3}{s^3} \end{bmatrix} \tag{23}$$

The structure at infinity of $G(s)$ is $\{f_1, f_2, f_3\} = \{1, 1, 3\}$. From this, the degree of an interactor of $G(s)$ is $w = f_3 = 3$, i.e., all interactors can be writen as

$$L(s) = L_3 s^3 + L_2 s^2 + L_1 s + L_0, \tag{24}$$

where $L_i \in R^{3 \times 3}$ is a coefficient matrix.

It is easy to verify that the following polynomial matrix is the interactor of $G(s)$.

$$L_A(s) = \begin{bmatrix} s & 0 & 0 \\ -s^3 & -s^3 & s^2 \\ -s^2 & s^2 & 0 \end{bmatrix} \tag{25}$$

And, it is also readily checked that

$$\begin{aligned} degL_A(s) &= f_m = f_3 = 3 \\ deg\det L_A(s) &= f_1 + f_2 + f_3 = 5. \end{aligned} \tag{26}$$

These are the properties that all interactors have. But, since the coeffinient matrix of the maxmum row degree terms of $L_A(s)$ is

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \tag{27}$$

which is not nonsingular, $L_A(s)$ is not row proper. In fact, row degrees of $L_A(s)$ are $\{r_1, r_2, r_3\} = \{1, 3, 2\}$ which do not coincide with the structure at infinity of $G(s)$.

On the other hand, the following $L_B(s)$ is one example of regular interactors of $G(s)$.

$$L_B(s) = \begin{bmatrix} s & s & 0 \\ s & 0 & s \\ -s^3 & s^3 & -s^2 \end{bmatrix} \tag{28}$$

$L_B(s)$ satisfies eq.(26), and its coefficient matrix of the maximum row degree terms is

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \tag{29}$$

which is nonsingular, and hence, $L_B(s)$ is row proper. In fact, the row degrees of $L_B(s)$, $\{r_1, r_2, r_3\} = \{1, 1, 3\}$ coincide with the structure at infinity of $G(s)$.

It should be noted that this system does not have a lower left regular interactor.

But, if the second and the third rows of $G(s)$ are exchanged, i.e.,

$$WG(s) = \begin{bmatrix} \frac{s^2+1}{s^3} & \frac{1}{s} & \frac{1}{s} \\ \frac{1}{s} & \frac{2}{s} & \frac{3s^2+3}{s^3} \\ \frac{s+1}{s^2} & \frac{s^2+2s+2}{s^3} & \frac{s+3}{s^2} \end{bmatrix} \tag{30}$$

where W is the row permutation matrix, then $WG(s)$ has the following lower left and regular interactor.

$$L_1(s) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ -s^3 & -s^2 & s^3 \end{bmatrix} \tag{31}$$

It should be noted that once the diagonal elements of the rower left triangular and regular matrix are fixed, its off diagonal elements are determined uniquely. This is an important property of the lowe left and regular interactor, which is very useful for the adaptive control.

IV. Conclusions

In this paper, we considered the problem of finding the relationship between the structure at infinity of a linear multivariable system and the structure of degrees of its interactor matrix. For this purpose, the regular interactor was defined as the interactor whose row degrees coincide with the structure at infinity of the plant. It was shown that the interactor is regular if and only if the interactor is row proper.

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