GLOBAL FINITE-TIME POSITIONING OF ROBOT MANIPULATORS WITH BOUNDED INPUTS

Yuxin Su
School of Electro-Mechanical Engineering, Xidian University, China
CINVESTAV-Saltillo, Mexico
yxsu@mail.xidian.edu.cn

Peter C. Müller
Safety Control Engineering
University of Wuppertal
Germany
mueller@srm.uni-wuppertal.de

Abstract
This paper addresses the global finite-time positioning problem for robot manipulators in the presence of actuator constraints, and a very simple bounded proportional-derivative (PD) control plus desired gravity compensation is formulated as solution to this problem. The global finite-time stability of the closed-loop system is proved with Lyapunov's direct method and finite-time stability theory. Simulations on a two-degrees-of-freedom (2-DOF) manipulator demonstrate the effectiveness of the proposed approach.

Key words
Robot control, finite-time stability, nonlinear proportional-derivative (NPD) control, actuator constraints.

1 Introduction

Robot control is a key problem in both theory and practice. Several control techniques that stabilize arbitrary positions of robot manipulators can be found in the literature. However, it is only in recent years that the research community has started to address the problem of torque/force limitation within the control law design for these systems [Morabito, Teel, and Zaccarian, 2004]. In particular, Kelly, Santibanez, and Berghuis (1997) proposed a global asymptotic regulating controller that is composed of a saturated PD feedback loop plus an exact model knowledge feedforward gravity compensation term. More recently, Zavala-Rio and Santibanez (2007) extended this approach to involve only one saturation function at each joint, and showed that the proposed approach can be conceived within the framework of the energy shaping plus damping injection methodology. Morabito *et al.* (2004) integrated a static nonlinear controller into an available PD control plus exact gravity compensation to guarantee the global setpoint control for the Euler-Lagrange system with input saturation. Zergeroglu, Dixon, Behal, and Dawson (2000) proposed adaptive regulation control to remove the requirement of exact knowledge of the gravity terms. Dixon (2007) formulated an adaptive regulation control scheme to solve the bounded regulation of uncertain robot manipulators in both kinematics and dynamics. To remove the requirement of the velocity measurements, Colbaugh, Barany, and Glass (1997) designed a global asymptotic regulating controllers that compensate for uncertainty; however, the control strategy switches between one controller that is used to drive the setpoint error to a small value, and another controller that is used to drive the setpoint error to zero. Recently, Alvarez-Ramirez, Kelly, and Cervantes (2003) formulated a saturated linear PID controller by resorting an additional saturated integral term to avoid the evaluation of the gravity term and obtained a semiglobal stability result.

The mentioned above schemes regulate robot manipulators to arbitrary position asymptotically. Asymptotic stability implies that the system trajectories converge to the equilibrium as time goes to infinity. It is now known that finite-time stabilization offers an effective alternative, which yields, in some senses, fast response, high-precision and disturbance rejection properties [Bhat and Bernstein, 1998, 2000; Hong, Xu, and Huang, 2002; Perruquetti and Drakunov, 2000]. In particular, for robot manipulators, Barambones and Etxebarria (2002) formulated a terminal sliding-mode adaptive control scheme for zero trajectory-tracking error in finite time. Gruyitch and Kokosy (1999) designed a sliding-mode controller to guarantee the robust global stability and attraction with a finite time. Parra-Vega, Rodriguez-Angeles, and Hirzinger (2001) proposed a dynamic sliding controller to implement the perfect tracking defined as the performance of zero tracking errors of position and force in finite time. Yu, Yu, Shirinzadeh, and Man (2005) formulated a continuous finite-time tracking controller for robot manipulators by using a new form of terminal sliding modes and showed the faster and high-precision tracking.

While these finite-time control schemes are elegant, and intuitively appealing, there is an implicit assumption in the development of these schemes that the manipulator actuators are able to provide any requested torque. This assumption can lead to difficulties in practical implementation since the available torque amplitude is limited in actual manipulators. To the best of our knowledge, the only previous work which targets at the finite-time regulation is given in [Hong, Xu, and Huang, 2002]. Specially, Hong et al. (2002) formulated a nonsmooth PD plus gravity compensation scheme and achieved a local finite-time result. As pointed by Kasac, Novakovic, Majetic, and Brezaket (2006), it is often difficult to explicitly characterize the domain of attraction that could be much smaller than the robot workspace. This means a global result is always more useful for both theoretical analysis and practical implementation.

In this paper the global finite-time positioning of robot manipulators under actuator constraints is addressed, which aims at designing a simple bounded PD plus gravity compensation control law such that the position of the robot can be regulated into a desired position in finite time. Compared with the bounded PD plus gravity compensation regulator proposed by Kelly et al. (1997), the developed approach gives a finite-time stabilization result and thus fast transit is obtained. Simulations performed on a two-DOF manipulator demonstrate the effectiveness of the proposed approach.

Throughout this paper, we use the notation A_m and A_M to indicate the smallest and largest eigenvalues, respectively, of a symmetric positive definite bounded matrix A. The norm of a vector x is defined as $\|x\| = \sqrt{x^T x}$ and that of a matrix A is defined as the corresponding induced norm $\|A\| = \sqrt{\lambda_M(A^T A)}$, and I denotes an identity matrix of the appropriate dimension.

2 Preliminaries

2.1 Robot Manipulator Model and Properties

In the absence of disturbances, the dynamics of an n-DOF robot manipulator can be written as [Arimoto, 1996; Sciavicco and Siciliano, 2000]

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + G(q) = \tau \tag{1}$$

where $q, \dot{q}, \ddot{q} \in \Re^n$ denote the link position, velocity, and acceleration, respectively, $M(q) \in \Re^{n\times n}$ represents the symmetric inertia matrix, $C(q,\dot{q}) \in \Re^{n\times n}$ denotes the centrifugal-Coriolis matrix, $D \in \Re^{n\times n}$ stands for the matrix composed of damping friction coefficients for each joint, $G(q) = \partial U(q)/\partial q \in \Re^n$ is a gravitational force, $U(q)$ is the potential energy, and $\tau \in \Re^n$ denotes the torque input vector. By recalling the robot manipulators considered, the following properties can be established [Arimoto, 1996; Sciavicco and Siciliano, 2000].

Property 1. The matrix D is a diagonal positive definite matrix.

Property 2. The matrix $M(q)$ is positive definite and bounded by

$$0 < M_m \le \|M(q)\| \le M_M \tag{2}$$

where M_m and M_M are some positive constants.

Property 3. The matrix $C(q,\dot{q})$ is defined using Christoffel symbols, and $\dot{M}(q) - 2C(q,\dot{q})$ is skew-symmetric, i.e.

$$\zeta^T\left(\dot{M}(q) - 2C(q,\dot{q})\right)\zeta = 0, \ \forall \zeta \in \Re^n \tag{3}$$

where $\dot{M}(q)$ is the time derivative of the inertia matrix $M(q)$.

Property 4. The matrix $C(q,\dot{q})$ satisfies the following relationship:

$$C(q,\xi)\upsilon = C(q,\upsilon)\xi, \qquad \forall \xi,\ \upsilon \in \Re^n \tag{4}$$

and is bounded by

$$0 < C_m\|\dot{q}\|^2 \le \|C(q,\dot{q})\dot{q}\| \le C_M\|\dot{q}\|^2, \ \forall q, \dot{q} \in \Re^n \tag{5}$$

where C_m and C_M are some positive constants.

Property 5. There exists a positive definite diagonal matrix A such that the following two inequalities, with specified constant $a > 0$, are satisfied simultaneously for any fixed q_d and any q

$$U(q) - U(q_d) - \Delta q^T G(q_d) + \frac{1}{2}\Delta q^T A \Delta q \ge a\|\Delta q\|^2 \tag{6}$$

$$\Delta q^T[G(q) - G(q_d)] + \Delta q^T A \Delta q \ge a\|\Delta q\|^2 \tag{7}$$

where $\Delta q = q - q_d$ denotes the position error, and q and q_d denote the actual and desired coordinates, respectively.

Property 6. The gravitational force vector $G(q)$ is bounded for all $q \in \Re^n$. That is, there exist finite constants $\gamma_i \ge 0$ such that $\sup_{q\in\Re^n}\{|G(q)|\} \le \gamma_i, \forall i = 1,2,\ldots,n$.

We also assume that each joint actuator has a maximum torque $\tau_{i,\max}$ satisfying

$$\tau_{i,\max} > \gamma_i \qquad \forall i = 1,2,\ldots,n \tag{8}$$

2.2 Finite-Time Stability

We begin the review of the concepts of finite-time stability and stabilization of nonlinear systems following the treatment in [Bhat and Bernstein, 1998; Hong, Xu, and Huang, 2002]. Consider the system

$$\dot{\zeta} = f(\zeta), \quad f(0) = 0, \quad \zeta(0) = \zeta_0, \quad \zeta \in \Re^n \tag{9}$$

with $f: U_0 \to \Re^n$ continuous on an open neighborhood U_0 of the origin. Suppose that system (9) possesses unique solutions in forward time for all initial conditions.

Definition 1. The equilibrium $\zeta = 0$ of system (9) is (locally) finite-time stable if it is Lyapunov stable and finite-time convergent in a neighborhood $U \subset U_0$ of the origin. The finite-time convergence means the existence of a function $T : U \setminus \{0\} \to (0, \infty)$, such that, $\forall\ \zeta_0 \in U \subset \Re^n$, the solution of (9) denoted by $s_t(\zeta_0) = 0$ with ζ_0 as the initial condition is defined, and $s_t(\zeta_0) \in U \setminus \{0\}$ for $t \in [0, T(\zeta_0))$, and $\lim_{t \to T(\zeta_0)} s_t(\zeta_0) = 0$ with $s_t(\zeta_0) = 0$ for $t > T(\zeta_0)$. When $U = \Re^n$, we obtain the global finite-time stability.

Definition 2. Let $(r_1, \ldots, r_n) \in \Re^n$ with $r_i > 0, i = 1, \ldots, n$. Let $V : \Re^n \to \Re$ be a continuous function. V is said to be homogeneous of degree $\sigma > 0$ with respect to $(r_1, \ldots, r_n)$, if, for any given $\varepsilon > 0$,

$$V(\varepsilon^{r_1}\zeta_1, \ldots, \varepsilon^{r_n}\zeta_n) = \varepsilon^{\sigma} V(\zeta), \quad \forall \zeta \in \Re^n \tag{10}$$

Definition 3. Let $f(\zeta) = f(f_1(\zeta), \ldots, f_n(\zeta))^T$ be a continuous vector field. $f(\zeta)$ is said to be homogeneous of degree $\kappa \in \Re$ with respect to $(r_1, \ldots, r_n)$, if, for any given $\varepsilon > 0$,

$$f_i(\varepsilon^{r_1}\zeta_1, \ldots, \varepsilon^{r_n}\zeta_n) = \varepsilon^{\kappa + r_i} f_i(\zeta), \ i = 1, \ldots, n, \forall \zeta \in \Re^n \tag{11}$$

System (9) is said to be homogeneous if $f(\zeta)$ is homogeneous. Some of the results on finite-time stability of a nonlinear system in [Hong, Xu, and Huang, 2002] that will be used in this paper are summarized by the following two lemmas.

Lemma 1. Consider the following system

$$\dot{\zeta} = f(\zeta) + \hat{f}(\zeta), f(0) = 0, \hat{f}(0) = 0, \ \zeta \in \Re^n \tag{12}$$

where $f(\zeta)$ is a continuous homogeneous vector field of degree $\kappa < 0$ with respect to $(r_1, \ldots, r_n)$. Assume $\zeta = 0$ is an asymptotically stable equilibrium of the system $\dot{\zeta} = f(\zeta)$. Then $\zeta = 0$ is a locally finite-time stable equilibrium of the system (12) if

$$\lim_{\varepsilon \to 0} \frac{\hat{f}_i(\varepsilon^{r_1}\zeta_1, \ldots, \varepsilon^{r_n}\zeta_n)}{\varepsilon^{\kappa + r_i}} = 0, \ i = 1, \ldots, n, \forall \zeta \neq 0 \tag{13}$$

Lemma 2. Global asymptotic stability and local finite-time stability of the closed-loop system imply global finite-time stability.

3 Control Design

3.1 Finite-time Stability

First, we define a class of scalar potential function as follows:

$$S(x) = \begin{cases} \dfrac{|x|^{\alpha+1}}{\alpha+1}, & |x| < \beta \\ \beta^{\alpha}|x| - \dfrac{\alpha\beta^{\alpha+1}}{\alpha+1}, & |x| \geq \beta \end{cases} \tag{14}$$

where $\alpha \in (0\ 1)$ and $\beta > 0$ are design parameters. The first derivative of $S(x)$ with respect to x gives a saturated function, and can be expressed as

$$s(x) = \begin{cases} |x|^{\alpha} \operatorname{sgn}(x), & |x| < \beta \\ \beta^{\alpha} \operatorname{sgn}(x), & |x| \geq \beta \end{cases} \tag{15}$$

where $\operatorname{sgn}(\cdot)$ being the standard signum function.

Lemma 3. The functions $S(x)$ and $s(x)$ in (14) and (15) have the following properties:

(1) $S(x) > 0$ for $x \neq 0$ and $S(x) = 0$ and $s(x) = 0$ for $x = 0$;

(2) $S(x)$ is twice continuously differentiable, and $s(x)$ is strictly increasing in x for $|x| < \beta$ and saturated for $|x| \geq \beta$. Moreover $s(x)$ is a homogeneous function for $|x| < \beta$;

(3) There is a constant $\kappa_1 > 0$ such that

$$S(x) \geq \kappa_1 s^2(x) > 0 \quad \text{for} \quad x \neq 0 \tag{16}$$

(4) There is a constant $\kappa > 0$ such that

$$x s(x) \geq \kappa s^2(x) > 0 \quad \text{for} \quad x \neq 0 \tag{17}$$

Proofs of these properties can be completed by simple calculation. ■

To aid the subsequent control design and analysis, we define the vectors $\operatorname{Tanh}(\cdot), S(\cdot), s(\cdot) \in \Re^n$ and the diagonal matrix $\operatorname{Sech}(\cdot) \in \Re^{n \times n}$ as follows:

$$\operatorname{Tanh}(\xi) = [\tanh(\xi_1), \cdots, \tanh(\xi_n)]^T \tag{18}$$

$$S(\xi) = [S(\xi_1), \ldots, S(\xi_n)]^T \tag{19}$$

$$s(\xi) = [s(\xi_1), \ldots, s(\xi_n)]^T \tag{20}$$

$$\operatorname{Sech}(\xi) = \operatorname{diag}(\operatorname{sech}(\xi_1), \cdots, \operatorname{sech}(\xi_n)) \tag{21}$$

where $\xi = [\xi_1, \cdots, \xi_n]^T \in \Re^n$, and $\operatorname{diag}(\cdot)$ denotes a diagonal matrix with zeros everywhere except for the main diagonal. Based on the definitions of (15), (18), (20), and (21), it can easily be shown that the following expressions hold:

$$\operatorname{Tanh}^T(\xi)\operatorname{Tanh}(\xi) \leq \operatorname{Tanh}^T(\xi)s(\xi) \leq s^T(\xi)s(\xi) \tag{22}$$

$$(\operatorname{Sech}^2(\xi))_M = 1 \tag{23}$$

Now the bounded PD plus desired gravity compensation for global finite-time positioning of robot manipulator is formulated as follows:

$$\tau = G(q_d) - K_p s(\Delta q) - K_d s(\dot{q}) \tag{24}$$

where K_p and K_d are positive definite constant diagonal proportional and derivative matrices, respectively, and $\alpha, \delta \in (0\ 1)$ and $\beta = 1$.

Substituting (24) into (1), the closed-loop dynamics becomes

$$\begin{aligned} M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + G(q) - G(q_d) \\ + K_p s(\Delta q) + K_d s(\dot{q}) = 0 \end{aligned} \tag{25}$$

whose origin $[\Delta q^T\ \dot{q}^T]^T = 0 \in \Re^{2n}$ is the unique equilibrium.

3.1 Finite-Time Stability Analysis

Theorem 1: Given the robot manipulator system (1) under the actuator constraints (8), with the proposed bounded PD plus desired gravity compensation controller (24), the closed-loop system (25) is globally finite-time stable, provided the control gains are chosen as follows:

$$D > \lambda_0\left(\sqrt{n}C_M + M_M\right)I \tag{26}$$

$$2K_0K_d > \lambda_0 K_{dM} I \tag{27}$$

$$K_1K_p > 2\lambda_0^2 M_M I \tag{28}$$

$$U(q) - U(q_d) - \Delta q^T G(q_d) + \frac{1}{2}\sum_{i=1}^{n} k_{pi} S(\Delta q_i) \ge a\|\mathrm{Tanh}(\Delta q)\|^2 \tag{29}$$

$$\mathrm{Tanh}^T(\Delta q)\left(G(q) - G(q_d)\right) + \mathrm{Tanh}^T(\Delta q)K_p s(\Delta q) \ge \left(a + \frac{1}{2}K_{dM}\right)\|\mathrm{Tanh}(\Delta q)\|^2 \tag{30}$$

where k_{pi} denotes the ith diagonal elements of matrices K_p, $K_0 = \mathrm{diag}(\kappa_1, \ldots, \kappa_n)$ and $K_1 = \mathrm{diag}(\kappa_{11}, \ldots, \kappa_{1n})$, respectively, which can be easily determined by using the inequalities (16) and (17) in Lemma 3, and $S(\Delta q_i)$ represents the ith elements of the vector $S(\Delta q) \in \Re^n$ defined in (19), and λ_0 and a are all small positive constants.

Note that the inequalities (29) and (30) correspond to inequalities (6) and (7) of Property 5, respectively, and the existence of such a matrix K_p is confirmed by the same argument given in proposing (6) and (7), and (21) [Arimoto, 1996; Su, Sun, Ren, and Mills. 2006; Su, Müller, and Zheng, 2007].

Remark 1. Condition (26) in Theorem 1 is not excessively restrictive and limitative, due to the fact that the small positive constant λ_0 does not use in the control law formulation. This implies that λ_0 always exists and can be selected as so arbitrarily small.

Proof. The proof proceeds in the following two steps. First, the global asymptotic stability is proved with Lyapunov's direct method. Second the local finite-time stability is shown using Lemma 1 and Lemma 2 is involved to guarantee the global finite-time stability.

1) Global asymptotic stability: To this end, we propose the following Lyapunov-like function candidate

$$V = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \lambda_0 \mathrm{Tanh}^T(\Delta q)M(q)\dot{q} + U(q) - U(q_d) - \Delta q^T G(q_d) + \sum_{i=1}^{n} k_{pi} S(\Delta q_i) + \lambda_0 \sum_{i=1}^{n} d_i \ln(\cosh(\Delta q_i)) \tag{31}$$

where d_i and k_{di} denote the ith diagonal elements of matrices D and K_d respectively.

We first consider the following

$$\begin{aligned}
&\frac{1}{4}\dot{q}^T M(q)\dot{q} + \frac{1}{2}\sum_{i=1}^{n} k_{pi} S(\Delta q_i) + \lambda_0 \mathrm{Tanh}^T(\Delta q)M(q)\dot{q} \\
&= \frac{1}{4}(\dot{q} + 2\lambda_0 \mathrm{Tanh}(\Delta q))^T M(q)(\dot{q} + 2\lambda_0 \mathrm{Tanh}(\Delta q)) \\
&\quad - \lambda_0^2 \mathrm{Tanh}^T(\Delta q)M(q)\mathrm{Tanh}(\Delta q) + \frac{1}{2}\sum_{i=1}^{n} k_{pi} S(\Delta q_i) \\
&\ge \frac{1}{2}\sum_{i=1}^{n} k_{pi} S(\Delta q_i) - \lambda_0^2 \mathrm{Tanh}^T(\Delta q)M(q)\mathrm{Tanh}(\Delta q) \\
&\ge \frac{1}{2}\sum_{i=1}^{n}(k_{pi}\kappa_{1i} - 2\lambda_0^2 M_M)\mathrm{Tanh}^2(\Delta q_i)
\end{aligned} \tag{32}$$

where (2) of Property 2, and (16) and (22) have been used.

Substituting (32) into (31), we have

$$\begin{aligned}
V &\ge \frac{1}{4}\dot{q}^T M(q)\dot{q} + \frac{1}{2}\sum_{i=1}^{n}(k_{pi}\kappa_{1i} - 2\lambda_0^2 M_M)\mathrm{Tanh}^2(\Delta q_i) \\
&\quad + U(q) - U(q_d) - \Delta q^T G(q_d) \\
&\quad + \frac{1}{2}\sum_{i=1}^{n} k_{pi} S(\Delta q_i) + \lambda_0 \sum_{i=1}^{n} d_i \ln(\cosh(\Delta q_i))
\end{aligned} \tag{33}$$

From (28) and (29) and (16) in Lemma 3, we obtain

$$V \ge \frac{1}{4}\dot{q}^T M(q)\dot{q} + a\|\mathrm{Tanh}(\Delta q)\|^2 + \lambda_0 \sum_{i=1}^{n} d_i \ln(\cosh(\Delta q_i)) > 0 \tag{34}$$

for $\left[\Delta q^T \; \dot{q}^T\right]^T \ne 0$.

Hence, we can conclude that V is a positive definite Lyapunov function with respect to $\Delta q, \dot{q}$.

Differentiating V with respect to time, we have

$$\begin{aligned}
\dot{V} &= \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + \dot{q}^T M(q)\ddot{q} + \lambda_0(\mathrm{Sech}^2(\Delta q)\Delta\dot{q})^T M(q)\dot{q} \\
&\quad + \lambda_0 \mathrm{Tanh}^T(\Delta q)\left[\dot{M}(q)\dot{q} + M(q)\ddot{q}\right] + \Delta\dot{q}^T K_p s(\Delta q) \\
&\quad + \dot{q}^T G(q) - \Delta\dot{q}^T G(q_d) + \lambda_0 \mathrm{Tanh}^T(\Delta q)D\Delta\dot{q}
\end{aligned} \tag{35}$$

Substituting $M(q)\ddot{q}$ from (25) into (35), and using (3) of Property 3, it follows that

$$\begin{aligned}
\dot{V} &= -\dot{q}^T D\dot{q} - \dot{q}^T K_d s(\dot{q}) - \lambda_0 \mathrm{Tanh}^T(\Delta q)K_d s(\dot{q}) \\
&\quad + \lambda_0 \mathrm{Tanh}^T(\Delta q)\left[C^T(q,\dot{q})\dot{q} + (\mathrm{Sech}^2(\Delta q)\Delta\dot{q})^T M(q)\dot{q}\right] \\
&\quad - \lambda_0\left[\mathrm{Tanh}^T(\Delta q)\left(G(q) - G(q_d)\right) + \mathrm{Tanh}^T(\Delta q)K_p s(\Delta q)\right]
\end{aligned} \tag{36}$$

By using (4) and (5) of Property 4 and (23), the fourth term of the right-hand side of (36) can be upper bounded by

$$\lambda_0 \mathrm{Tanh}^T(\Delta q)\left[C^T(q,\dot{q})\dot{q} + (\mathrm{Sech}^2(\Delta q)\Delta\dot{q})^T M(q)\dot{q}\right] \le \lambda_0(\sqrt{n}C_M + M_M)\|\dot{q}\|^2 \tag{37}$$

Note that the derivation of the first term of (37) we utilized $\|\mathrm{Tanh}(\Delta q)\| \le \sqrt{n}$ according to $|\tanh(\Delta q_i)| \le 1$.

Substituting (30) and (37) into (36), we obtain

$$
\begin{aligned}
\dot V \le & -\dot q^T D\dot q - \dot q^T K_d s(\dot q) - \lambda_0 \mathrm{Tanh}^T(\Delta q) K_d s(\dot q) \\
& + \lambda_0(\sqrt{n}C_M + M_M)\|\dot q\|^2 - \lambda_0\left(a + \frac{1}{2}K_{dM}\right)\|\mathrm{Tanh}(\Delta q)\|^2 \\
\le & -s(\dot q)^T K_0 K_d s(\dot q) + \frac{\lambda_0}{2}K_{dM}\|s(\dot q)\|^2 \\
& -\dot q^T\left[D - \lambda_0\left(\sqrt{n}C_M + M_M\right)I\right]\dot q \\
& + \frac{\lambda_0}{2}K_{dM}\|\mathrm{Tanh}(\Delta q)\|^2 - \lambda_0\left(a + \frac{1}{2}K_{dM}\right)\|\mathrm{Tanh}(\Delta q)\|^2 \\
\le & -\frac{1}{2}s(\dot q)^T(2K_0K_d - \lambda_0 K_{dM} I)s(\dot q) \\
& -\dot q^T\left[D - \lambda_0\left(\sqrt{n}C_M + M_M\right)I\right]\dot q - \lambda_0 a\|\mathrm{Tanh}(\Delta q)\|^2
\end{aligned} \tag{38}
$$

where $K_0 = \mathrm{diag}(\kappa_1, \ldots, \kappa_n)$, which was defined in (27). Note that the derivation of (38) we have utilized the inequality $bc \le \frac{1}{2}(b^2 + c^2)$ with $b = \|\mathrm{Tan}(\Delta q)\|$ and $c = \|s(\dot q)\|$.

From (26) and (27), we conclude that $\dot V < 0$. In fact, $\dot V = 0$ means $\mathrm{Tanh}(\Delta q) = 0$ and $\dot q = 0$. From the definition of hyperbolic tangent function, we have $\Delta q(t) \to 0$, $\dot q(t) \to 0$, as $t \to \infty$ for any initial state $(q(0), \dot q(0))$. Hence, we have the global asymptotic stability about the point $(\Delta q = 0, \dot q = 0)$.

2) Local finite-time stability: Following the idea presented by Hong et al. (2002), the local finite-time stability is proved using Lemma 1. To this end, let $x_1 = \Delta q$, $x_2 = \dot x_1 = \dot q$, and $x = (x_1^T, x_2^T)^T$. The state equation of the closed-loop system of the robot manipulator with the proposed control law (24) is

$$
\begin{cases}
\dot x_1 = x_2 \\
\dot x_2 = -M^{-1}(x_1 + q_d)\left[C(x_1 + q_d, x_2)x_2 + Dx_2 \right.\\
\qquad \left. + G(x_1 + q_d) - G(q_d) + K_p s(x_1) + K_d s(x_2)\right]
\end{cases} \tag{39}
$$

Clearly, $x = 0$ is the equilibrium of (39). It can be seen that the closed-loop system is not homogeneous. To use Lemma 1, we rewrite (39) as follows:

$$
\begin{cases}
\dot x_1 = x_2 \\
\dot x_2 = -M^{-1}(q_d)K_p s(x_1) + \hat f_2(x)
\end{cases} \tag{40}
$$

with

$$
\begin{aligned}
\hat f_2 = & -M^{-1}(x_1 + q_d)\left[C(x_1 + q_d, x_2)x_2 + G(x_1 + q_d)\right. \\
& \left. - G(q_d) + Dx_2 + K_d s(x_2)\right] - \widetilde M(x_1, q_d)K_p s(x_1)
\end{aligned} \tag{41}
$$

$$
\widetilde M(x_1, q_d) = M^{-1}(x_1 + q_d) - M^{-1}(q_d) \tag{42}
$$

For $\|x_1\| < \sqrt{n}$, i.e., $|x_{1i}| < 1$, by using Lemma 1 it can be easily verified that the following system

$$
\begin{cases}
\dot x_1 = x_2 \\
\dot x_2 = -M^{-1}(q_d)K_p s(x_1)
\end{cases} \tag{43}
$$

is homogeneous of degree $\kappa = \frac{\alpha - 1}{\alpha + 1} < 0$ for $0 < \alpha < 1$ with respect to $(r_{11}, r_{12}, \ldots, r_{1n}, r_{21}, r_{22}, \ldots, r_{2n})$ with $r_{1i} = r_1 = 2/(\alpha + 1)$ and $r_{2i} = r_2 = 1$. Note that $f(0) = 0$ and $\hat f(0) = 0$ from (43), and (41) and Lemma 3, respectively.

Next we will involve Lemma 1 to show the local finite-time stability of the closed-loop system (39). To this end, first note that from (16) for $\|x_1\| < \sqrt{n}$ and $\|x_2\| < \sqrt{n}$ (i.e., $|x_{1i}| < 1$, and $|x_{2i}| < 1$),

$$
s(\varepsilon^{r_1}x_1) = o(\varepsilon^{r_1}), s(\varepsilon^{r_2}x_2) = o(\varepsilon^{r_2}) \tag{44}
$$

Since $M^{-1}(x_1 + q_d)$ and $C(x_1 + q_d, x_2)$ are smooth and $\kappa < 0$ [Hong et al., 2002], we have

$$
\begin{aligned}
& \lim_{\varepsilon \to 0} - \left\{ \frac{M^{-1}(\varepsilon^{r_1}x_1 + q_d)}{\varepsilon^{\kappa + r_2}} \left[C(\varepsilon^{r_1}x_1 + q_d, \varepsilon^{r_2}x_2)\varepsilon^{r_2}x_2 + D\varepsilon^{r_2}x_2 \right.\right. \\
& \qquad \left.\left. + G(\varepsilon^{r_1}x_1 + q_d) - G(q_d) + K_d s(\varepsilon^{r_2}x_2) \right] \right\} \\
& = -M^{-1}(q_d)\left[C(q_d, 0) + D\right]x_2 \lim_{\varepsilon \to 0}\varepsilon^{-\kappa} - K_d \lim_{\varepsilon \to 0} o(\varepsilon^{-\kappa}) \\
& = 0
\end{aligned} \tag{45}
$$

Applying the mean value theorem to each entry of $\widetilde M(x_1, q_d)$, it follows that [Hong et al., 2002]

$$
\widetilde M(\varepsilon^{r_1}x_1, q_d) = M^{-1}(\varepsilon^{r_1}x_1 + q_d) - M^{-1}(q_d) = o(\varepsilon^{r_1}) \tag{46}
$$

As a result, for $\|x_1\| < \sqrt{n}$ (i.e., $|x_{1i}| < 1$), we have

$$
\lim_{\varepsilon \to 0} - \frac{\widetilde M(\varepsilon^{r_1}x_1, q_d)K_p s(\varepsilon^{r_1}x_1)}{\varepsilon^{\kappa + r_2}} = -\lim_{\varepsilon \to 0} o(\varepsilon^{2r_1 - \kappa - r_2}) = 0 \tag{47}
$$

Note that the derivations of (45) and (47) we have utilized the fact that $\kappa = \frac{\alpha - 1}{\alpha + 1} < 0$, $2r_1 - \kappa - r_2 = \frac{2(2 - \alpha)}{\alpha + 1} > 0$ for $0 < \alpha < 1$.

Thus, for any fixed $x = (x_1^T, x_2^T)^T$, we get

$$
\lim_{\varepsilon \to 0} \frac{\hat f_2(\varepsilon^{r_1}x_1, \varepsilon^{r_2}x_2)}{\varepsilon^{\kappa + r_2}} = 0 \tag{48}
$$

Therefore, according to Lemma 1, we have the local finite-time stability of the closed-loop system in the region of the neighbourhood of zero defined by $D = \left\{x \in \Re \,\middle|\, \|x\| < 2\sqrt{n}\right\}$.

Finally, by invoking Lemma 2, we get the global finite-time stability. This completes the proof. ■

Remark 2. The above results still hold true when the desired gravity compensation is replaced by the real gravity compensation, i.e., the control law becomes

$$
\tau = G(q) - K_p s(\Delta q) - K_d s(\dot q) \tag{49}
$$

4 Illustrative Example

Simulations on a two-DOF robot were conducted to illustrate the effectiveness of the proposed controller. The entries to model the robot manipulator are, respectively [21]

$$
\begin{aligned}
M &= \begin{bmatrix} \theta_1 + 2\theta_2 \cos(q_2) & \theta_3 + \theta_2 \cos(q_2) \\ \theta_3 + \theta_2 \cos(q_2) & \theta_3 \end{bmatrix} \\
C &= \begin{bmatrix} -2\theta_2 \sin(q_2)\dot{q}_2 & -\theta_2 \sin(q_2)\dot{q}_2 \\ \theta_2 \sin(q_2)\dot{q}_1 & 0 \end{bmatrix} \\
G &= \begin{bmatrix} \theta_4 \sin(q_1) + \theta_5 \sin(q_1 + q_2) \\ \theta_5 \sin(q_1 + q_2) \end{bmatrix}
\end{aligned} \tag{50}
$$

Furthermore, a Coulomb friction is also considered in the simulations. To keep the notation used for model (1), it is defined $D = \mathrm{diag}(\theta_6, \theta_7)$, and

$$
f_c(\dot{q}) = \begin{bmatrix} \theta_8 \,\mathrm{sgn}(\dot{q}_1) \\ \theta_9 \,\mathrm{sgn}(\dot{q}_2) \end{bmatrix} \tag{51}
$$

where the parameters are given in SI units and summarized in Table I.

TABLE I PARAMETERS OF THE ROBOT MANIPULATOR

Notation	Value	Notation	Value
θ_1	2.351	θ_6	2.288
θ_2	0.084	θ_7	0.175
θ_3	0.102	θ_8	7.170 if $\dot{q}_1 > 0$ and 8.049 if $\dot{q}_1 < 0$
θ_4	38.465		
θ_5	1.825	θ_9	1.724

The hyperbolic tangent saturated PD plus gravity compensation method (Tangent PD) proposed by Kelly *et al.* [2] is used to comparison, and can be expressed as

$$
\tau = G(q) - K_p \tanh(\Delta q) - K_d \tanh(\dot{q}) \tag{52}
$$

From (50), it is straightforward to have

$$
\gamma_1 = 40.29, \gamma_2 = 1.825 \tag{53}
$$

The final desired positions were $q_d = \left[\frac{\pi}{4}, \frac{\pi}{2}\right]^T$ (rad) . The sampling period was $T = 1\,\mathrm{ms}$. All the initial parameters were set as zero. The actuator constraints were assumed as $\tau_{\max} = [150, 15]\,\mathrm{Nm}$. The gains for the proposed controller were chosen in accordance with stability conditions (26)-(30) and $k_{pi} + k_{di} \le \tau_{i,\max} - \gamma_i$ as: $\alpha = 0.75$, $\beta = 1$, $K_p = \mathrm{diag}(3.4, 8.3)$, $K_d = \mathrm{diag}(6.2, 2.7)$. The parameters of the Tangent PD controller were $K_p = \mathrm{diag}(7.0, 3.0)$ and $K_d = \mathrm{diag}(6.7, 1.0)$.

Figures. 1 and 2 illustrate the position errors and requested input torques of the proposed approach and the hyperbolic tangent PD plus gravity compensation scheme. It can be seen that the robot targeted at the final desired position correctly, and after a transient due to errors in initial condition, the position errors tend asymptotically to zero. Moreover, the fast response of the proposed controller is achieved in comparison with the conventional hyperbolic tangent PD plus control scheme, especially for the second link. The reason behind there is not large fast transient improvement of the first link maybe that the gravity term dominates the control action. Furthermore, the better results is obtained without much requested input torques.

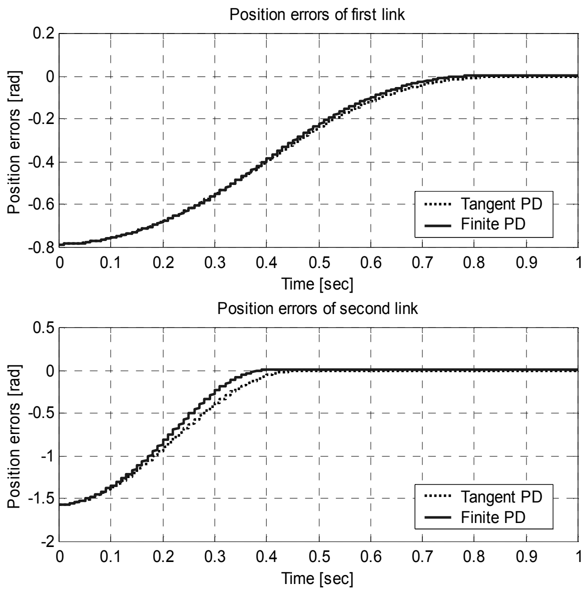

Figure 1. Position errors.

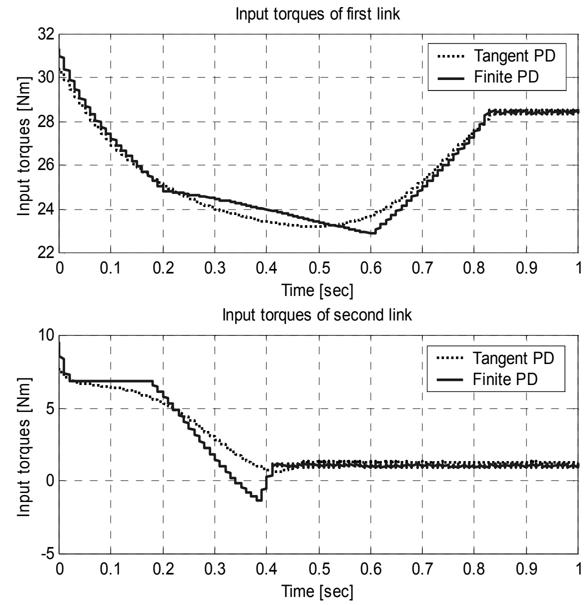

Figure 2. Input torques.

5 Conclusion

We have proven the global finite-time regulation of robot manipulators in the presence of actuator constraints, with nonsmooth but continuous PD plus gravity compensation scheme with Lyapunov's direct method and finite-time stability theory. The developed approach offers an alternative approach for improving the design of the robot regulator with bounded inputs, and also solves the global finite-time bounded control problem for a large class of nonlinear systems in the prensence of actuator constraints. The simulations performed on a two-DOF robot manipulator demonstrated the fast response of the proposed controller over the conventional hyperbolic tangent PD plus scheme.

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References

Alvarez-Ramirez, J., Kelly, R.and Cervantes, I. (2003). Semiglobal stability of saturated linear PID control for robot manipulators. *Automatica*, **39(6)**, pp. 989-995.

Arimoto, S. (1996). *Control Theory of Non-linear Mechanical Systems: A Passivity-based and Circuit-theoretic Approach*. Oxford: Clarendon Press.

Arteaga, M. A.and Kelly, R. (2004). Robot control without velocity measurements: new theory and experimental results. *IEEE Trans. Robot. Automat.*, **20(2)**, pp. 297-308.

Barambones, O. and Etxebarria, V. (2002). Energy-based approach to sliding composite adaptive control for rigid robots with finite error convergence time. *Int. J. Control*, **75(5)**, pp. 352–359.

Bhat, S. P. and Bernstein, D. S. (1998). Continuous finite-time stabilization of the translational and rotational double integrators. *IEEE Trans. Autom. Control*, **43(5)**, pp. 678–682.

Bhat, S. P.and Bernstein, D. S. (2000). Finite-time stability of continuous autonomous systems. *SIAM J. Control and Optimization*, **38(3)**, 751-766.

Colbaugh, R., Barany, E.and Glass, K. (1997). Global regulation of uncertain manipulators using bounded controls. *Proc. IEEE Conf. Robotics and Automation*, Albuquerque, New Mexico, (pp. 1148-1155).

Dixon, W. E. (2007). Adaptive regulation of amplitude limited robot manipulators with uncertain kinematics and dynamics. *IEEE Trans. Autom. Control*, **52(3)**, pp. 488-493.

Gruyitch, L. T. and Kokosy, A. (1999). Robot control for robust stability with finite reachability time in the whole. *J. Robot. Syst.*, **16(5)**, pp. 263-283.

Hong, Y., Xu, Y. and Huang, J. (2002). Finite-time control for robot manipulators. *Syst. Control Lett.*, **46(4)**, pp. 243–253.

Kasac, J., Novakovic, B., Majetic, D. and Brezak, D. (2006). Global positioning of robot manipulators with mixed revolute and prismatic joints. *IEEE Trans. Autom. Control*, **51(6)**, pp. 1035-1040.

Kelly, R. Santibanez, V. and Berghuis, H. (1997). Point-to-point control under actuator constraints. *Control Engineering Practice*, **5(11)**, pp. 1555-1562.

Morabito, F., Teel, A. R. and Zaccarian, L. (2004). Nonlinear antiwindup applied to Euler-Lagrange systems. *IEEE Trans. Robot. Automat.*, **20(3)**, pp. 526-537.

Parra-Vega, V., Rodriguez-Angeles, A. and Hirzinger, G. (2001). Perfect position/force tracking of robots with dynamical terminal sliding mode control. *J. Robot. Syst.*, **18(9)**, pp. 517-532.

Perruquetti, W. and Drakunov, S. (2000). Finite time stability and stabilization. *Proc. 39th IEEE Conf. Decision and Control*, Sydney, Australia, (pp. 1894-1899).

Sciavicco, L.and Siciliano, B. (2000). *Modeling and Control of Robot Manipulators*, 2nd ed. London, U. K.: Springer-Verlag.

Su, Y. X., Sun, D., Ren, L., and Mills, J. K. (2006). Integration of saturated PI synchronous control and PD feedback for control of parallel manipulators. *IEEE Trans. Robot.*, **22(1)**, pp. 202-207.

Su, Y. X., Müller, P. C.and Zheng, C. H. (2007). A global asymptotic stable output feedback PID regulator for robot manipulators. *Proc. IEEE Int. Conf. on Robotics and Automation*, Rome, Italy, (pp. 4484-4489).

Yu, S. H., Yu, X. H., Shirinzadeh, B. and Man, Z. H. (2005). Continuous finite-time control for robotic manipulators with terminal sliding mode. *Automatica*, **41(11)**, pp. 1957-1964.

Zavala-Rio, A. and Santibanez, V. (2007). A natural saturating extension of the PD-with-desired-gravity-compensation control law for robot manipulators with bounded inputs. *IEEE Trans. Robotics*, **23(2)**, pp. 386-391.

Zergeroglu, E., Dixon, W. E., Behal, A. and Dawson, D. M. (2000). Adaptive set-point control of robotic manipulators with amplitude-limited control inputs. *Robotica*, **18(2)**, pp. 171-181.