Curvature Index Failures in the Dynamics of an Electric Coupled Oscillator

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Abstract—In this work, the dynamics of an electric coupled oscillator is studied. The attention is centered on the first and second curvature indexes or Lyapunov coefficients of a Hopf bifurcation. The analysis is performed using the frequency domain method to calculate analytical expression of the indexes, and then determine the location of the first two curvature singularities. Numerical continuations are used to determine the dynamical behavior of the system in the vicinity of these points. A relatively complex structure of nested limit cycles was found near these degeneracies.

I. INTRODUCTION

The curvature indexes, also known as Lyapunov coefficients or focal values, define the stability of a limit cycle emerging from a Hopf bifurcation. There are several ways to compute these coefficients, most of them reduce the system to the normal form of the Hopf bifurcation using different techniques (Poincaré normal forms, Lyapunov functions, power series expansion, etc., as shown in [1]–[4]). This task sometimes is cumbersome, especially in high dimensional systems [3]. An alternative method is given by the frequency domain approach which uses some well known techniques like harmonic balance, Nyquist stability criterion and state space formulations [5]–[7] to compute these indexes. These tools are somewhat standard in electrical and electronics engineering.

In this article we analyze the dynamics of an electric oscillator using bifurcation theory. The attention is focused on computing Hopf bifurcations curves and the conditions leading to the first and second curvature index failures. The dynamical scenario emerging from these degeneracies (thoroughly analyzed in [2] and [8] using singularity theory) are obtained using AUTO [9]. Some global phenomena, including connections of cyclic fold bifurcations and cyclic cusp points are also described.

This work is organized as follows. In Section II a brief introduction to the frequency domain approach to analyze oscillations is given. In Section III the state space formulation of the electric circuit and its corresponding frequency domain representation are provided. The dynamics associated to the first and second curvature index failures is studied in Sections IV and V, respectively. Finally, some concluding remarks are given in Section VI.

II. HOPF BIFURCATION IN THE FREQUENCY DOMAIN

This section presents the basic concepts for the analysis of Hopf bifurcations using the frequency domain approach. The rigorous analysis is formulated in [5], [6].

Let us consider an n-dimensional nonlinear dynamical system described by

\[ \dot{x} = F(x; \eta), \]  

where \( x \in \mathbb{R}^n \) is the state vector, \( \eta \in \mathbb{R}^l \) is the parameter vector and \( F : \mathbb{R}^n \times \mathbb{R}^l \rightarrow \mathbb{R}^n \) is a smooth nonlinear vector field that satisfies \( F(0; \eta_0) = 0 \).

The autonomous system (1) can be represented in feedback form with the linear part in the forward path and the nonlinear part in the feedback path, as

\[
\begin{align*}
\dot{x} &= Ax + BDy + Bu, \\
y &= Cx, \\
u &= g(y; \eta) - Dy,
\end{align*}
\]

where \( A \in \mathbb{R}^{n \times n} \) is the linear part of the original system, \( g : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}^l \) is the nonlinear part, \( B \in \mathbb{R}^{n \times l} \), \( C \in \mathbb{R}^{m \times n} \) and \( D \in \mathbb{R}^{l \times m} \). Matrix \( D \) is arbitrary, revealing that there exist infinitely many equivalent feedback representations of (1). In addition, this extra degree of freedom can be used to obtain more suitable representations. Notice that all matrices may depend on \( \eta \). The resulting multivariable feedback system is represented in Fig. 1a, where \( v \in \mathbb{R}^l \) is the input vector, \( d \in \mathbb{R}^m \) is a disturbance input and both are set to zero, \( G(s; \eta) \) is the transfer function matrix of the forward path defined as

\[ G(s; \eta) = C[sI - (A + BDC)]^{-1}B, \]

where \( f(e; \eta) \) is the nonlinear part given by

\[ f(e; \eta) = g(y; \eta) - Dy, \]

and \( e = -y \).

The equilibrium point of the system is obtained solving

\[ G(0; \eta)f(e; \eta) + e = 0. \]

The Jacobian matrix of the nonlinear function \( f(e; \eta) \) evaluated at the equilibrium results

\[ J(\eta) = \left. \frac{\partial f(e; \eta)}{\partial e} \right|_{e = e}. \]

A schematic representation of the linearized feedback scheme is shown in Fig. 1b.

Thus, if the open loop transfer function matrix \( G(\imath \omega; \eta)J(\eta) \) has an eigenvalue at the critical point \(-1 + \imath 0\)
for \( \eta = \eta_0 \) and \( \omega = \omega_0 \) (\( \omega_0 \neq 0 \)), then a pair of eigenvalues of the closed loop system linearization assume the values \( \pm i\omega_0 \), setting the defining condition of a Hopf bifurcation. Then, generically, as the main bifurcation parameter \( \eta \) is varied, the equilibrium changes stability at \( \eta = \eta_{10} \) and a limit cycle emerges from it. The remaining parameters \( \eta_j \), \( j \neq i \) are auxiliary and fixed. In this framework, the graphical Hopf bifurcation method [5], [6] can be used to obtain an approximate solution of the limit cycle using well known techniques like harmonic balance method and Nyquist stability criterium. In addition, the method provides a way to derive analytical expressions for computing curvature indexes. The first and second curvature indexes, namely \( \sigma_1 \) and \( \sigma_2 \), formulas are given in the Appendix.

In the following we will use the notation \( H_{10} \) and \( H_{20} \) to indicate the first and second curvature index failures, respectively. The use of the notation \( H_{ij} \), with \( i, j = 0 \ldots n \), is very common in the literature, where the subindex \( i \) indicate the number of curvature indexes that are zero and \( j \) denotes if there is a transversally degeneration of the Hopf bifurcation. By definition, \( j = 0 \) means that a simple pair of complex conjugate eigenvalues effectively cross the imaginary axis transversally.

### III. Coupled electric oscillator

In this section the dynamics of a coupled electric circuit is analyzed using the frequency domain method. The schematic diagram of the oscillator is shown in Fig. 2 and is similar to the one studied in [10]–[12]. The mathematical model is easily obtained by simply applying the Kirchoff laws and setting the state variables: \( x_1 = v_{C_1}, x_2 = i_{L_1}, x_3 = v_{C_2} \) and \( x_4 = i_{L_2} \). In addition, the nonlinear element is characterized by the current-voltage \( (i_G - v_G) \) relation \( i_G = \frac{1}{2}v_G - \frac{1}{2}v_G^3 + \frac{3}{5}v_G^5 \). The values of the remaining elements are: \( C_1 = 1/\eta_1, C_2 = 1/(1 + \sqrt{2}), L_1 = 1/\eta_3, L_2 = 1/(2 - \sqrt{2}) \) and \( R = \eta_2 \). The bifurcation parameters used here are \( \eta_1, \eta_2 \) and \( \eta_3 \), while the values of the rest of the components are chosen such that there is an irrational relation between the frequencies of oscillation. Therefore, the mathematical model of the electric oscillator is given by

\[
\begin{align*}
\dot{x}_1 &= \eta_1 \left[ \frac{1}{2}x_1 - \frac{3}{2}x_2^2 - x_3^3 + x_2 - x_4 \right], \\
\dot{x}_2 &= -\eta_3 x_1, \\
\dot{x}_3 &= (1 + \sqrt{2}) x_4, \\
\dot{x}_4 &= (2 - \sqrt{2}) [x_1 - x_3 - \eta_2 x_4].
\end{align*}
\]

(7)

By inspection of (7) it is easy to see that the nonlinearity of the system is restricted to the first equation and depends only on \( x_1 \). As will be shown next, this fact greatly simplifies the representation of the system in the frequency domain, since the matrix \( G(s; \eta) \) results one-dimensional, and hence there is a unique eigenvalue. The feedback representation considered for (7) is

\[
\begin{bmatrix}
0 & \eta_1 & 0 & -\eta_1 \\
-\eta_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 + \sqrt{2} \\
2 + \sqrt{2} & 0 & -2 + \sqrt{2} & (2 - \sqrt{2}) \eta_2
\end{bmatrix}
\]

(8)

\[
B = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T, \\
C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \\
D = \begin{bmatrix} 0 \end{bmatrix}, \\
u = g(y; \eta) - Dy = \eta_1 \begin{bmatrix} 1 & \frac{1}{2}x_1 - \frac{3}{5}x_2^2 - x_3^3 \end{bmatrix}
\]

(9)

(10)

(11)

In this case, the output is \( y = -e = x_1 \). The resulting expressions for the transfer function \( G(s; \eta) \) and the nonlinear function \( f(e; \eta) \) are

\[
G(s; \eta) = \frac{s \left( s^2 + \rho \eta_2 s^3 + \sqrt{2} \right)}{s^4 + \rho \eta_2 s^3 + \sqrt{2} + (\rho + \eta_3) \eta_1 \left( s^2 + \rho \eta_2 \eta_3 s + \sqrt{2} \eta_3 \eta_1 \right)},
\]

(13)

\[
f(e; \eta) = \eta_1 \left[ -\frac{1}{2}e - \frac{3}{5}e^2 + e^3 \right].
\]

(14)

where \( \rho = 2 - \sqrt{2} \). From (5), the equilibrium point is \( \hat{e} = 0 \), since \( G(s; \eta) \) has a zero at \( s = 0 \). The Jacobian results in

\[
J(\eta) = \frac{\partial f(e; \eta)}{\partial e} \bigg|_{e=0} = -\frac{1}{2} \eta_1.
\]

(15)

Since \( G(s; \eta) \) and \( J(\eta) \) are scalar quantities, the open loop transfer function \( G(s; \eta)J(\eta) \) has an unique eigenvalue given by

\[
\lambda(\omega; \eta) = G(s; \eta)J(\eta) \bigg|_{s = i\omega},
\]

(16)
The singularity is analyzed in the next section.

Therefore, the Hopf bifurcation points are obtained solving

\[
\begin{align*}
\Re\{\lambda\} &= -1, \\
3\{\lambda\} &= 0.
\end{align*}
\]  

(17)

In this particular case, it is easy to obtain the critical values of \(\eta_1\) and the frequency of the oscillation \(\omega_0\) in terms of \(\eta_2\) and \(\eta_3\). The resulting formulas are very extensive to be included in this paper, but they are not difficult to obtain using any symbolic math software. The solution obtained for \(\eta_3 = 0.65\) are shown in Fig. 3. As seen on Fig. 3a, the Hopf bifurcation curve is divided in two branches, namely \(H_1\) and \(H_2\) and both intersect at a Hopf-Hopf bifurcation (also called double Hopf) point \((HH)\) at \(\eta_1 = 2.17571\) and \(\eta_2 = 1.85708\). A plot of the frequency of the emerging limit cycle from the Hopf bifurcation is shown on Fig. 3b, where \(\omega_{H_1}\) is associated with \(H_1\) and \(\omega_{H_2}\) with \(H_2\). Notice that both frequencies coincide at the maximum point of the Hopf curve and are different for the Hopf-Hopf point.

Hopf-Hopf bifurcation explains the appearance of quasiperiodic oscillations in this circuit. At least four cases related to the unfolding of the truncated normal form [13] of the Hopf-Hopf bifurcation can be recognized in the parameter plane \(\eta_1 - \eta_2\) for different values of \(\eta_3\). Three of them belong to the simple cases (2D quasiperiodic oscillations or tori) and the remaining one to the complex cases (2D and 3D tori). These phenomena as well as the location of a fold-flip bifurcation of periodic orbits, for a similar circuit can be seen in [12]. The transition from simple to complex scenarios is related to the failure of the first curvature index of \(H_1\). This singularity is analyzed in the next section.

IV. FIRST CURVATURE INDEX FAILURE, \(H_{10}\)

The \(H_{10}\) singularity occurs when the first curvature index of a Hopf bifurcation becomes zero, this situation is also known in the literature as generalized Hopf or Bautin bifurcation [13]. The common techniques used to study this singularity are Poincaré normal forms, Lyapunov functions, power series expansion, etc. In this work we use the frequency domain method described in Section II. The formula to compute \(\sigma_1\) is given in the Appendix. Then, using (18) and replacing the parameter values and frequency corresponding to the Hopf bifurcation curve [i.e. the solution of (17)], the value of the curvature index is obtained. The plot of \(\sigma_1\) versus \(\eta_2\) is shown in Fig. 4. Notice that there are two crossings by zero, which means that there is a couple of \(H_{10}\) singularities. The first curvature failure \(H_{10}^a\) is located at \((\eta_1, \eta_2) = (2.16308, 1.90484)\), while the second failure \(H_{10}^b\) is at \((\eta_1, \eta_2) = (2.30433, 1.65203)\). The results obtained with the frequency domain method were confirmed using the continuation programs AUTO and MATCONT [14].

It is well known that a cyclic fold bifurcation curve emerges from this singularity [13]. The curves arising in both generalized Hopf points are connected through a complex structure of cyclic fold (CF) bifurcations and cuspidal points (C), as shown in the continuation diagram of Fig. 5 for \(\eta_3 = 0.65\). The local scenarios around \(H_{10}^a\) and \(H_{10}^b\) are shown in Figs. 6 and 7, respectively. The dynamical behavior of the system around the point \(H_{10}^a\) (the situation is analogous for \(H_{10}^b\)) is better described if a couple of cross-sections above and below the singularity are made. The bifurcation diagrams varying \(\eta_1\) for \(\eta_2 = 1.92\) (above \(H_{10}^a\)) and \(\eta_2 = 1.88\) (below \(H_{10}^b\)) are shown in Figs. 8a and 8b, respectively. There, the dashed lines represent unstable equilibria and unstable cycles and the solid lines are stable equilibria and stable cycles.

Starting with the bifurcation diagram for \(\eta_2 = 1.92\) (Fig. 8a), the equilibrium undergoes a supercritical Hopf bifurcation \(H_1^a\) (the first curvature index is negative), then the limit cycle is stable and increases its amplitude towards...
the left (decreasing values of $\eta_1$). The stable limit cycle collapses with an unstable cycle at $CF_2$. This phenomenon is not associated directly with the first curvature index failure.

The situation for $\eta_2 = 1.88$ (Fig. 8b) is quite different. The equilibrium point exhibits a subcritical Hopf bifurcation $H^+_1$, since the curvature index becomes positive due to $H_{10}^a$. Then, the limit cycle is unstable and increases its amplitude towards the right until it collapses with a stable limit cycle at $CF_1$. This cyclic fold bifurcation is in direct relation with $H_{10}^a$. The stable limit cycle increases its amplitude and coalesces with an unstable cycle at $CF_2$. Notice that there is a range of values of $\eta_1$ where three cycles coexist. This situation is similar to that occurring in the vicinity of the second curvature index failure, but as will be shown next, $CF_2$ is not related to $H_{20}$.

![Fig. 6. Detailed view of the singularity $H_{10}^a$.](image)

![Fig. 7. Detailed view of the singularity $H_{10}^b$.](image)

V. SECOND CURVATURE INDEX FAILURE, $H_{20}$

The second curvature index failure is obtained when both Lyapunov coefficients are simultaneously zero, i.e., $\sigma_1 = \sigma_2 = 0$ and the singularity is denoted as $H_{20}$. In this complex co-dimension three phenomenon, a curve of cyclic cusp bifurcations emerges from the point $H_{20}$, instead of the cyclic fold curve arising in the $H_{10}$ bifurcation, so there are expected to exist up to three limit cycles in a neighborhood of this singularity. Furthermore, in this case, three bifurcation parameters need to be varied ($\eta_1$, $\eta_2$, and $\eta_3$) to locate this degeneracy. To detect the second index failure it is necessary to obtain the points of the parameter space where $\sigma_1 = 0$. This task was performed numerically using LOCBIF [15], resulting in the curve of $H_{10}$ bifurcations shown in Fig. 9. Obviously these points satisfy the conditions of the Hopf bifurcations and thus the corresponding frequency can be obtained from (17). Then $\sigma_2$ is computed using (19) for the parameter values where $\sigma_1 = 0$, and a $H_{20}$ singularity was detected at $(\eta_1, \eta_2, \eta_3) = (2.235, 1.978, 0.631)$.

The unfolding of the $H_{20}$ singularity on the parameter plane $\eta_2 - \eta_3$ is shown in Fig. 10 and the corresponding bifurcation diagrams that describe the dynamical behavior of the system around this singular point are shown in Fig. 11.
Figure 10 is obtained projecting the $H_{10}$ curve in the plane $\eta_2 - \eta_3$. The $H_{20}$ point divides the $H_{10}$ curve in two, namely $H_{10}^+$ and $H_{10}^-$ according to the sign of the second curvature index. Basically this singularity splits the parameter space in three regions with different dynamical phenomena. In order to describe the dynamical behavior of the system in a neighborhood of the $H_{20}$ point, continuation diagrams are obtained varying $\eta_1$ for values of $\eta_2$ and $\eta_3$ fixed.

Setting $\eta_2 = 1.980$ and $\eta_3 = 0.629$ (region 1 in Fig. 10) the diagram of Fig. 11a is obtained. An unstable limit cycle, born at the Hopf bifurcation $H_{10}^+$, collides with a stable cycle at $CF_1$. This stable cycle suffers another cyclic fold bifurcation at $CF_2$ where it collapses with an unstable cycle. This phenomenon is global and does not belong to the local unfolding.

The dynamics associated to region 2 is shown in Fig. 11b. The Hopf bifurcation becomes supercritical since the curve $H_{10}^-$ is crossed, so the limit cycle is now stable and grows its amplitude towards the left until it suffers the cyclic fold bifurcation $CF_0$ with an unstable limit cycle. This unstable cycle collides with a stable limit cycle at $CF_1$. This last cycle collapses with the global bifurcation $CF_2$ as before. Therefore, in this region at least three nested limit cycle surrounding the equilibrium point do exist (actually there are more cycles due to global bifurcations not directly associated with the $H_{20}$ singularity).

The situation becomes far more simple in region 3 (Fig. 11c), since both cyclic fold bifurcations $CF_0$ and $CF_1$ disappear in a cusp point when crossing the curve $C$ from region 2 to 3 (see Fig. 10). Therefore, in region 3 there is only a stable limit cycle that emanates towards the left from the supercritical Hopf bifurcation $H_{10}^-$. Again, this cycle disappears due to the global cyclic fold $CF_2$.

The transition from region 3 back to region 1 is characterized by a change in the stability of the limit cycle emerging from the Hopf bifurcation, that becomes unstable (when crossing $H_{10}^+$) and grows towards the right until it vanishes at the cyclic fold bifurcation $CF_1$, as explained before.

It is interesting to notice the change in the bifurcation structure in the neighborhood of $H_{10}$ due to the second curvature index failure. Toward this end, the continuation diagram in the parameter plane $\eta_1 - \eta_2$ for $\eta_3 = 0.626$ is derived and is shown in Fig. 12. Notice that the scenarios for $\eta_2 = 1.985, 1.9875$ and 1.99 can be associated to those on Figs. 11a, b and c, respectively.

Finally, the location of the second curvature index failure was confirmed using MATCONT, since this program calculates numerically the value of the second Lyapunov coefficient every time a generalized Hopf bifurcation is found.
VI. Conclusions

The frequency domain method provides analytical closed form expressions to calculate the curvature indexes. These expressions might become more or less complicated depending on the system nonlinearity. In the case studied here, the formulas have been easily obtained, since the nonlinearity depends only on one variable of the system.

In this work we combine the analytic power of the frequency domain approach of the Hopf bifurcation with the numerical continuation results to describe the complexity of multiple curvature failures involving up to three limit cycles in the unfolding of certain organizing centers of the dynamics.

APPENDIX

CURVATURE INDEX FORMULAS

The expressions of the first and second curvature indexes in the frequency domain are [6], [7]

\[
\sigma_1 = -\Re \left\{ \frac{w^T G(\omega)p_1}{w^T G'(\omega)J V_{11}} \right\} = -\Re \{ \gamma_1 \},
\]

\[
\sigma_2 = -\Re \left\{ \frac{w^T}{w^T G'(\omega)J V_{11}} \left[ -\gamma_1 G'(\omega)(J V_{13} + p_1) \right] \right\}
- \gamma_1 G(\omega)p_1 + \frac{1}{2} \gamma_i^2 G''(\omega)J V_{11} + G(\omega)p_2 \right\},
\]

where

\[
G'(\omega) = \left( \frac{dG(\omega)}{ds} \right)_{s=\omega}, \quad G''(\omega) = \left( \frac{d^2G(\omega)}{ds^2} \right)_{s=\omega},
\]

\[
p_1 = D_2 \left[ \frac{1}{2} V_{11} V_{22} + V_{11} V_{02} \right] + \frac{1}{8} D_3 V_{11}^2 V_{11},
\]

\[
p_1' = \frac{dp_1(s)}{ds},
\]

\[
p_2 = \frac{D_2}{2} \left[ 2V_{11} V_{04} + 2V_{02} V_{13} + V_{22} V_{33} + V_{11} V_{24} + V_{13} V_{22} \right] + \frac{D_3}{8} [4V_{11} V_{02}^2 + 2V_{11} V_{22} V_{22} + 2V_{11} V_{11} V_{13} ]
+ \frac{D_3}{4} \left[ V_{11}^2 V_{22} + 6V_{11}^2 V_{11} V_{02} + 3V_{11} V_{11} V_{11} V_{22} \right] + \frac{D_5}{192} V_{11}^4 V_{11},
\]

\[
H(s) = [I + G(s)]^{-1} G(s),
\]

\[
V_{02} = -\frac{H(0)}{4} D_2 V_{11} V_{11},
\]

\[
V_{22} = -\frac{H(2i\omega)}{4} D_2 V_{11},
\]

\[
V_{33} = -\frac{H(3i\omega)}{4} \left[ 2D_2 V_{11} V_{22} + \frac{1}{6} D_3 V_{11}^3 \right],
\]

\[
V_{04} = -\frac{H(0)}{4} \left[ D_2 \left[ 2V_{02}^2 + V_{22} V_{22} \right]
+ V_{11} V_{13} + V_{11} V_{13} \right] + \frac{D_3}{4} [2V_{11} V_{22} + V_{11} V_{11} V_{02} + \frac{D_4}{16} V_{11}^2 V_{11}^2 ],
\]

\[
V_{24} = -\frac{H(2i\omega)}{4} \left[ 2D_2 \left[ 2V_{02} V_{22} + V_{11} V_{33} \right] + D_3 \left[ V_{11} V_{02} + V_{11} V_{11} V_{22} \right] + \frac{D_4}{12} V_{11}^3 V_{11} \right],
\]

where \( V_{11} \) and \( w \) are the right and left eigenvectors of the matrix \( G(\omega)J \), \( \omega \) is the frequency at the Hopf bifurcation, \( D_i \) indicate the \( i \)th partial derivative evaluated at the equilibrium \( \bar{e} \), and \( V_{13} \) is calculated solving

\[
(I - V_{11} V_{11}^T) [I + G(\omega)J] V_{13} = -(I - V_{11} V_{11}^T) G(\omega)p_1,
\]

with the restriction \( V_{11} \perp V_{13} \).

Notice that in this particular case \( V_{11} = 0 \) since \( V_{11} = 1 \). In addition, \( V_{02} = V_{04} = 0 \) since \( G(0) = 0 \) (then \( H(0) = 0 \)). These facts simplify greatly the calculations of the remaining vectors used to obtain the expressions of the curvature indexes.

REFERENCES


