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RELATIVISTIC QUANTUM CONTROL THEORY

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Abstract
This is an attempt to develop ideas about a possible Relativistic Quantum Control Theory that could be valuable to enhance the performance of particle detection and collision. In order to achieve this goal, a precise formulation of Optimal Control Systems in terms of the Classical Field Theory is proposed in such a way that the resulting dynamics, as well the constraint relations that systems should obey, must be invariant to Lorentz transformations, i.e., must be expressed in a covariant manner. A quantization procedure of the whole Optimal Control System is carried out, which has been made Lorentz-invariant. The resulting Optimal Control System could be studied and applied to problems involving quantum particles in the relativistic regime allowing for minimizing times, average distances and/or energy costs of the processes.

Key words
Canonical quantization, covariant formulation, field theory, optimal control, quantum systems.

1 Introduction

Along the 20^{th} century, Physics and Cybernetics experienced a strong development giving decisive contributions to modern science in spite of the weak interactions between them. This fact, may be is due to the different approaches with Physics being a descriptive science and Cybernetics, a prescriptive one [Fradkov, 2007].

However, it can not be denied that automatic systems play an important role concerning physical experiments, but in most cases, the control theory is considered to be secondary with no effective contributions to explain physical phenomena [Fradkov, 2007].

In the late 1980s, with the development of ultra fast lasers, methods based on optimal control were developed to control molecular systems [Krempl et. al, 1992]. The interest in this kind of problem increased in the early 1990s and the concept of classical and quantum approaches were developed [Dahleh et. al, 1996; Brumer and Shapiro, 2003; Gonzalez-Henao et. al, 2015].

Additionally, researches on quantum computation hardware can be implemented by manipulating the quantum state of trapped ions via laser or electrical fields [Hangan et. al, 2004]. Consequently, it seems to be important to formulate optimal control strategies to quantum particles in the relativistic regime allowing, for example, for minimizing times, average distances and/or energy costs of the processes meant to be controlled in such a physical domain.

Besides, a Relativistic Quantum Control Theory could be valuable to enhance the performance of particles detection and collision processes, as well in the purpose of promoting controlled annihilation of particles and antiparticles. Here some ideas about the Relativistic Quantum Control Theory are developed, starting with Optimal Control basic points.

The optimal control problem is formulated by using a classical field theory approach. An expression of a functional is obtained, in a covariant notation. The next step is the quantization of the covariant optimal control system by using canonical procedures.

Next, the main section of the paper is presented with a possible formulation of optimal control theory in Quantum Electrodynamics (QED). A few remarks and conclusions complete the work.

2 General Ideas About Optimal Control

The general optimal control problem, object of the conventional Control Theory, is to look for an admissible control $u^*(t)$ that causes the dynamical system of interest to follow an admissible trajectory $x^*(t)$ that minimizes a certain performance or cost functional measure.

Consequently, this problem can be formulated as follows: given a set X of state vector-valued functions $x(t)$, with $t \in \Re : [t_o, t_f]$, and a set U of control functions $u(t) : \Re \to \Re^m$, find the functions $x \in X$ and $u \in U$ for the dynamical system $\dot{x} = f(x, u(t))$, with f being a smooth vector field, which minimizes a cost functional $J : XxU \to \Re$ given by:

$$J(u) = h(x(t_f), t_f) + \int_{t_0}^{t_f} g(x(t), u(t), t)dt, \quad (1)$$

with h and g being smooth functions depending on the variables $x(t)$, $u(t)$ and on the value of $x(t_f)$, such that $h : \Re^n x \Re \rightarrow \Re$ and $g : \Re^n x \Re^m X \Re \rightarrow \Re$, which are judiciously chosen regarding the particular optimal control problem to be solved.

Expression (1) defines the so-called "Bolza's problem" in Optimal Control Theory, which comprises the two particular cases, namely "Mayer's problem" and "Lagrange's problem", corresponding to set $g(x(t), u(t))$ or $h(x(t_f))$ to zero, respectively [Kirkov, 1970; Sieniutycz, 2014].

Concerning to non relativistic quantum evolution, examples of Mayer's problems are minimum time problems, in which the control goal is to drive and initial state to a given target state in minimum time.

Lagrange's problem describes a situation in which the control cost accumulates with time. This is the case, for example, of minimizing the energy used during the control action and/or the average distance of the trajectory in a quantum evolution process, starting in a given state.

Finally, Bolza's problem, being a combination of both, arises when there is a cumulative cost which increases during the control action, but special emphasis is placed on the situation at the final time in a quantum dynamics evolution.

To generalize this ideas to relativistic quantum systems, the time must be included as a state coordinate and the formulation must be modified with new definitions for the functionals to be optimized, regarding to the electric and magnetic fields operating the control actions.

3 Optimal Control: A Classical Field Theory approach

Expression (1) can be written in the framework of Classical Field Theory as:

$$I(u(x_\mu)) = h(\Phi_\alpha(B)) + \int_A^B d^4x g(\Phi_\alpha(x_\mu), u(x_\mu), x_\mu)\,, \quad (2)$$

where $x_\mu = (x_0, x_1, x_2, x_3) = (ct, x, y, z)$ is the space-time covariant four-vector, defined in the Minkowski space; A and B the initial and the final event in the four-dimensional space-time; $h(\Phi_\alpha(B))$ the smooth Mayer function evaluated on $\Phi_\alpha(B)$, i.e., the value assumed by the classical field of interest in the space-time coordinates given by B.

Here, the control action $u(x_\mu)$ depends explicitly on the space-time coordinates and the Lagrange function $g(\Phi_\alpha(x_\mu), u(x_\mu), x_\mu)$ depends on $\Phi_\alpha(x_\mu)$, (x_μ), and $u(x_\mu)$. $I(u(x_\mu))$ represents the cost functional to be minimized [Campos et. al, 2010].

Expression (2) is composed of a scalar function and a volumetric integral in the space-time of other scalar quantities and, consequently, is Lorentz-invariant by Jacobian transformations of coordinates. The cost functional is thus invariant by Lorentz transformations.

It is worth observing that the functional proposed presents certain generality concerning the Classical Field Theory, so that field Φ_α can be taken either as the real or complex scalar field, Dirac field, or the electromagnetic field, adequately expressed in their covariant form [Barut, 1980].

As an example and to clarify some points about optimal control in the classical field theory context, some facts about the covariant formulation of the electromagnetic field are presented. In order to do this, a notation is established:

$A_\mu = (\phi, \mathbf{A}) = (A_0, A_1, A_2, A_3)$: electromagnetic four potential;
$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$: electromagnetic field tensor,

with $F_{\mu\nu} = -F_{\nu\mu}$, i.e., $F_{\mu\nu}$ is antisymmetric with rank 2.

Electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ are related to potentials $\mathbf{A}$ and Φ as:

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \Phi;$$

$$\mathbf{B} = \nabla x \mathbf{A},$$

with the covariant gauge condition: $A'_\mu = A_\mu + \partial_\mu \Lambda$,, which accounts for the invariance of A_μ, with Λ being a smooth scalar function, and $\partial_\mu = \dfrac{\partial}{\partial x^\mu}$ corresponding to the covariant derivative.

The Lagrangian Density of the electromagnetic field in the presence of charged particles and currents is given by:

$$L = -\frac{F_{\mu\nu}F^{\mu\nu}}{4} - \frac{1}{c} J^\mu A_\mu, \quad (3)$$

where J^μ is the current four-vector, with $J^\mu = (c\rho, J^1, J^2, J^3)$, which satisfies the covariant continuity equation $\partial_\mu J^\mu = 0$.

Therefore, considering equation (3), the electromagnetic action is expressed as:

$$S(A^\mu, \partial_\mu A^\mu) = \int_a^b L\, d^4x = -\int_a^b (\frac{F_{\mu\nu}F^{\mu\nu}}{4} + \frac{1}{c} J^\mu A_\mu)\, d^4x. \quad (4)$$

Consequently, imposing the first variation of the action equal to zero, ($\delta S = 0$), the dynamic equation of the electromagnetic field becomes [Barut, 1980; Greiner and Reinhart, 1996]:

$$\partial_\mu F^{\mu\alpha} = \frac{J^\alpha}{4\pi}, \qquad (5)$$

that represents Maxwell's non-homogeneous equations in Gaussian unities.

Here, the classical field Φ_α corresponds to A_μ, so that action functional S depends on this field and on its space-time derivative $\partial_\mu A^\mu$ [Jackson, 1999].

Thus, in order to define a suitable control action in terms of a four-vector u_μ it is possible to set the time-component of the four-potential A_μ as the time-dependent control action in direct analogy with the conventional $u(t)$ found in the standard Optimal Control Theory.

Therefore the electric potential, once made time-dependent, could work as control action over the system dynamics. Considering the same type of argumentation, the cost functional (2) can be rewritten as [Campos et. al, 2010]:

$$I(\Phi(x_0)) = h(A_\mu(B)) + \int_A^B g(A_\mu, \Phi(x_0), x_\mu)\, d^4x. \qquad (6)$$

4 Quantization of Covariant Optimal Control Systems

In the context of the Non-Relativistic Quantum Mechanics, Canonical Quantization can be regarded as the process that associates Classical Mechanics quantities to Hermitian operators in Quantum Mechanics. These operators act on the state vectors defined in the Hilbert Space associated to a given quantum system. Such a process allows replacing the classical description by the quantum one.

Here, this procedure is applied to a simple example, deriving the quantum Hamiltonian from the classical counterpart allowing the construction of a functional to be applied in quantum optimal control.

Considering the harmonic oscillator, i.e., a system with a Lagrangian L, which classical Hamiltonian expressed in the conjugated variables p and q is given by: $H(p,q) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$.

The corresponding quantum Hamiltonian is given by: $\hat{H}(p,q) = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$, with $\hat{x}$ and $\hat{p}$ being the quantum operators position and momentum, respectively.

As known from Quantum Mechanics, these operators must obey the canonical commutation relation: $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker symbol.

In the context of the Quantum Field Theory, it is necessary to define the canonical momentum density conjugated to a field $\Phi(\mathbf{x}, t)$ as:

$$\pi(\mathbf{x}, t) = \frac{\partial L}{\partial\dot{\Phi}(\mathbf{x}, t)} = \partial_0\Phi(\mathbf{x}, t), \qquad (7)$$

where L stands for the covariant classical Lagrangian, and with the canonical commutation given by:

$$[\pi(\mathbf{x}, t), \Phi(\mathbf{x}', t)] = [\partial_0(\mathbf{x}, t), \Phi(\mathbf{x}', t)] = i\delta^D(\mathbf{x} - \mathbf{x}'), \qquad (8)$$

where $\delta^D(\mathbf{x} - \mathbf{x}')$ is the D-dimensional Dirac's function with respect to two points in the four-dimensional space-time [Greiner and Reinhart, 1996].

Regarding the physical quantities defined and their physical meaning, it must be stressed that position is not an operator here, but rather an argument of quantum field Φ, that plays the role of the position operator from Quantum Mechanics.

Therefore, a generic Bolza's cost functional for a field Φ, by means of the canonical quantization procedure, can be written as [Dahleh et. al, 1996]:

$$I(u(\mathbf{x}, t)) = h(\Phi(\mathbf{x_F}, t_F)) + \int_A^B dx^4 g(\Phi(\mathbf{x}, t), u(\mathbf{x}, t), (\mathbf{x}, t)), \qquad (9)$$

considering that the field $\Phi(\mathbf{x}, t)$ obeys the canonical commutation relation (8). Moreover, the control action is provided by an external field, imposed by an external source.

5 Optimal Control in Quantum Electrodynamics

Quantum Electrodynamics (QED) is an Abelian gauge theory with symmetry group $U(1)$, that is, the multiplicative group of all complex numbers with absolute value 1, i.e., the unit circle in the complex plane.

To study the control of the charged spin-1/2 fields (electron-positron field) it must be considered that the gauge field that mediates their interaction is the electromagnetic field. The QED Lagrangian density for a spin-1/2 field interacting with the electromagnetic field is given, in natural unities ($\hbar = 1; c = 1$), by the real part of:

$$L = \bar{\Psi}(i, \gamma^\mu D_\mu - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \qquad (10)$$

In equation (10), γ^μ is a Dirac matrix, Ψ is a bispinor field of spin-1/2 particle, and $\bar{\Psi} = \Psi^\dagger\gamma_0$ is the Dirac adjoint.

Additionally, $D_\mu = \partial_\mu + ieA_\mu + ieB_\mu$ is the gauge covariant derivative, e is the electric charge of the bispinor field, A_μ is the covariant four-potential of the electromagnetic field generated by the electron itself, B_μ is the external field imposed by the external source, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor.

Applying the expression of the gauge covariant derivative D_μ to the Lagrangian (10):

$$L = i\bar{\Psi}\gamma^\mu\Psi - e\bar{\Psi}\gamma_\mu(A^\mu + B^\mu)\Psi - m\bar{\Psi}\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \quad (11)$$

Expression (11) allows inserting Ł in the Lagrange equation of motion for a covariant field:

$$\partial_\mu[\frac{\partial L}{\partial(\partial_\mu\Psi)}] - \frac{\partial L}{\partial\Psi} = 0, \quad (12)$$

allowing the derivation of the field equations for QED. Considering the calculations [Butkovskiy and Samoilenko, 1990; D'Alessandro, 2003]:

$\partial_\mu[\frac{\partial Ł}{\partial(\partial_\mu\Psi)}] = \partial_\mu(i\bar{\Psi}\gamma^\mu)$;
$\frac{\partial Ł}{\partial\Psi} = -e\bar{\Psi}\gamma_\mu(A^\mu + B^\mu)\Psi - m\bar{\Psi}\Psi$,

equation (12) results:

$$i\partial_\mu\bar{\Psi}\gamma^\mu - m\bar{\Psi} = e\bar{\Psi}\gamma_\mu(A_\mu + B_\mu) = 0. \quad (13)$$

It can be observed that the left hand side of (13) is like the Dirac equation for a free particle, and the right hand side stands for the interaction with the electromagnetic field.

Consequently, equations (12) and (13) are the motion equations of the field $\bar{\Psi}$ and Ψ, respectively.

The equation for field A^μ can be obtained by considering:

$\partial_\nu[\frac{\partial Ł}{\partial(\partial_\nu A^\mu)}] = \partial_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu)$;
$\frac{\partial Ł}{\partial A_\mu} = -e\bar{\Psi}\gamma^\mu\Psi$,

in the Lagrange equation for A^{mu}, obtaining:

$$\partial_\nu F^{\nu\mu} = e\bar{\Psi}\gamma^\mu\Psi. \quad (14)$$

Finally, imposing the Lorentz gauge condition, results:

$$\partial^\mu\partial_{mu} = e\bar{\Psi}\gamma^\mu\Psi. \quad (15)$$

Considering the points explained, a cost functional for Optimal Control in the context of Quantum Electrodynamics for the Bolza type problem can be [Butkovskiy and Samoilenko, 1990; D'Alessandro, 2003]:

$$I(B_\mu(\mathbf{x}, t) = h(\Psi(\mathbf{x}_F, t_F) + \int_A^B d^4x g(\Psi(\mathbf{x}, t), B_\mu(\mathbf{x}, t), (\mathbf{x}, t)), \quad (16)$$

where h depends on field Ψ (electron-positron field) final state, as well on the final state itself. Function g has dependence on the field (Ψ), on the state (space-time coordinates) and on the external field B_μ due to an external source.

Consequently, expression (16) can be used as a general expression of a function to be minimized when a control action must be designed over a quantum electro-dynamical system. As the control is represented by the external field B_μ, the current distribution that generates it can be correctly built.

6 Conclusions

Some ideas about how to formulate an Optimal Quantum Control Theory starting with ideas about optimal control for classical systems is observed herein. By using Lorentz gauge condition for covariant variables, a function is proposed representing the quantum Bolza type problem in the context of quantum electrodynamics.

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