Feedback classification of single-input systems over von Neumann regular rings

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Abstract

This work deals with linear systems with scalars in a commutative ring R with the property of being "von Neumann regular", i.e. R is zero-dimensional and has no nonzero nilpotents. We prove that every single-input, n-dimensional system over R is feedback equivalent to a special normal form, whose existence actually characterizes the class of von Neumann regular rings. This normal form, which captures completely the structure of the reachable submodule of the system, is associated to a collection of n principal ideals generated by idempotent elements $f_1, \dots, f_n$, each dividing the following one. The normal form can be obtained by an explicit algorithm, which is implemented in PARI-GP in the case $R = \mathbb{Z}/(d\mathbb{Z})$, where d is a squarefree integer.

1. Introduction and notation

Let R be a commutative ring with 1. An m-input, n-dimensional system (or a system of size (n,m)) over R is a pair of matrices (A,B), with $A \in R^{n\times n}$ and $B \in R^{n\times m}$. See the motivation for studying linear systems over commutative rings in [1]. The pair (A,B) can be regarded as the control process with states $(x_i)_{i\geq 0} \in R^n$ and inputs $(u_i)_{i\geq 1} \in R^m$:

$$x_0 = 0, \quad x_i = Ax_{i-1} + Bu_i, \text{ for } i \geq 1.$$

Then, the set of all states reachable from the origin is the submodule of R^n given by the image of the matrix $A^*B = [B|AB|\cdots|A^{n-1}B]$. The system (A,B) is reachable if the columns of A^*B generate R^n.

More generally, for all $i = 1,\dots,n$, one can define N_i^Σ (see [4]), the submodule of R^n given by $\mathrm{im}([B|AB|\cdots|A^{i-1}B])$. These submodules N_i^Σ are invariant under feedback equivalence. Recall that two systems (A,B) and (A',B') are feedback equivalent if there exist invertible matrices $P \in GL_n(R), Q \in GL_m(R)$ and a matrix $K \in R^{m\times n}$ such that $(A',B') = (PAP^{-1} + PBK, PBQ)$, i.e. (A',B') can be obtained from (A,B) by a combination of basis changes and state feedback.

The present article is motivated by the following situation. In [2], single input systems (A,b) over a Bézout domain R are studied, with the condition that the system (A,b) is weakly reachable, i.e. the square matrix A^*b has nonzero determinant. It is shown that any such system is feedback equivalent to a reduced form

$$\left(\tilde{A} = \begin{bmatrix} * & * & * & \cdots & * \\ d_2 & * & * & \cdots & * \\ 0 & d_3 & * & \cdots & * \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & d_n & * \end{bmatrix}, \tilde{b} = \begin{bmatrix} d_1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \right),$$

where the d_i's are nonzero, and the $*$'s in row i of $\tilde{A}$ can be "adjusted" modulo d_i to derive a canonical form. If all the d_i's are equal to 1, then there are no $*$'s, i.e. the system is reachable and the reduced form coincides with the classical controller canonical form.

The purpose of this work is to generalize that reduced form to systems which are not weakly reachable, and also to allow Bézout rings with zero-divisors. If each d_i is idempotent, then the $*$'s in that row can be easily transformed elements orthogonal to d_i by performing elementary transformations. We prove that the class of commutative von Neumann regular rings is characterized precisely as those rings for which any single-input system is equivalent to such a normal form with each d_i idempotent and orthogonal to the $*$'s in the corresponding row. An algorithm is given to compute explicitely the normal form of a system. Next, for a given single-input system Σ, we prove that the structure of each submodule N_i^Σ can be recovered in terms of $d_1,\dots,d_i$. We show with two examples that the principal ideals d_iR do not characterize the feedback equivalence class of a system, at least not trivially, some further work must still be done. Finally, the algorithm is illustrated with a numerical example.

2. Main results

All rings will be commutative and with 1. A von Neumann regular ring, also called absolutely flat ring, is a ring R such that for any $a \in R$ there exists x such that $a = a^2x$. There are many equivalent conditions, for example R is zero-dimensional and has no nonzero nilpotent elements, or every finitely generated ideal is principal (i.e. R is Bézout) and generated by an idempotent (see [5]), or every element is the product of a unit with an idempotent. See [3, Lemma 10]. An important fact we need to know is that R is an elementary divisor ring, i.e. for any $n \times m$ matrix B over R there exist invertible matrices $P \in GL_n(R)$ and $Q \in GL_m(R)$ such that PBQ is diagonal, with diagonal entries $d_1|d_2|\cdots|d_r$, where $r = \min\{n,m\}$ (see [3, Theorem 11]). Also, R has Bass stable range 1: if $(a,b) = R$, there exists k such that $a+bk$ is a unit of R (if u,v are units such that $a^2 = ua$ and $b^2 = vb$, we can take $k = \frac{u-a}{v}$).

We are now ready to prove our main result.

Theorem 1 (Normal form for single-input systems) *For a commutative ring R, the following statements are equivalent:*

(i) R is von Neumann regular.

(ii) Any single input system (A,b) of size $(n,1)$ over R is feedback equivalent to one in the form:

$$\left(\tilde{A} = \begin{bmatrix} * & * & * & \cdots & * \\ d_2 & * & * & \cdots & * \\ 0 & d_3 & * & \cdots & * \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & d_n & * \end{bmatrix}, \tilde{b} = \begin{bmatrix} d_1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \right),$$

with each d_i idempotent, and all the $*$'s in row i of $\tilde{A}$ orthogonal to d_i.

Proof. (i) $\Rightarrow$ (ii) Let (A,b) be a single-input system over a von Neumann regular ring R, with $A \in R^{n\times n}$, $b \in R^n$. The proof will be done by induction on n. If $n = 1$, the system is just a pair of scalars (a,b). Put $b = p^{-1}d$, with p a unit and d idempotent. Then, (a,b) is feedback equivalent to the system

$$(pap^{-1} + pb(-a), pb) = ((1-d)a, d)$$

in the required normal form.

Let $n > 1$. As R is an elementary divisor ring, there exists an invertible matrix P such that $b' = Pb = \begin{bmatrix} d_1 \\ 0 \end{bmatrix}$. Consider the matrix $A' = PAP^{-1}$ partitioned as $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & A_{22} \end{bmatrix}$, with $A_{22} \in R^{(n-1)\times(n-1)}$, $d_1 \in R$ and the remaining blocks of appropriate sizes. Applying induction to the system (A_{22}, a_{21}) of dimension $n-1$, there exists an $(n-1)\times(n-1)$ invertible matrix P_1 and a matrix $K_1 \in R^{1\times(n-1)}$ such that

$$(\widetilde{A_{22}}, \widetilde{a_{21}}) = (P_1 A_{22} P_1^{-1} + P_1 a_{21} K_1, P_1 a_{21})$$

is in the required normal form, with associated idempotent elements $d_2, \ldots, d_n$, and each d_i orthogonal to all the $*$'s in row $(i-1)$ of $\widetilde{A_{22}}$:

$$\left(\widetilde{A_{22}} = \begin{bmatrix} * & * & * & \cdots & * \\ d_3 & * & * & \cdots & * \\ 0 & d_4 & * & \cdots & * \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & d_n & * \end{bmatrix}, \widetilde{a_{21}} = \begin{bmatrix} d_2 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \right).$$

Now, define the matrix $P' = \begin{bmatrix} 1 & -K_1P_1 \\ 0 & P_1 \end{bmatrix}$, with inverse $P'^{-1} = \begin{bmatrix} 1 & K_1 \\ 0 & P_1^{-1} \end{bmatrix}$ and consider the system $(P'A'P'^{-1}, P'b')$ of the form

$$\left(\begin{bmatrix} * & * \\ P_1 a_{21} & P_1 A_{22} P_1^{-1} + P_1 a_{21} K_1 \end{bmatrix}, \begin{bmatrix} d_1 \\ 0 \end{bmatrix} \right).$$

At this point, $(P'A'P'^{-1}, P'b')$ is almost in reduced form. Finally, denoting by v the first row of $P'A'P'^{-1}$, we see that $v - d_1 v$ is orthogonal to d_1, hence we can define $K' = -v$ and it follows that the system

$$(\tilde{A}, \tilde{b}) = (P'A'P'^{-1} + P'b'K', P'b')$$

is feedback equivalent to (A', B') (and thus equivalent to (A,B)) and has the desired form, which proves (ii).

(ii) $\Rightarrow$ (i). Conversely, we will prove that any finitely generated ideal is principal and generated by an idempotent element. Let I be an ideal generated by the entries of some column vector $b \in R^n$, and consider any system of size $(n,1)$ of the form (A,b). By (ii), there exists an invertible matrix P such that Pb is the column vector $[d_1, 0, \ldots, 0]'$, with d_1 idempotent. But Pb also generates I, so we are finished. □

This normal form is similar to that obtained for Bézout domains in [2]. Also, if the first r elements d_i are equal to 1, we recover the reduced form associated to the residual rank r obtained in [6, Proposition 2.5] for rings such that unimodular rows can be completed to invertible matrices. Finally, if all the d_i's are equal to 1, the system is reachable and the normal form is simply the classical controller canonical form.

Now, observe that the proof of (i) $\Rightarrow$ (ii) is in some sense constructive, which gives rise to an effective algorithm, if the ring R is such that Hermite normal forms are computable.

Algorithm 2 (NormalForm) .-

- *INPUT: matrices* $A \in R^{n\times n}, b \in R^n$.
- *OUTPUT: matrices* $\tilde{P}, \tilde{K}$ *such that* $(\tilde{P}A\tilde{P}^{-1} + \tilde{P}b\tilde{K}, \tilde{P}b)$ *is in normal form.*
- *STEP 1: Find P such that* $Pb = \left[\begin{smallmatrix} d_1 \\ 0 \end{smallmatrix}\right]$ *is in Hermite normal form. If necessary, multiply P by a unit to obtain* d_1 *idempotent.*
- *STEP 2: if* $n = 1$*, return with output* $(\tilde{P} = P, \tilde{K} = -A)$*. If not, continue with STEPS 3 . . . 6.*
- *STEP 3: Extract* (A_{22}, a_{21}) *from* $PAP^{-1} = \left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & A_{22} \end{smallmatrix}\right]$.
- *STEP 4: Recursive call with input* (A_{22}, a_{21}) *and output* (P_1, K_1).
- *STEP 5: Define* $P' = \left[\begin{smallmatrix} 1 & -K_1P_1 \\ 0 & P_1 \end{smallmatrix}\right]$ *and* $K' = -$*(first row of* $P'A'P'^{-1}$*).*
- *STEP 6: Return with output* $(\tilde{P} = P'P, \tilde{K} = K')$.

Next, we determine how the d_i's are related to the structure of the system.

Proposition 3 (The d_i's and the system's structure) *Let* $\Sigma = (A, B)$ *be a system of size* $(n, 1)$ *over a ring R in the normal form of Theorem 1:*

$$A = \begin{bmatrix} * & * & * & \cdots & * \\ d_2 & * & * & \cdots & * \\ 0 & d_3 & * & \cdots & * \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & d_n & * \end{bmatrix}, B = \begin{bmatrix} d_1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

with d_i *idempotents and the* $*$*'s in row i of A orthogonal to* d_i*. If we denote by* $f_i = d_1 \cdots d_i$*, for* $i = 1, \ldots, n$*, one has:*

(i) The reachability matrix A^*B *is diagonal. Concretely:*

$$A^*B = \begin{bmatrix} f_1 & 0 & \cdots & 0 \\ 0 & f_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_n \end{bmatrix}$$

(ii) For each $i = 1, 2, \ldots, n$*, one has*

$$N_i^\Sigma \cong f_1R \oplus f_2R \oplus \cdots \oplus f_iR$$

Proof. Since all the $*$'s in row i of A are orthogonal to d_i, the proof of (i) is immediate, and then for any $i = 1, \ldots, n$, (ii) follows by looking at the first i columns of A^*B, which generate N_i^Σ. □

Unfortunately, although the $d_i's$ are closely related with the system's structure, they do not characterize the feedback equivalence, nor do the R-modules N_i, as will be seen in the next examples.

Example 4 The systems $(2,3)$ and $(4,3)$ over the von Neumann regular ring $R = \mathbb{Z}/(6\mathbb{Z})$ are in reduced form with the same associated element $d_1 = 3$ and the same submodule N_1, but are not feedback equivalent. An equivalence would imply an equality of the form $p2p^{-1} + p3k = 4$, i.e. $2 = 3pk$, which is impossible.

Example 5 Let R be a von Neumann regular ring and $e^2 = e$ any idempotent which is not a unit (if no such e exists, then R is a field). The system

$$\Sigma : \left(A = \begin{bmatrix} 0 & 0 \\ e & e-1 \end{bmatrix}, B = \begin{bmatrix} e \\ 0 \end{bmatrix} \right)$$

is in reduced form with associated elements $d_1 = d_2 = e$. Consider the matrices

$$P = \begin{bmatrix} e & 1-e \\ 1-e & 2e-1 \end{bmatrix}, P^{-1} = \begin{bmatrix} 1 & 1-e \\ 1-e & e \end{bmatrix}.$$

Then, Σ is feedback equivalent to $\Sigma' = (PAP^{-1}, PB)$, which is of the form:

$$\left(\begin{bmatrix} e-1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} e \\ 0 \end{bmatrix} \right).$$

Therefore, Σ' is in reduced form with associated elements $\{d'_1 = e, d'_2 = 1\}$, which cannot be obtained from those of Σ by multiplying with units.

We conclude with a numerical example.

Example 6 We have used the PARI-GP calculator to implement Algorithm 2 for systems over von Neumann regular rings $R = \mathbb{Z}/(d\mathbb{Z})$, where d is a squarefree integer. Here it is shown how the algorithm works on a randomly generated example of fixed dimension $n = 6$ and working modulo $d = 30$:

$$A = \begin{bmatrix} 12 & 24 & 13 & 6 & 0 & 17 \\ 2 & 25 & 16 & 11 & 6 & 28 \\ 13 & 27 & 29 & 7 & 15 & 8 \\ 6 & 7 & 6 & 28 & 20 & 3 \\ 28 & 2 & 17 & 22 & 2 & 17 \\ 21 & 5 & 26 & 25 & 24 & 28 \end{bmatrix}, \quad b = \begin{bmatrix} 22 \\ 9 \\ 7 \\ 7 \\ 1 \\ 5 \end{bmatrix}.$$

Running the algorithm, we obtain

$$
\begin{aligned}
P &= \begin{bmatrix} 12 & 4 & 10 & 12 & 2 & 29 \\ 13 & 11 & 28 & 21 & 7 & 3 \\ 28 & 12 & 17 & 12 & 23 & 26 \\ 6 & 6 & 23 & 2 & 14 & 21 \\ 28 & 1 & 4 & 24 & 19 & 0 \\ 24 & 26 & 11 & 7 & 7 & 13 \end{bmatrix}, \\
K &= \begin{bmatrix} 12 & 15 & 5 & 21 & 14 & 26 \end{bmatrix},
\end{aligned}
$$

which yield the normal form $(PAP^{-1}+PbK, Pb) =$

$$
\left(\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 16 & 15 & 0 & 15 & 0 & 15 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 25 & 25 \\ 0 & 0 & 0 & 0 & 25 & 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right).
$$

The above system is in the normal form of Theorem 1, because $1, 16, 6, 25$ are idempotents, and the following orthogonalities hold: $16 \cdot 15 = 0 = 6 \cdot 25$. Finally, the reachabilty matrix of the normal form is computed:

$$
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 16 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},
$$

thus obtaining the structure of all the modules $N_i^{(A,b)}$.

3. Conclusion

In this paper we have been able to give a precise algebraic characterization (von Neumann regular rings) to a linear system's property (the equivalence of any single-input system to a normal form). The normal form obtained is associated to a collection of principal ideals which allow the computation of all the submodules N_i associated to a system. The following question still remains open: How can the normal form be transformed into a canonical form?

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