ESTIMATION OF THE DOMAIN CONTAINING ALL COMPACT INVARIANT SETS OF THE VIRAL INFECTION MODEL

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Abstract

In this paper we examine the localization problem of compact invariant sets of nonlinear time-varying systems with the differentiable right-side. We extend our results respecting the localization problem obtained earlier for time-invariant systems and apply them to a viral infection model.

Dynamical properties of the viral infection model is nowadays an intensively studied topic, see e.g. in [1] and references therein. In this paper we examine equations of the viral infection model

$$\begin{aligned} \dot{x} &= \lambda - dx - \beta xy, \\ \dot{y} &= \beta xy - ay - p(t)yz, \\ \dot{z} &= cy - bz, \\ & x(0) > 0,\ y(0) > 0,\ z(0) > 0, \end{aligned} \tag{1}$$

see [1]. Here $x(t)$ describes susceptible host cells, $y(t)$ is the virus population and $z(t)$ is a CTL response. The coefficient $p(t)$ expresses the strength of the lytic component, which is a general continuous positive periodic nonzero function of time. The main purpose of this paper is to extend previous results of authors on the localization of all compact invariant sets of nonlinear time-invariant systems and apply them to a viral infection model. Here by a localization we mean a description of a set containing all compact invariant sets of the system (1) in terms of equalities and inequalities defined in the state space $\mathbf{R}^3 = \{(x_1, x_2, x_3)^T\}$. Our approach is based on using the first order extremum conditions.

Consider the nonlinear C^∞ - differentiable time-varying system

$$\dot{x} = f(x,t), \quad f(x,t) = (f_1(x,t), \ldots, f_n(x,t))^T, \tag{2}$$

where $x \in \mathbf{R}^n$ is a state vector. Let $\psi(t, t_0, x)$ be a solution of the system (2), $\psi(t_0, t_0, x) = x$.

We define a (locally) invariant set $G \subset \mathbf{R}^n$ of the system (2) as a family of sets $\{G(t) \mid t \in \mathbf{R}\} \subset \mathbf{R}^n$, $G = \cup_{t \in \mathbf{R}} G(t)$, such that for any t_0 and any $x \in G(t_0)$ exists $\delta > 0$ such that: $\psi(t, t_0, x) \subset G(t)$, with $t \in (t_0 - \delta, t_0 + \delta)$ for some $\delta = \delta(x, t_0) > 0$.

Our goal is to obtain results on localizing of compact invariant sets of the system (2). We talk that we get a localization set $K \subset \mathbf{R}^n$ if it contains all compact invariant sets of the system (2). We introduce a C^∞ - differentiable function $h(x)$, $x \in \mathbf{R}^n$, which is considered as a function defined on the space $\mathbf{R}^n \times \mathbf{R} = \{(x, t)\}$ as well. We assume that $h(x)$ is not the first integral of (2). The function $h(x)$ is called localizing. By $h|_M$ we denote the restriction of the function $h(x)$ on a set M. By $\mathcal{L}_f h$ we denote the Lie derivative of the function h with respect to the vector field f,

$$\mathcal{L}_f h|_{(x,t)} = \sum_{i=1}^{n} f_i(x, t) \frac{\partial h(x)}{\partial x_i}.$$

By S_h we denote the set $\{(x, t) \in \mathbf{R}^n \times \mathbf{R} | \mathcal{L}_f h|_{(x,t)} = 0\}$. Let $\pi : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ be the projection $\pi(x, t) = x$, and $s_h = \pi(S_h)$. It means that $x \in s_h$ iff exists $t \in \mathbf{R}$ such that $(x, t) \in S_h$. We fix any function $h(x)$ and propose to use numbers $h_{\inf} := \inf\{h(x) | x \in s_h\}$; $h_{\sup} := \sup\{h(x) | x \in s_h\}$ for studying a location of compact invariant sets of the system (2). Namely, we have

Theorem 1 *Each compact invariant set of (2) is contained in the set*

$$K(h) := \{x \in \mathbf{R}^n | h_{\inf} \leq h(x) \leq h_{\sup}\}.$$

Suppose that we are interested in the localization of all compact invariant sets located in some subset Q of the state space $\mathbf{R}^n$. We state

Theorem 2 *Each compact invariant set of (2) from the set $Q \subset \mathbf{R}^n$ is contained in the set*

$$K(h, Q) := \{x \in \mathbf{R}^n | h_{\inf}(Q) \leq h(x) \leq h_{\sup}(Q)\} \cap Q,$$

where $h_{\inf}(Q) := \inf\{h(x) | x \in s_h \cap Q\}$; $h_{\sup}(Q) := \sup\{h(x) | x \in s_h \cap Q\}$.

Proposition 3 *If $Q \cap s(h) = \emptyset$ then the system (2) has no compact invariant sets located in Q.*

Our results are obtained by using these results. It was established in [1] the following assertion:

Theorem 4 *All solutions of (1) are positive for $t > 0$ and there exists constant M such that all solutions satisfy for all large t the inequality $x(t) < M; y(t) < M; z(t) < M$*

This result can be refined for compact invariant sets in the following way.

Theorem 5 *All compact invariant sets of (1) are located in the set*

$$K_1 = \{ (x, y, z) \in \mathbf{R}^3_+ | \; x + y + \frac{a}{2c} z \leq \frac{\lambda}{m} \},$$

with $m := \min(d; \frac{a}{2}; b)$.

Theorem 6 *All compact invariant sets of (1) are located in the set*

$$K_2 = \{ (x, y, z) \in \mathbf{R}^3_+ | \; x \leq \frac{\lambda}{d}; z \leq \frac{c\lambda}{bm} \}.$$

Theorem 7 *Let* $a \geq d$. *All compact invariant sets of (1) are located in the set*

$$K_3 = \{ (x, y, z) \in \mathbf{R}^3_+ | \; x + y \leq \frac{\lambda}{2b} \}.$$

Finally we establish localizing bounds depended on bounds for the function $p(t)$.

Theorem 8 *1. Suppose that* $\max(a; d) < 2b$. *Then all compact invariant sets are contained in*

$$K_{4a}(\xi) = \{ (x, y, z) \in \mathbf{R}^3_+ | \; \xi z^2 + x + y \geq \frac{\lambda}{2b} \}$$

with a parameter ξ *satisfying the inequality*

$$\xi > \frac{\max p(t)}{2c} = \frac{p^*}{2c}.$$

2. Suppose that $\min(a; d) > 2b$. *Then all compact invariant sets are contained in*

$$K_{4b}(\xi) = \{ (x, y, z) \in \mathbf{R}^3_+ | \; \xi z^2 + x + y \leq \frac{\lambda}{2b} \}$$

with a positive parameter ξ *satisfying the inequality*

$$\xi < \frac{\min p(t)}{2c} = \frac{p_*}{2c}.$$

Calculating intersection of the sets $K_{4a}(\xi)$ we obtain the localizing set

$$K_{4a} = \cap K_{4a}(\xi) = \{ (x, y, z) \in \mathbf{R}^3_+ | \; \frac{p^*}{2c} z^2 + x + y \geq \frac{\lambda}{2b} \}$$

in the case $\max(a; d) < 2b$ and the localizing set

$$K_{4b} = \cap K_{4b}(\xi) = \{ (x, y, z) \in \mathbf{R}^3_+ | \; x + y \leq \frac{\lambda}{2b}, \quad \frac{p_*}{2c} z^2 + x + y \leq \frac{\lambda}{2b} \}$$

that is the intersection of the sets $K_{4b}(\xi)$ in the case $\max(a; d) > 2b$.

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References

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