A SLIDING MODE TECHNIQUE FOR RECONSTRUCTION OF UNKNOWN CONTROL INPUT IN LINEAR SYSTEMS

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Abstract

The unknown input estimation problem for linear dynamic systems exposed to the disturbances is studied in the paper. A novel approach to designing sliding mode observer is suggested for systems not satisfying matching, minimum phase, and detectability conditions. To apply this approach, the reduced-order model of the original system is used. Such a model allows to reduce the dimension of the sliding mode observer, besides the restrictions placed on the original system are relaxed.

Key words

Linear systems, unknown input, estimation, sliding motion.

1 Introduction

This work investigates the problem of unknown input estimation in linear control systems. The unknown input can be interpreted as a function describing faults in engineering systems, and the problem is reduced to the fault diagnosis one. This problem was intensively studied for the past 30 years; see, for example, [Blanke et al.(2006); Ding(2014); Samy et al.(2011); Witczak(2014)]. Different tools for solving the problem have been developed: identification, diagnostic observers, parity relations. The interesting method of identification is based on sliding mode observers (SMO), it uses features of sliding motion studied in [Utkin(1992)].

To solve the problems of unknown input estimation and fault reconstruction, SMOs are used in linear systems [Edwards et al.(2000); Edwards and Spurgeon(1994); Tan and Edwards(2003); Tan and Edwards(2009); Zhirabok et al.(2021); Zhirabok et al.(2019)], in nonlinear systems [Fridman et al.(2008); Yan and Edwards(2007)], and in singular systems [Chan et al.(2017)]. Besides, SMOs are used to ensure fault-tolerant properties [Alwi and Edwards(2008); Edwards et al.(2012); Edwards et al.(2010)]. To design SMO, two conditions should be satisfied: the minimum phase condition and the matching condition [Edwards et al.(2000); Floquet et al.(2007); Defoort et al.(2016)]; they restrict the possibility of SMO application.

To relax the matching condition, two methods are used. The first one is based on a high-order sliding mode differentiator [Bejarano and Fridman(2010); Fridman et al.(2007); Floquet et al.(2007); Fridman et al.(2008); Yang et al.(2013)]; such a differentiator generates the derivatives of the outputs to transform the original system into a system which satisfies the matching condition. Multiple SMOs in cascade are used in the second method [Tan and Edwards(2009)]. Both methods proved good results but assume that the system is a minimum phase and the estimation scheme has a complex structure.

The method suggested by [Alwi et al.(2009)] relaxes the matching condition but the unknown input estimate is corrupted by the fault derivative. The method suggested in [Rios et al.(2014)] does not ensure asymptotic convergence, as a result, the estimation errors are only bounded. The minimum phase condition is relaxed in [Bejarano et al.(2009)] for systems where the unknown inputs are at the dynamics and output, and sufficient and necessary conditions for unknown input and state reconstruction are received. These conditions are less limiting than strong detectability. The method in [Bejarano(2011)] under some restrictions solved the problem of partial unknown input estimation. In [Hmidi et al.(2020)] and [Wang et al.(2017)] the detectability condition is used instead of the minimum phase one.

Note that all cited above papers consider vector unknown inputs and vector functions describing faults. Besides, the dimension of SMOs in [Edwards et al.(2000)] and similar papers coincides with that of the original system.

The main contribution of the present paper is that SMOs are designed for systems which do not satisfy common conditions: minimum phase, matching, and detectability ones. This is explained by that SMO is designed not for the full order model of the original system but for its reduced order model. Such a model has not some special features of the full order model which prevent the possibility for SMO design. As an example, the full order model is non-detectable while the reduced order model has the detectability property. Besides, the canonical form of matrices describing such a model is used. As a result, the limitations placed on the full order model, are relaxed and the dimension of the observer decreases.

Consider a control system

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bu(t) + Zz(t) + Dd(t), \\ &y(t) = Cx(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the vector of control, $y(t) \in \mathbb{R}^l$ is the output vector, A, B, C, Z, and D are known matrices, $z(t) \in \mathbb{R}$ is the unknown bounded input, $\|z(t)\| \leq \beta$, $d(t) \in \mathbb{R}^p$ is the unmatched disturbance, one assumes that $d(t)$ is an unknown bounded function.

Remark 1. The relation $z(t) \in \mathbb{R}$ corresponds to the most probable single faults. If $z(t) \in \mathbb{R}^s$, $s > 1$, the suggested method should be applied to each component of the vector function $z(t)$ presenting multiple faults.

The method suggested in [Yan and Edwards(2007)] assumes that system (1) meets two conditions: 1) $rank(C[Z\ D]) = rank([Z\,D])$, 2) system is a minimum phase; the methods suggested in [Hmidi et al.(2020); Wang et al.(2017)] assume that (1) is detectable. In our paper, the unknown input estimation problem will be solved without these conditions. The method suggested in the paper is based not on the full order model of the original system but on its reduced order model, besides, the identification canonical form of such a model is used.

Note that another type of matching condition with respect to the disturbance was considered in [Antipov et al.(2021); Utkin et al.(2022)]: $rank(B) = rank(B\ D)$. In our paper we assume that the disturbance is unmatched.

2 Reduced order model design

One proposes that (A, C) is non-detectable that is $Ker(V^{(n)}) \neq \emptyset$, where

$$V^{(n)} = \begin{pmatrix} C \\ CA \\ \dots \\ CA^{n-1} \end{pmatrix},$$

besides, unobservable part of (1) is unstable.

Assumption. $Im(Z) \not\subseteq Ker(V^{(n)})$.

Let r_z be minimal relative degree of y with respect to $z(t)$. The output $y_*(t)$ corresponds to r_z and the matrix R_* exists satisfying the condition $R_*y(t) = y_*(t)$. Assumption means that $r_z < \infty$ and the unknown input can be identified.

Remark 2. If $Im(Z) \subseteq Ker(V^{(n)})$, the unknown input cannot be estimated because it lies in the unobservable part of system (1). If $Im(Z) \not\subseteq Ker(V^{(n)})$, the output $y_*(t)$ is affected by the unknown input which can be estimated.

The problem is solved on the basis of the reduced order model of (1) generally described as follows:

$$\begin{aligned} \dot{x}_*(t) &= A_*x_*(t) + B_*u(t) + G_*y(t) \\ &\quad + Z_*z(t) + D_*d(t), \\ y_*(t) &= C_*x_*(t), \end{aligned} \tag{2}$$

where $x_*(t) \in \mathbb{R}^k$ is the vector of state A_*, B_*, G_*, C_*, D_*, and Z_* are matrices to be determined. One assumes that $x_*(t) = \Phi x(t)$ for the matrix Φ to be determined. As is known [Zhirabok et al.(2017a); Zhirabok et al.(2017b)], that matrices R_* and Φ meet the conditions

$$\begin{aligned} &\Phi A = A_*\Phi + G_*C, \quad R_*C = C_*\Phi, \\ &\Phi B = B_*, \quad \Phi D = D_*, \quad \Phi Z = Z_*. \end{aligned} \tag{3}$$

Our purpose is to develop the method to construct the model (2) insensitive to the disturbance that allows solving the problem of unknown input estimation.

We specify the matrices A_* and C_* in the identification canonical form

$$A_* = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad C_* = (\,1\ 0\ 0\ \dots\ 0\,). \tag{4}$$

Remark 3. This form exists if (A_*, C_*) in (2) is observable. Otherwise, system (2) can be transformed into the observable canonical form [Kwakernaak and Sivan(1972)], and then the observable part of this form can be transformed into (4) of less dimension.

Based on (4), one obtains from (3) the following relations:

$$\begin{aligned} &\Phi_1 = R_*C, \ \Phi_i A = \Phi_{i+1} + G_{*i}C, \ i = 1, ..., k-1, \\ &\Phi_k A = G_{*k}C. \end{aligned} \tag{5}$$

Here Φ_i and G_{*i} are i-th rows of the matrices Φ and J_*, $i = 1, ..., k$, respectively. One can show [Zhirabok et al.(2017b)] that (5) are transformed into the equation

$$(\,1\; -G_{*1}\; ...\; -G_{*k}\,)W^{(k)} = 0, \tag{6}$$

where

$$W^{(k)} = \begin{pmatrix} R_*CA^k \\ CA^{k-1} \\ ... \\ C \end{pmatrix}.$$

Insensitivity to the disturbance in the form of condition $\Phi D = 0$ one can take into consideration as $(\,1\; -G_{*1}\; ...\; -G_{*k}\,)D^{(k)} = 0$ [Zhirabok et al.(2017a); Zhirabok et al.(2017b)] where

$$D^{(k)} = \begin{pmatrix} R_*CD & R_*CAD & ... & R_*CA^{k-1}D \\ 0 & CD & ... & CA^{k-2}D \\ ... & ... & ... & ... \\ 0 & 0 & ... & 0 \end{pmatrix}.$$

Clearly, this equation and (6) can be written as

$$(\,1\; -G_{*1}\; ...\; -G_{*k}\,)(W^{(k)}\;\; D^{(k)}) = 0. \tag{7}$$

Solve this equation for minimal $k \geq r_z$ and design the model (8) as

$$\begin{aligned} \dot{x}_*(t) &= A_*x_*(t) + B_*u(t) + G_*y(t) + Z_*z(t), \\ y_*(t) &= C_*x_*(t). \end{aligned} \tag{8}$$

Note that assumption $Im(Z) \not\subseteq Ker(V^{(n)})$ means that $Z_* \neq 0$.

If (7) has not solution for all $k < n$, the model is sensitive to the disturbance and the problem of exact unknown input estimation has no solution. In this case, the problem of approximate unknown input estimation can be solved [Zhirabok et al.(2019)]. Another approach is to estimate the disturbance $d(t)$ and then use such an estimate in (2) [Alvarez-Sanchez et al.(2013)].

Rewrite (7) as

$$\begin{aligned} \dot{x}_{*1}(t) &= x_{*2}(t) + B_{*1}u(t) + G_{*1}y(t) + Z_{*1}z(t), \\ \dot{x}_{**}(t) &= A_{*2}x_{**}(t) + B_{*2}u(t) + G_{*2}y(t) + Z_{*2}z(t), \\ y_*(t) &= x_{*1}(t), \end{aligned} \tag{9}$$

where A_{*2} is submatrix of the matrix A_* corresponding to the vector $x_{**} = (x_{*2}, ..., x_{*k})^T$,

$$\begin{pmatrix} B_{*1} \\ B_{*2} \end{pmatrix} = B_*, \quad \begin{pmatrix} G_{*1} \\ G_{*2} \end{pmatrix} = G_*, \quad \begin{pmatrix} Z_{*1} \\ Z_{*2} \end{pmatrix} = Z_*.$$

Transform the above model into the form

$$\dot{x}_*(t) = \begin{pmatrix} \dot{x}_{*1}(t) \\ \dot{x}_{**}(t) \end{pmatrix} = \begin{pmatrix} 0 & A_{*1} \\ 0 & A_{*2} \end{pmatrix} \begin{pmatrix} x_{*1}(t) \\ x_{**}(t) \end{pmatrix} + B_*u(t) + G_*y(t) + Z_*z(t),$$

where $A_{*1} = (1\; 0 \ldots 0)$.

Remark 4. The model (9) corresponds to that in [Edwards et al.(2000); Wang et al.(2017)] with stable matrix A_{*2} since the original system is a minimum phase or detectable by assumption. In contrast to this, A_{*2} in our approach may be unstable; stability of SMO is provided below by feedback.

3 Sliding mode observer design

SMO is sought in the from

$$\begin{pmatrix} \dot{\hat{x}}_{*1}(t) \\ \dot{\hat{x}}_{**}(t) \end{pmatrix} = \begin{pmatrix} 0 & A_{*1} \\ 0 & A_{*2} \end{pmatrix} \begin{pmatrix} \hat{x}_{*1}(t) \\ \hat{x}_{**}(t) \end{pmatrix} + B_*u(t) + G_*y(t) - Kv(t) - \begin{pmatrix} l_1 & 0 \\ L_2 & 0 \end{pmatrix} e(t), \tag{10}$$

where $v(t) = sign(e_1)$,

$$e(t) = \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} = \begin{pmatrix} \hat{y}_*(t) - R_*y(t) \\ \hat{x}_{**}(t) - x_{**}(t) \end{pmatrix}, \;\; K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix},$$

the coefficients l_1 and L_2 provide the observer stability; they exist due to the canonical form (4).

The error $e(t)$ is described by

$$\begin{aligned} \dot{e}(t) &= \begin{pmatrix} -l_1 & A_{*1} \\ -L_2 & A_{*2} \end{pmatrix} e(t) - Kv(t) - Z_*z(t) \\ &= A_{**}e(t) - Kv(t) - Z_*z(t), \end{aligned} \tag{11}$$

or

$$\begin{aligned} \dot{e}_1(t) &= -l_1e_1(t) + A_{*1}e_2(t) - k_1v(t) - Z_{*1}z(t), \\ \dot{e}_2(t) &= -L_2e_1(t) + A_{*2}e_2(t) - k_2v(t) - Z_{*2}z(t), \end{aligned} \tag{12}$$

where A_{**} is stable matrix.

Remark 5. It is known that if $e(t)$ meets the equation $\dot{e}(t) = \bar{F}e(t) + g(t)$ with stable matrix $\bar{F}$ and the function $g(t)$ is bounded, then $e(t)$ is bounded as well that is $\|e(t)\| \leq \gamma$ for some scalar γ.

Theorem. *The observer (10) generates the following estimates for the function $z(t)$: if $Z_{*2} = 0$, then*

$$\hat{z}(t) = -Z_{*1}^{+}k_1v_{eq}(t), \tag{13}$$

and

$$\hat{z}(t) = -Z_{*2}^{+}k_2v_{eq}(t) \tag{14}$$

when $Z_{*2} \neq 0$, where $Z_{*1}^{+} = (Z_{*1}^{T}Z_{*1})^{-1}Z_{*1}^{T}$ and $Z_{*2}^{+} = (Z_{*2}^{T}Z_{*2})^{-1}Z_{*2}^{T}$, $v_{eq}(t)$ is the continuous approximation of the discontinuous function $v(t)$ [Edwards et al.(2000)]:

$$v_{eq}(t) = \frac{e_1(t)}{|e_1(t)| + \varepsilon},$$

ε is a small positive scalar.

Proof. Note that proof is similar to that developed in [Wang et al.(2017)]. Firstly one proves that $\|e(t)\| \le \delta$ for some δ. Because $\|v(t)\| = 1$ and the function $z(t)$ is bounded, then $\|Z_* z(t) + Kv(t)\| \le g_0$ for some scalar g_0. It is evident from (12) and Remark 5 that the function $e(t)$ is bounded and $\|e(t)\| \le \delta$ for some δ.

Secondly, one proves that $e_1 = 0$ in finite time due to choices of the observer gains; as a result, sliding motion is achieved. Introduce Lyapunov candidate function $V_1 = e_1^2$ and using (12) take its derivative:

$$\dot{V}_1 = 2e_1\dot{e}_1 = 2e_1(-l_1 e_1 + A_{*1}e_2 - k_1 v - Z_{*1}z).$$

Since $v = sign(e_1)$, then $e_1 k_1 v = k_1|e_1|$ and

$$\begin{aligned}\dot{V}_1 &\le -2l_1 e_1^2 + 2|e_1|(-k_1 + \|A_{*1}\|\|e_2\| + \|Z_{*1}\|\|z\|) \\ &\le -2l_1 e_1^2 + 2|e_1|(-k_1 + \delta\|A_{*1}\| + \beta\|Z_{*1}\|).\end{aligned}$$

If k_1 satisfies

$$k_1 > \delta\|A_{*1}\| + \beta\|Z_{*1}\|, \tag{15}$$

then $\dot{V}_1 < 0$, and sliding motion ($e_1 = \dot{e}_1 = 0$) happens in finite time.

Thirdly, one proves that $e_2 = 0$ in finite time due to choices of the observer gains, and sliding motion is achieved. Because A_{**} is stable matrix, there exist positive definite symmetric matrices P and W such that $A_{**}^T P + PA_{**} = -W$. Write down the matrix P in the form $P = \begin{pmatrix} p_1 & P_2 \\ P_2^T & P_3 \end{pmatrix}$ where P_3 is positive definite symmetric matrix. Consider Lyapunov candidate function $V_2 = e^T Pe$ and take its derivative using (11):

$$\dot{V}_2 = e^T(A_{**}^T P + PA_{**})e - 2e^T P(Z_* z + Kv).$$

It is evident from (12) and $e_1 = \dot{e}_1 = 0$ that $A_{*1}e_2 = k_1 v + Z_{*1}z$. Using $k_2 = P_3^{-1}A_{*1}^T k_3$ for some $k_3 > 0$ and $e_1 = 0$, we obtain

$$\begin{aligned}\dot{V}_2 &= -e^T We - 2e^T PZ_* z \\ &\quad -2(0 \quad e_2^T)\begin{pmatrix} p_1 & P_2 \\ P_2^T & P_3 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} v \\ &= -e^T We - 2e^T PZ_* z \\ &\quad -2(e_2^T P_2^T \quad e_2^T P_3)\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} v \\ &= -e^T We - 2e^T PZ_* z \\ &\quad -2(e_2^T P_2^T k_1 + e_2^T P_3 k_2)v \\ &= -e^T We - 2e^T PZ_* z \\ &\quad -2(e_2^T P_2^T k_1 + (A_{*1}e_2)^T k_3)v \\ &= -e^T We - 2e^T PZ_* z \\ &\quad -2(e_2^T P_2^T k_1 v + (k_1 v + Z_{*1}z)^T k_3 v) \\ &\le -e^T We + 2\delta\|PZ_*\|\beta + 2k_1\delta\|P_2\| \\ &\quad -2k_1 k_3 + 2k_3\beta\|Z_{*1}\|.\end{aligned}$$

If one chooses k_3 and k_1 as

$$k_3 > 2\delta\|P_2\|, \quad k_1 > 2\beta\|Z_{*1}\| \tag{16}$$

and then k_3 is chosen as

$$k_3 > \frac{2\beta\delta\|PZ_*\|}{k_1 - 2\beta\|Z_{*1}\|}, \tag{17}$$

then $\dot{V}_2 < 0$. Theorem has been proved.

Based on (15), (16), and (17), the coefficients k_1 and k_3 should be chosen as

$$\begin{aligned}&k_1 > max\{\delta\|A_{*1}\| + \beta\|Z_{*1}\|,\ 2\beta\|Z_{*1}\|\}, \\ &k_3 > max\left\{2\delta\|P_2\|,\ \frac{2\beta\delta\|PZ_*\|}{k_1 - 2\beta\|Z_{*1}\|}\right\}.\end{aligned}$$

It is evident from (12) that if $Z_{*2} = 0$, the unknown input $z(t)$ is estimated based on the first expression in (12) by (13); otherwise, one uses the second expression in (12) and obtains (14).

4 Examples

4.1 Practical example

Consider the control system

$$\begin{aligned}&\dot{x}_1(t) = \tfrac{1}{i_p}x_2(t) + d(t), \\ &\dot{x}_2(t) = -\tfrac{K_d}{J_H}x_2(t) + \tfrac{K_M}{J_H}x_3(t), \\ &\dot{x}_3(t) = -\tfrac{K_\omega}{L_m}x_2(t) - \tfrac{R_m}{L_m}x_3(t) + \tfrac{K_U}{L_m}u(t) + z(t), \\ &y_1(t) = x_1(t), \quad y_2(t) = x_3(t).\end{aligned} \tag{18}$$

Equations (18) constitute a model of electric servoactuator where x_1 is the output rotation angle at the reducer output shaft; x_2 is the output rotation velocity at the motor output shaft; x_3 is the current through the servoactuator windings; i_p is the reducing ratio of the reducer; J_H is the moment of inertia of the electric servoactuator and of the rotating parts of the reducer; K_ω and K_M are the respective coefficients of the counter EMF and of the torque; K_d is the torques of the Coulomb friction at the motor output shaft; R_m and L_m are the active and inductive resistances of the electric servoactuator windings, respectively; K_U is the amplification factor.

One assumes that the unknown input $z(t) = -\frac{\tilde{R}(t)}{L_m}x_3(t)$ is caused by the deviation $\tilde{R}(t)$ of the active resistances from its nominal value R_m.

Note that the unknown input estimation problem for this example can be solved by known methods (for example, [Edwards et al.(2000)]) but we use our approach to demonstrate its efficiency.

Denote $\gamma_1 = 1/i_p, \gamma_2 = -K_d/J_H, \gamma_3 = K_M/J_H, \gamma_4 = -K_\omega/L_m, \gamma_5 = -R_m/L_m, \gamma_6 = K_U/L_m$. System (18) is described by the matrices

$$A = \begin{pmatrix} 0 & \gamma_1 & 0 \\ 0 & \gamma_2 & \gamma_3 \\ 0 & \gamma_4 & \gamma_5 \end{pmatrix},\ B = \begin{pmatrix} 0 \\ 0 \\ \gamma_6 \end{pmatrix},\ C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$Z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ D = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

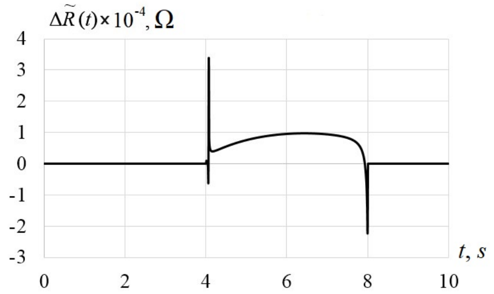

Figure 2. Behavior of fault estimation error $\hat{R}_m(t) - R_m(t)$.

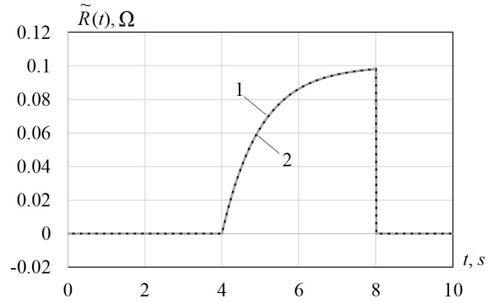

Figure 1. Behavior of function $\tilde{R}(t)$.

Design the model insensitive to $d(t)$. One can show that (7) is solvable with $k = 2$ and

$$(\, R_* \; -G_{*1} \; -G_{*2}\,) = (\, 0 \; 1 \; 0 \; -\gamma_2 - \gamma_5 \; 0 \; -\gamma_3\gamma_4 + \gamma_2\gamma_5\,)$$

which gives $R_* = (0 \;\; 1)$,

$$G_* = \begin{pmatrix} 0 & \gamma_2 + \gamma_5 \\ 0 & \gamma_3\gamma_4 - \gamma_2\gamma_5 \end{pmatrix}, \; \Phi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \gamma_4 & -\gamma_2 \end{pmatrix},$$
$$B_* = \begin{pmatrix} \gamma_6 \\ -\gamma_2\gamma_6 \end{pmatrix}, \; Z_* = \begin{pmatrix} 1 \\ -\gamma_2 \end{pmatrix}.$$

The model is given by

$$\begin{aligned} \dot{x}_{*1}(t) &= x_{*2}(t) + (\gamma_2 + \gamma_5)y_2(t) + \gamma_6 u(t) + z(t), \\ \dot{x}_{*2}(t) &= (\gamma_3\gamma_4 - \gamma_2\gamma_5)y_2(t) - \gamma_2\gamma_6 u(t) - \gamma_2 z(t). \end{aligned}$$

Choosing $\lambda_1 = -1$ and $\lambda_2 = -2$ as eigenvalues of the matrix A_{**}, one obtains $l_1 = 3$, $L_2 = 2$; SMO is given by

$$\begin{aligned} \dot{\hat{x}}_{*1}(t) &= \hat{x}_{*2}(t) + (\gamma_2 + \gamma_5)y_2(t) + u(t) \\ &\quad -k_1 v(t) - 3e_1(t), \\ \dot{\hat{x}}_{*2}(t) &= (\gamma_3\gamma_4 - \gamma_2\gamma_5)y_2(t) - \gamma_2\gamma_6 u(t) \\ &\quad -k_2 v(t) - 2e_1(t), \\ \hat{y}_*(t) &= \hat{x}_{*1}(t), \end{aligned} \tag{19}$$

where $e_1(t) = \hat{y}_*(t) - y_2(t)$, $v(t) = sign(e_1(t))$. The function $z(t)$ is estimated as $\hat{z}(t) = k_1 v_{eq}$

For simulation, consider system (18) and the observer (19) with $u(t) = sin(t)$, $d(t) = 2sin(2t)$, $k_1 = 10$, $k_2 = 20$, $\beta = 1$, and $|e_1(0)| = 0$. Simulation results are presented in Figs. 1 and 2; behavior of function $\tilde{R}(t)$ and the estimation error $\Delta\tilde{R}(t) = \hat{R}_m(t) - R_m(t)$ for the function $R_m(t)$ are shown.

4.2 Illustrative example

Consider non-minimum phase and non-detectable system control system:

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + x_2(t) + u(t), \\ \dot{x}_2(t) &= -x_2(t) + x_4(t) + z(t), \\ \dot{x}_3(t) &= x_3(t) + x_4(t) + d(t), \\ \dot{x}_4(t) &= -x_4(t) + d(t), \\ y_1(t) &= x_1(t), \quad y_2(t) = x_4(t). \end{aligned} \tag{20}$$

The system is described by the following matrices:

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \; B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$
$$H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; Z = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \; D = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

It can be shown that the system is not minimum phase and detectable. Really, Rozenbrock matrix $R(s)$ for system (20) is given by

$$R(s) = \begin{pmatrix} sI - A & -[D \; Z] \\ C & 0 \end{pmatrix}$$
$$= \begin{pmatrix} s+1 & -1 & 0 & 0 & 0 & 0 \\ 0 & s+1 & 0 & -1 & 0 & -1 \\ 0 & 0 & s-1 & -1 & -1 & 0 \\ 0 & -1 & 0 & s+1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Since $rank(R(s)) = 5$ under $s = 1$, then $s = 1$ is invariant zero of $(A, [D \; Z], C)$, therefore the system is no minimum phase. Besides, one can shown that $Ker(V^{(3)}) = (0 \;\; 0 \;\; 1 \;\; 0)^T$ and the unobservable part of the system presented by x_3 is unstable, therefore the system is non-detectable. Then the methods suggested in [Tan and Edwards(2003); Wang et al.(2017)] cannot be used in this case but SMO can be designed by our method.

Clearly, $Im(Z) \not\subseteq Ker(V^{(n)})$, $r_z = 2$, $y_* = y_1$, and $R_* = (1 \;\; 0)$. The solution of (7) is $G_{*1} = (-2 \;\; 0)$ and $G_{*2} = (-1 \;\; 1)$; then

$$\Phi_1 = (1 \;\; 0 \;\; 0 \;\; 0), \quad \Phi_2 = (1 \;\; 1 \;\; 0 \;\; 0),$$
$$Z_* = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad B_* = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus, the model (8) is of the form

$$\begin{aligned} \dot{x}_{*1}(t) &= x_{*2}(t) - 2y_1(t) + u(t), \\ \dot{x}_{*2}(t) &= -y_1(t) + y_2(t) + u(t) + z(t), \\ y_*(t) &= x_{*1}(t) = y_1(t), \end{aligned}$$

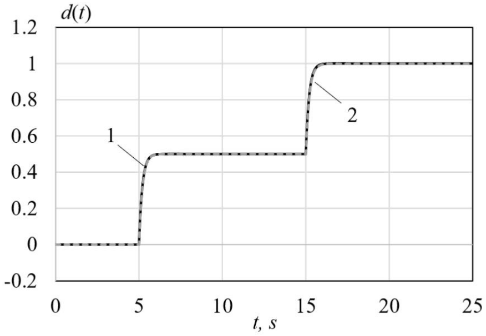

Figure 3. Behavior of step-shaped function $z(t)$.

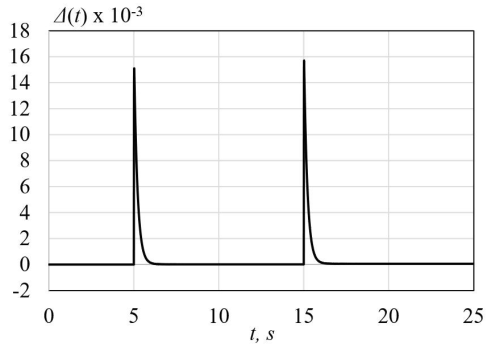

Figure 4. Behavior of fault estimation error $\Delta(t) = \hat{z}(t) - z(t)$.

where $x_{*1} = x_1$ and $x_{*2} = x_1 + x_2$.

One chooses $\lambda_1 = -1$ and $\lambda_2 = -2$ as eigenvalues of the matrix A_{**} and obtains $l_1 = 3$, $L_2 = 2$; SMO is described by equations

$$\begin{aligned} \dot{\hat{x}}_{*1}(t) &= \hat{x}_{*2}(t) - 2y_1(t) + u(t) - k_1 v(t) - 3e_1(t), \\ \dot{\hat{x}}_{*2}(t) &= -y_1(t) + y_2(t) + u(t) - k_2 v(t) - 2e_1(t), \\ \hat{y}_*(t) &= \hat{x}_{*1}(t), \end{aligned} \tag{21}$$

$e_1(t) = \hat{y}_*(t) - y_1(t)$, $v(t) = sign(e_1(t))$. The function $z(t)$ is estimated as $\hat{z}(t) = k_2 v_{eq}$.

For simulation, consider system (20) and the observer (21) with $u(t) = sin(t)$, $d(t) = 20sin(2t)$, $k_1 = 2$, $k_2 = 3$, $\beta = 1.5$, and $|e_1(0)| = 0$. Simulation results are shown in Figs. 3 and 4; graphics of the function $z(t)$, its estimation $\hat{z}(t)$ and the estimation error $\Delta(t) = \hat{z}(t) - z(t)$ for step-shaped type of the function $z(t)$ are presented.

5 Conclusion

The problem of unknown input estimation for systems which do not satisfy the matching, minimum phase, and detectability conditions is studied. The suggested methods is based on the reduced order model of the original system insensitive to the disturbance. The practical and illustrative examples show the effectiveness of the proposed method. Future work is developing algorithms for unknown input and parameter estimation for nonlinear systems such that considered in [Furtat et al.(2022)] to compare the results with [Furtat and Orlov(2020)].

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