Phase-flip bifurcation: Theory and Experiment

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Phase synchronization [1] is ubiquitous in nature. It is common in weakly coupled nonlinear systems. Basically two types of phase synchronization, *in-phase* and *antiphase* or *out-of-phase* are seen in both instantaneously coupled [2-4] and delay coupled [5] systems. A transition to either of the synchronization states can be induced by changing the coupling strength [3, 4] in instantaneously coupled systems or by changing delay time [6] in delay coupled systems. However, the transition from *inphase* to *antiphase* state or *vice versa* is found intercepted by a desynchronization regime [3,4] in instantaneously coupled systems. On the contrary, a sharp transition from *inphase* to *antiphase* is observed [6] in two delay coupled oscillators when the time delay is varied above a critical value. Obviously, the phase difference of the coupled oscillators jumps from 0 to π at the critical time delay. This phenomenon of sharp transition from *inphase* to *antiphase* with time delay is defined [6] as phase-flip bifurcation. The phase-flip is accompanied by a large change in oscillator frequency from a lower to higher value respectively. Similar large change in oscillation frequency is also noted in *antiphase* and *inphase* states for instantaneously coupled oscillators too but no such sharp transition between the antiphase to inphase states is found as mentioned above. A close analogy of phase-flip bifurcation is found in sharp transition from antiphase pattern to inphase pattern in coordinated rhythmic movements of limbs [7] when a gradual increase in the frequency of movements are made. The frequency of limb cycle is the control parameter there.

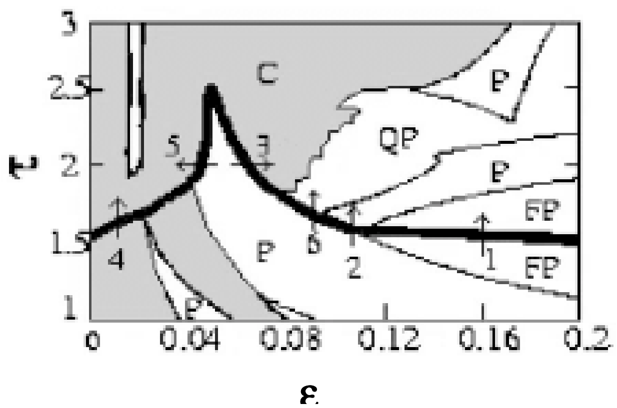

Fig.1. Phase diagram (ε-τ) for identical coupled Rössler oscillator. The thick line indicates the locus of phase-flip bifurcation and the numbered arrows indicate the transtion from inphase to antiphase.

We describe here our results both theoretical and experimental on phase-flip bifurcation. First we present the theoretical framework of phase-flip bifurcation using two identical Rössler oscillators and then its experimental evidence in electronic circuit of two delay coupled Chua oscillators.

It is to be noted that the phase-flip bifurcation is found [6] in different dynamical regimes like *amplitude death*, *periodic to periodic, quasiperiodic to quaisperiodic and chaotic to chaotic* transition states of many delay coupled nonlinear oscillators. For simplicity, we take two identical Rössler oscillators and coupled them via the y-variable as given in eq.(1).

$$\begin{aligned}
\frac{\mathrm{d}\, x_{1,2}(t)}{\mathrm{d}\, t} &= -y_{1,2}(t) - z_{1,2}(t) \\
\frac{\mathrm{d}\, y_{1,2}(t)}{\mathrm{d}\, t} &= x_{1,2}(t) - a y_{1,2}(t) + \varepsilon[y_{2,1}(t-\tau) - y_{1,2}(t) \\
\frac{\mathrm{d}\, z_{1,2}(t)}{\mathrm{d}\, t} &= b + z_{1,2}(t)(x_{1,2}(t) - c)) \qquad (1)
\end{aligned}$$

Subscripts (1, 2) denote individual oscillators. The uncoupled (ε=0) subsystems are chaotic for selected a=b=0.1, c=14. We study the coupled system as a function of coupling strength ε and the time delay τ. For instantaneous coupling (τ=0) this system has been studied in great details and a variety of interesting dynamical phenomena are known to occur. We report here the existence of phase-flip bifurcation in time-delay-coupled Rössler system. Figure 1 schematically shows the different dynamical states that arise for a range of parameters ε and τ. The shaded region, marked by C, corresponds to chaotic states while the white region shows regions of regular behaviors. These include the fixed point (FP) regime corresponding to amplitude death, periodic (P) and quasiperiodic (QP) dynamics. The phase-flip bifurcation takes place across the bold line in the whole range of ε and for selected range of τ. The arrows denote the direction of the transition from *inphase* to *antiphase* (or *out-of-phase*) motion. The attached numbers with the arrows label different dynamics before and after this bifurcation: 1 has amplitude death (or FP) both before and after the transition, 2 is from periodic to periodic dynamics, 3 and 5 are from periodic to chaotic, 4 is from chaotic to chaotic motion. While 6 denotes transition from periodic to quasiperiodic motion.

The largest lyapunov exponents as a function of the time-delay parameters are shown for the coupled Rössler system in Fig.2 across different settings of the phase-flip bifurcation. Figure 2(a) is for the transition along the arrow 1 at coupling strength, ε=0.16. Since the largest lyapunov exponent λ_1 is negative, the dynamics moves to a FP attractor. The inset in Fig.2(a) are the representative trajectories at τ=1.52 before the bifurcation and at τ=1.63 after the bifurcation. The trajectories of the coupled oscillators are *inphase* for τ=1.52 before the bifurcation and *out-of-phase* for τ=1.63 after the bifurcation. The bifurcation is clearly seen as the largest lyapunov exponent changes its slope at a critical time delay value (marked by an downward arrow) and close to $\tau \approx 1.6$. Note that the largest lyapunov exponent remains negative before and after the change in slope. Hence the coupled systems remain in amplitude death situation before and after the bifurcation. The relative phase of the coupled Rossler oscillators in Fig.2(b) jumps from 0 to 2π at the transition or bifurcation point. The frequency of the locked oscillators also jumps in Fig.2(c) from a lower value to a larger value which is even larger than the frequency of the uncoupled oscillators. This large increase in oscillation frequency in antiphase states is also recorded compared to inphase states [4] in instantaneously coupled oscillators although no sharp transition can be seen. The mechanism of such increase in oscillator frequency is yet to be explained.

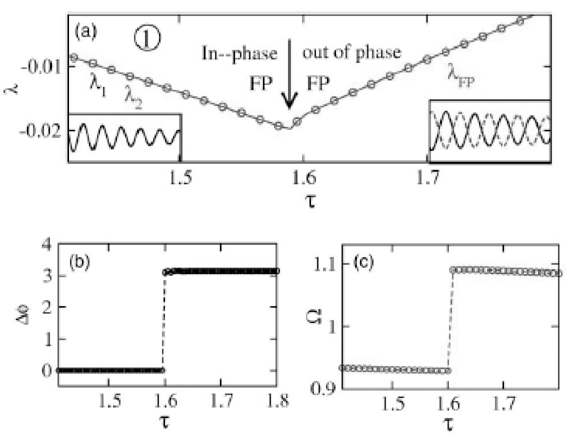

Fig.3. Phase-flip bifurcation in two identical coupled Rössler oscillators: (a) largest lyapunov exponents (λ_1, λ_2) plotted with time delay; inset on the left shows inphase motion of the oscillators, inset on the right shows antiphase motion, (b) relative with time delay, (locked frequency of the oscillators with time delay. Largest lyapunov exponents are both negative before and after the bifurcation that indicate the motion of the coupled oscillators are in amplitude death (or FP) regime as marked by the number 1 corresponding to Fig.1.

The phase-flip bifurcation in other dynamical regimes of two coupled Rössler oscillators are indicated by upward arrows in Fig.3. Trajectories before and after the bifurcation for the regions marked by 2, 3 and 4 in Fig.1 are shown in Fig.3(a)-(c) in the insets. When the transition is from one limit cycle to another, λ_1, remains zero across the transition but λ_2 shows a discontinuity in Fig.3(a). In the transition from a limit cycle to a chaotic attractor λ_1 and λ_4 are discontinuous while λ_2 $=\lambda_3$ remain equal to 0 in Fig.3(b). In chaotic to chaotic transition, λ_1 and λ_2 are positive discontinuous while λ_4 has a maximum at the phase–flip bifurcation point as shown in Fig.3(c). The signature of phase-flip bifurcation is thus noted as a sharp transition in relative phase from 0 to π value with time delay in coupled oscillators accompanied by a jump in oscillator frequency to higher value.

We attempted for experimental evidence of phase-flip bifurcation with two nonidentical Chua oscillators since it is difficult to design two identical oscillators, in reality. As a consequence asymmetry is introduced in the coupled oscillators in terms of both coupling and time delay and found the existence of phase-flip bifurcation even in presence of both types of asymmetries. So far we are successful in finding experimental evidence of phase-flip bifurcation during the period-period transition, however, this, at least, provides the necessary experimental support to the theoretical frameworks of phase-flip bifurcation.

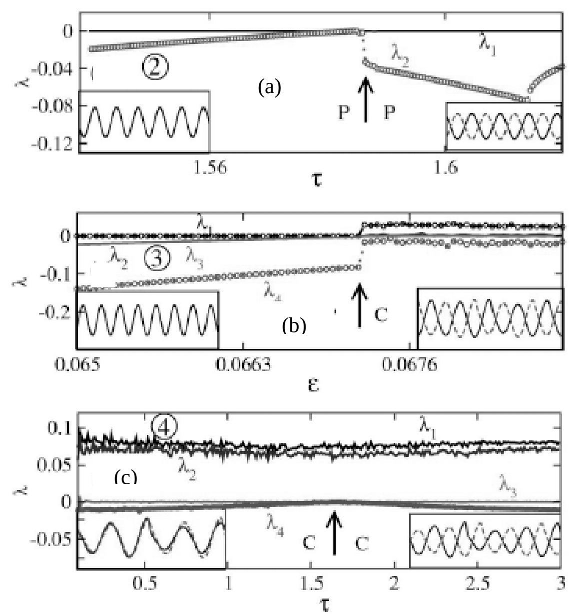

Fig.3. Phase-flip bifurcation in different regimes of coupled Rössler oscillators: largest lyapunov exponents for Rössler system in eq.(1) as a function of timer delay: (a) for fixed coupling strength ε=0.16, periodic to periodic transition marked 2, (b) for fixed coupling strength ε=2, periodic to chaotic transition marked by 3 (c) for fixed coupling strength ε=0.01, chaotic to chaotic transition marked by 4. All trajectories before and after the transitions are shown in the insets.

The coupled circuit is shown in Fig.4. Each Chua oscillator [7] consists of a passive resistor $R_{1,10}$, two capacitors $C_{1,3}$ and $C_{2,4}$ and, inductor $L_{1,2}$ with leakage resistance $R_{2,9}$ while the piecewise linear function $f(\bullet)$ is simulated by using two op-amp U1-U2 (or U2-U4) and associated resistances. All component values are noted against each component in the circuit diagram. The op-amp U5 is used for unidirectional current flow from the node of C_3 capacitor of one oscillator (numbered as 2) to the node of C_1 capacitor of another oscillator (numbered as 1). A delay network D2 (RC network with resistor R_{22} and capacitor C_5) with a series resistance R_{17} are used to introduce delay coupling from the oscillator 2 to oscillator 1. The resistor R_{22} is tuned to control time delay τ_1 while R_{17} controls the coupling strength ε_1. Similarly, the op-amp

U6 allows unidirectional current to flow from the node of C_1 capacitor of the oscillator 1 to the node of C_3 capacitor of the oscillator 2 via another delay network D1 (combination of resistor R_{21} and capacitor C_6) and the series resistance R_{18}. Accordingly, the resistor R_{17} controls the delay time τ_2 and R_{18} controls the coupling strength ε_2. A bi-directional delay coupling is thus established between the two Chua oscillators using delay networks and resistances (R_{17}, R_{18}).

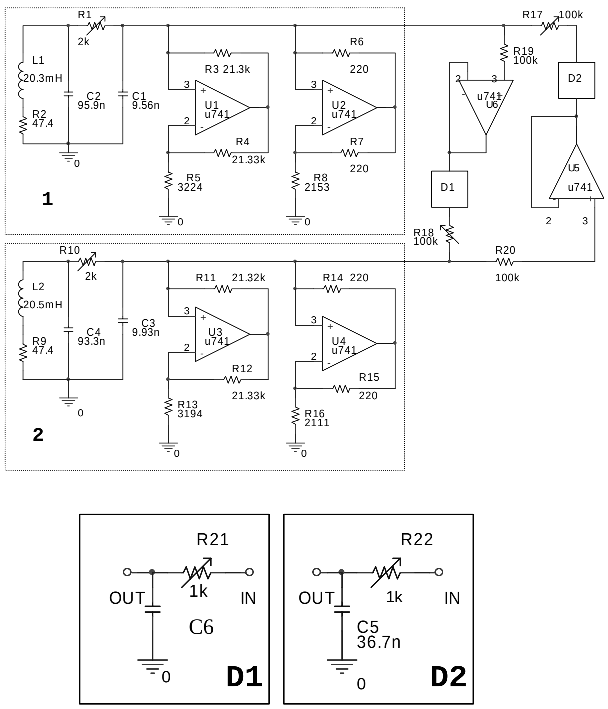

Fig.4. Two delay coupled Chua oscillators: two oscillators are drawn inside dotted boxes numbered by 1, 2. Delay networks D1 and D2 are separately drawn.

The model equations of the delay coupled Chua oscillator are given by

$$
\begin{aligned}
\frac{dV_{C1}}{dt} &= \frac{1}{R_1C_1}[(V_{C2}-V_{C1}) - f(V_{C1})] + \frac{1}{R_{17}C_1}(\delta V_{C3}(t-\tau_1) - V_{C1}(t))\\
\frac{dV_{C2}}{dt} &= \frac{1}{R_1C_2}(V_{C1}-V_{C2}+R_1 I_{L1})\\
\frac{dI_{L1}R_1}{dt} &= -\frac{1}{R_1L_1}(V_{C2}+R_2 I_{L1})\\
\frac{dV_{C3}}{dt} &= \frac{1}{R_{10}C_3}[(V_{C4}-V_{C3}) - f(V_{C3})] + \frac{1}{R_{18}C_3}(\delta V_{C1}(t-\tau_2) - V_{C3}(t))\\
\frac{dV_{C3}}{dt} &= \frac{1}{R_{10}C_3}(V_{C3}-V_{C4}+R_{10} I_{L2})\\
\frac{dI_{L2}R_{10}}{dt} &= -\frac{1}{L_2}(V_{C4}+R_9 I_{L2})
\end{aligned}
\qquad (2a)
$$

where

$$
f(V_{C_1,C_3}) = \begin{cases} b_{1,2}V_{C_1,C_3} + (b_{1,2}-a_{1,2}) & \text{if } V_{C_1,C_3} < -E \\ a_{1,2}V_{C_1,C_3} & \text{if } -E \le V_{C_1,C_3} \le E \\ b_{1,2}V_{C_1,C_3} + (a_{1,2}-b_{1,2}) & \text{if } V_{C_1,C_3} > E \end{cases} \qquad (2b)
$$

$$
a_{1,2} = \left(-\frac{1}{R_{5,16}} - \frac{1}{R_{8,15}}\right), \quad b_{1,2} = \left(\frac{1}{R_{3,11}} - \frac{1}{R_{8,15}}\right) \qquad (2c)
$$

and

$$
\tau_1 = R_{18}C_6;\ \tau_2 = R_{17}C_5;\ \varepsilon_1 = \frac{1}{R_{34}};\ \varepsilon_2 = \frac{1}{R_{32}};\ \delta = 1/\sqrt{2} \qquad (2d)
$$

Eq.(2a) is the governing equation of the two delay coupled oscillators while the piecewise linear function of the oscillators is defined in eq.(1b). The slopes of the piecewise linear function $a_{1,2}$ and $b_{1,2}$ are given in eq.(1c) and other parameters are defined in eq.1(d). The V_{Ci} denotes the node voltage at ith capacitor C_i (i=1,2,3,4) and I_{Lj} denotes current through jth inductor L_j (j=1,2).

In experiment, firstly, we decide the dynamics of the two uncoupled Chua oscillators by fixing the resistances R_1 =1525 Ω and R_{10} =1507 Ω. In uncoupled state, the first oscillators is period-2 for R_1 =1525 Ω while the second oscillator is period-4 for R_{10} =1507 Ω . After coupling, we fix the coupling strength $\varepsilon_1=1/R_{17}$ and $\varepsilon_2=1/R_{18}$ by appropriate selection of R_{17} =21.36kΩ and R_{18} =65.2kΩ and also fix one of the time delays $\tau_2=R_{22}C_5$ by the choice of R_{22} =523 Ω and C_5=100.8nf. The delay $\tau_1=R_{21}C_6$ is only varied by varying R_{21} while keeping the capacitor C_6=36.7nf fixed.

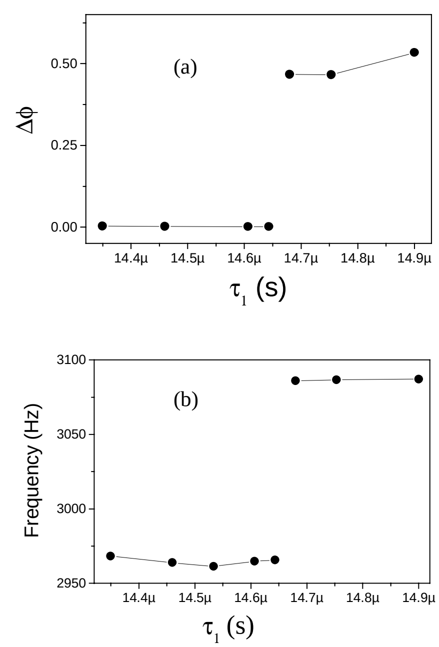

Fig.5 Phase slip bifurcation, (a) phase difference $\Delta\phi(\tau_1)$, (b) frequency of the locked oscillators with with time delay τ_1.: solid lines with black circles denote estimated phase difference in (a) and frequency of the locked oscillators in (b)

The voltage V_{C1} and V_{C3} at capacitor nodes C_1 and C_3 respectively are monitored using a 2-channel digital oscilloscope (TEKTRONIX, TDS 220, 100MHz) with a maximum sampling rate of 1.0GS/s and record length of 2500 data points in each snapshot. The instantaneous phases $\phi_i(t)$ of measured oscillatory voltages, V_{C1} and V_{C3}, are estimated using Hilbert transform [1] separately. The phase difference $\Delta\phi$ of the oscillatory voltages of the coupled oscillators is then plotted with delay time τ_1 as shown in Fig.5(a) which reveals a sharp transition from 0 to 2π value as indicated by the discontinuity of the solid line plots with black circles. This transition is accompanied by a sharp increase in frequency of the coupled system as shown in Fig.5(b). The frequency is estimated from the average rate of change in instantaneous phase $\langle d\phi_i(t)/dt \rangle$. The experimental results in Fig.5 confirm the existence of phase-flip bifurcation as defined above. In fact, the Chua oscillators become chaotic under bi-directional coupling with small time delay, however, the coupled oscillators move to periodic state when the delay time τ_1 is increased. The coupled Chua oscillators remain periodic with increase in delay τ_1 and maintain inphase synchrony until switches to antiphase synchrony above a critical time delay. The measured voltage time series of V_{C1} and V_{C3} are plotted in Fig.6(a) in solid and dotted lines respectively for τ_1=14.46μs (R_{21}=394Ω) which show *inphase* synchrony while they switch over to *antiphase* (or *out-of-phase*) for τ_1=14.75μs (R_{21}=402Ω) as shown in Fig.6(b). The phase flip bifurcation or the sharp transition from inphase to antiphase occurs above the critical time delay τ_1=14.64μs (R_{21}=399Ω) as shown in Fig.5.

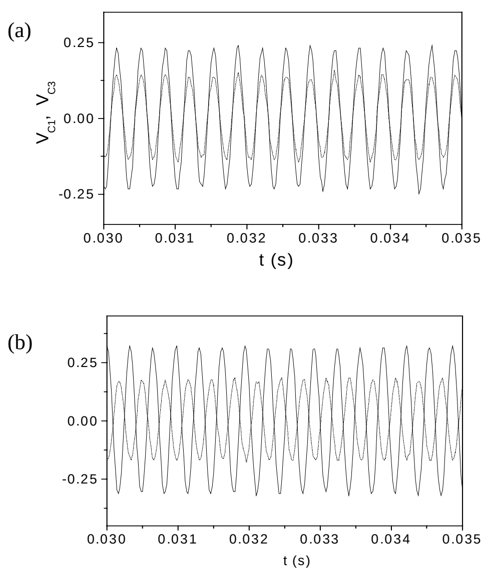

Fig.6. Experimental time series: (a) inphase synchrony: $V_{C1}(t)$, $V_{C3}(t)$ in solid and dotted lines respectively for τ_1=14.46μs (R_{21}=394Ω), (b) antiphase: $V_{C1}(t)$, $V_{C3}(t)$ in solid and dotted lines respectively for τ_1=14.75μs (R_{21}=402Ω).

It may be noted that the phase-flip bifurcation is observed here for two different time delay (τ_1, τ_2) and different strength (ε_1, ε_2) of bi-directional coupling. Such asymmetry of coupled system is unavoidable in experimental system due to natural parameter mismatch of similar oscillators. It confirms the existence of phase-flip bifurcation even in presence of asymmetry. We also observed phase flip bifurcation in amplitude death regime, however, we failed to record the transition point due to limitation in our measurement facilities in capturing the transient time series.

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