DIFFERENTIAL COALITIONAL GAME IN CONDITIONS OF UNCERTAINTY FOR NETWORKS

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Abstract
In the paper, a dynamic model of the game of two coalitions in conditions of uncertainty is constructed. As a solution of the game the "coalitional guaranteed equilibrium" is suggested. The performing of the equalities of collection rationality gives maximality of solution by Pareto (under suggestion of well-disposeness of players inside each coalition). The two-coalitional game in uncertainty conditions is transformed into the special three-person game (without uncertainty). It is proved that Pareto-Slater equilibrium of the new game is the coalitional guaranteed equilibrium in the initial game. For linear-quadratic game the sufficient conditions of optimality are obtained.

Keywords
Differential game, coalition, uncertainty.

1 Introduction

A variety of interacting manufacturing, computer and other systems forms a network structure. Each system can be regarded as an active component (agent). To achieve their goals some agents can create a new structure, which is called a coalition. Formation of coalitions is a way to configure virtual organizations of agents with coordinated strategies depending on dynamically changing conditions [Lewis, Zhang, Hengster-Movric and Das, 2014; Li, Duan, 2014].

The game approach [Engwerda, 2005; Gu, 2008] is useful for studying the dynamic interaction between coalitions. It is concerned with the possibility of adequate description by the game theory facilities the complex controlled systems and making in them optimum decisions. Whereas in an optimization model one is interested in location the best decision, that minimizes or maximizes a given objective function, in a game problem the objective is a function of arguments that can be chosen independently by multiple decision makers, possibly with conflicting individual objectives.

Coalitional game theory is a branch of game theory dealing with cooperative behavior. In a coalitional game, the key idea is to study the formation of cooperative groups, i.e., coalitions among a number of players. By cooperating, the players can strengthen their position in this particular game. In this context, coalitional game theory proves to be a powerful tool for modelling cooperative behavior in many networking applications [Saad, Han, Debbah, Hjorungnes and Basar, 2009; Niyato, Wang, Saad and Hjorungnes, 2010; Han, Niyato, Saad, Basar and Hjorungnes, 2011; Li, Xu, Wang and Guizani, 2011].

In this article, a differential game of two coalitions (each consisting of two players) is considered. Relations between the players inside a coalition are considered to be friendly and are built on the basis of the maximum by Pareto. Therefore in appropriate mathematical models a construction of both individual and collective prizes is possible. A collective prize is created on the basis of the Pareto principle.

It is assumed that the interaction between the coalitions may be of different nature, even antagonistic. Therefore it is handy to use a guaranteed approach based on the concept of threats and counterthreats.

Decision making by the members of the coalitions occurs in conditions of uncertainty (for example, errors in measurements, inexact definition of parameters, revolting influence of external forces, interference in the information transfer process etc.). As a "special kind" of uncertainty, the "information uncertainty" concerning with full or partial absence of information about the following "move" of the coalition-opponent can be selected. So each coalition has to construct its decision based on some predetermined rule. This may be given, f.i., by using the Slater principle.

As a solution of the game of two coalitions in conditions of uncertainty considered in the article the coalitional guaranteed equilibrium is suggested. Its properties are investigated, for linear-quadratic game the sufficient conditions of optimality are obtained.

2 Game theoretic problem

Let us consider a 4-person differential game in uncertainty conditions

$$\left\langle I=\{1,2,3,4\}, \Sigma, \{\bigcup_i\}_{i\in I}, \mathrm{Z}, \{F_i(U,Z,t_0,x_0)\}_{i\in I}\right\rangle \quad (1)$$

where the evolution of the dynamical system, Σ, is described by the differential equation

$$\dot{x}=A(t)x+\sum_{i\in I}B_i(t)u_i+z,\ x(t_0)=x_0,\ \|x_0\|\neq 0 \quad (2)$$

with the initial state $(t_0,x_0)\in[0,\vartheta]\times R^n$, where $\vartheta>0$ is the terminal period, $x=x(t)\in R^n$ is the state of the system at time t, the matrices $A(t)$, $B_i(t)$ are continuous and bounded, $u_i\in R^n$ – is the control variable of the i-th player, set of strategies of the i-th player is

$$\bigcup_i=\{U_i\div u_i(t,x)\,|\,u_i(t,x)=\Theta_i(t)x\},$$

(the sign "÷" means that the control variable u_i corresponds to the strategy U_i), the matrices $\Theta_i(t)$ are continuous and bounded, $z\in R^n$ is an uncertainty, the set of uncertainties is defined by Z,

$$\mathrm{Z}=\{Z\div z(t,x)\,|\,z(t,x)=P(t)x\},$$

the matrix $P(t)$ is continuous and bounded (the functional form of matrices $\Theta_i(t)$ and $P(t)$ is defined in Theorem 2).

Kit $U=(U_1,U_2,U_3,U_4)\in\bigcup_1\times\bigcup_2\times\bigcup_3\times\bigcup_4=\bigcup$ is called the situation of the game (1).

The strategies of coalitions 1 and 2 have the form

$$\begin{aligned} U_{K_1}&=(U_1,U_2)\in\bigcup_1\times\bigcup_2=\bigcup_{K_1},\\ U_{K_2}&=(U_3,U_4)\in\bigcup_3\times\bigcup_4=\bigcup_{K_2}.\end{aligned} \quad (3)$$

Then the situation of the game takes the form

$$U=(U_{K_1},U_{K_2})\in\bigcup.$$

Let the prize function of i-th player be defined as

$$F_i(U,Z,t_0,x_0),\quad i\in I.$$

Having associated for joint choice of strategies, players of each coalition aspire to maximize the sum of their functions, i.e., summarize the values of the prize functions of coalition members. So we introduce the functions

$$\Phi_1(U_{K_1},U_{K_2},Z,t_0,x_0)$$
$$=F_1(U_{K_1},U_{K_2},Z,t_0,x_0)+F_2(U_{K_1},U_{K_2},Z,t_0,x_0),$$
$$\Phi_2(U_{K_1},U_{K_2},Z,t_0,x_0)$$
$$=F_3(U_{K_1},U_{K_2},Z,t_0,x_0)+F_4(U_{K_1},U_{K_2},Z,t_0,x_0).$$

Each coalition aspires to maximize its own function Φ_i at the expense of the choice of strategy $U_{K_i}\in\bigcup_{K_i}$ with counting a realization of any uncertainty $Z\in\mathrm{Z}$.

The party of the game is unfolded as follows. Each player chooses and uses his own strategy $U_i\div u_i(t,x)$ from the set $\bigcup_i$. Then the situation $U=(U_1,U_2,U_3,U_4)$ is formed. Regardless of such choice, some uncertainty $Z\div z(t,x)$ from Z affects to the system Σ. Then the solution $x(t)$, $t\in[0,\vartheta]$, of the system (2) under $u_i=u_i(t)$, $z=z(t,x)$ is built. With this solution, the functions $u_i[t]=u_i(t,x(t))$ of strategies U_i chosen by players and the function $z[t]=z(t,x(t))$ of uncertainty Z affecting Σ are formed. Then the prize functions of players are calculated. The game is over.

3 Definition of the solution and it's properties

First, let us give a definition of threats and counterthreats of coalitions.

Definition 1. Let $U=(U_{K_1},U_{K_2})\subset\bigcup$ be some situation of the coalitional game (1)-(3). As a threat of the coalition K_1 to $\bigcup$ we call the possibility of changing the strategy $U_{K_1}=(U_1,U_2)\in\bigcup_{K_1}$ to $\tilde{U}_{K_1}=(\tilde{U}_1,\tilde{U}_2)\in\bigcup_{K_1}$ so that for all $Z\in\mathrm{Z}$

$$F_i(\tilde{U}_{K_1},U_{K_2},Z,t_0,x_0)$$
$$\geq F_i(U_{K_1},U_{K_2},Z,t_0,x_0),\quad i\in K_1,$$

and at least one of the inequalities is strict.

As a counterthreat of coalition K_2 (as a response to the threat of the coalition K_1) we call the ability of the members of the coalition K_2 to change the strategy $U_{K_2}=(U_3,U_4)\in\bigcup_{K_2}$ into $\tilde{U}_{K_2}=(\tilde{U}_3,\tilde{U}_4)\in\bigcup_{K_2}$ so that for all $Z\in\mathrm{Z}$, firstly

$$F_j(\tilde{U}_{K_1},\tilde{U}_{K_2},Z,t_0,x_0)$$
$$\geq F_j(\tilde{U}_{K_1},U_{K_2},Z,t_0,x_0),\quad j\in K_2,$$

and at least one of the inequalities is strict, secondly

$$F_j(\tilde{U}_{K_1},\tilde{U}_{K_2},Z,t_0,x_0)$$
$$> F_j(U_{K_1},U_{K_2},Z,t_0,x_0),\quad j\in K_2,$$

thirdly

$$F_i(\tilde{U}_{K_1},\tilde{U}_{K_2},Z,t_0,x_0)$$
$$< F_i(U_{K_1},U_{K_2},Z,t_0,x_0),\quad i\in K_1.$$

The threat of the coalition K_2 and the counterthreat of the coalition K_1 are formed in a similar way.

Now we define the coalitional guaranteed equilibrium of the game (1)-(3).

Definition 2. The pair (U^*,P^*), where $U^*=(U^*_{K_1},U^*_{K_2})$, is called the coalitional guaranteed equilibrium of the game (1)-(3) if for all initial conditions $(t_0,x_0)\in[0,\vartheta]\times R^n$ there exists such uncertainty $Z^*\in\mathrm{Z}$ that the following conditions are performed:

1a) the strategy $U^*_{K_1}$ of the coalition K_1 is maximal by Pareto in the 2-criterial problem

$$\left\langle \bigcup_{K_1},\ \{F_i(U_{K_1},U^*_{K_2},Z^*,t_0,x_0\}_{i\in K_1} \right\rangle,$$

i.e., for all $U_{K_1}\in\bigcup_{K_1}$ and for all $(t_0,x_0)\in[0,\vartheta]\times R^n$ the system of inequalities

$$F_i(U_{K_1},U^*_{K_2},Z^*,t_0,x_0)$$
$$\geq F_i(U^*_{K_1},U^*_{K_2},Z^*,t_0,x_0),\quad i\in K_1,$$

is incompatible and at least one of them is strict;

1b) the strategy $U^*_{K_2}$ of the coalition K_2 is maximal by Pareto in the 2-criterial problem

$$\left\langle \bigcup_{K_2},\ \{F_j(U^*_{K_1},U_{K_2},Z^*,t_0,x_0\}_{j\in K_2} \right\rangle,$$

i.e., for all $U_{K_2}\in\bigcup_{K_2}$ and for all $(t_0,x_0)\in[0,\vartheta]\times R^n$ the system of inequalities

$$F_j(U^*_{K_2},U_{K_2},Z^*,t_0,x_0)$$
$$\geq F_j(U^*_{K_1},U^*_{K_2},Z^*,t_0,x_0),\quad j\in K_2,$$

is incompatible and at least one of them is strict;

2) in response to the threat of any coalition, the other coalition has a counterthreat in the sense of definition 1;

3) the uncertainty Z^* is the minimum by Slater in the 4-criterial problem

$$\left\langle \mathrm{Z}, \{F_i(U^*_{K_1},U^*_{K_2},Z,t_0,x_0\}_{i\in I} \right\rangle,$$

i.e., for all $Z\in\mathrm{Z}$ and for all $(t_0,x_0)\in[0,\vartheta]\times R^n$ the system of inequalities

$$F_i(U^*_{K_1},U^*_{K_2},Z,t_0,x_0)$$
$$\geq F_i(U^*_{K_1},U^*_{K_2},Z^*,t_0,x_0),\quad i\in I,$$

is incompatible.

We say that the kit $(U^*_{K_1},U^*_{K_2},Z^*)$ forms equilibrium situation in the game (1)-(3) or affords the solution of this one.

Note the properties of the incorporated solution.

Property 1. If for some $Z^*\in\mathrm{Z}$ the situation $(U^*_{K_1},U^*_{K_2})$ is as follows:

1) the equality

$$\Phi_1(U^*_1,U^*_2,U^*_{K_2},Z^*,t_0,x_0)$$
$$=\max_{\substack{U_1\in\bigcup_1\\ U_2\in\bigcup_2}}\Phi_1(U_1,U_2,U^*_{K_2},Z^*,t_0,x_0) \tag{4}$$

is true, then the strategy $U^*_{K_1}=(U^*_1,U^*_2)$ will be Pareto-optimal in the game (1)-(3);

2) the equality

$$\Phi_2(U^*_{K_1},U^*_3,U^*_4,Z^*,t_0,x_0)$$
$$=\max_{\substack{U_3\in\bigcup_3\\ U_4\in\bigcup_4}}\Phi_2(U^*_{K_1},U_3,U_4,Z^*,t_0,x_0) \tag{5}$$

is true, then the strategy $U^*_{K_2}=(U^*_3,U^*_4)$ will be Pareto-optimal in the game (1)-(3).

Property 2. Let the triple $(U^*_{K_1},U^*_{K_2},Z^*)$ satisfy the definition 2, and equalities (4), (5) be true. If in the game (1)-(3) there exist "maxmins"

$$\Phi^g_1[t_0,x_0]=\max_{U_{K_1}}\min_{U_{K_2},Z}\Phi_1(U_{K_1},U_{K_2},Z,t_0,x_0)$$
$$=\min_{U_{K_2},Z}\Phi_1(U^g_{K_1},U_{K_2},Z,t_0,x_0),$$
$$\Phi^g_2[t_0,x_0]=\max_{U_{K_2}}\min_{U_{K_1},Z}\Phi_2(U_{K_1},U_{K_2},Z,t_0,x_0)$$
$$=\min_{U_{K_1},Z}\Phi_2(U_{K_1},U^g_{K_2},Z,t_0,x_0),$$

then the sum of prize functions of players inside either coalition in situation $(U^*_{K_1},U^*_{K_2})$ is no less then "maxmin", that is

$$\Phi_i(U^*_{K_1},U^*_{K_2},Z^*,t_0,x_0)\geq\Phi^g_i[t_0,x_0],\quad i=1,2.$$

Property 3. If for all $i=1,2,3,4$ in the game (1)-(3) there exist maxmins

$$\max_{U_i}\min_{U_{I\setminus i},Z}\Phi_j(U_i,U_{I\setminus i},Z,t_0,x_0)$$
$$=\min_{U_{I\setminus i},Z}\Phi_j(U^g_i,U_{I\setminus i},Z,t_0,x_0),\quad j=1,2,$$

the situation $(U^*_{K_1},U^*_{K_2})$ and uncertainty $Z^*\in\mathrm{Z}$, satisfying (4) and (5), then

$$\sum_{i\in K_j}F_i(U^*_{K_1},U^*_{K_2},Z^*,t_0,x_0)$$
$$\geq\sum_{i\in K_j}\max_{U_i}\min_{U_{I\setminus i},Z}F_i(U_{K_1},U_{K_2},Z,t_0,x_0),\quad j=1,2.$$

4 Transformation of the two-coalitional game into the 3-person one

Let us consider an auxiliary non-coalition positional 3-person game (the uncertainty performs the role of the 3-rd player)

$$\Gamma=\left\langle I=\{1,2,3\},\Sigma,\{\bigcup_{K_1},\bigcup_{K_2},\mathrm{Z}\},\ \{\Phi_i(U_{K_1},U_{K_2},Z,t_0,x_0\}_{i=1,2,3}\right\rangle,$$

where the functions Φ_1,Φ_2 were defined earlier,

$$\Phi_3(U_{K_1},U_{K_2},Z,t_0,x_0)$$
$$=-\sum_{i\in I}\gamma_iF_i(U_{K_1},U_{K_2},Z,t_0,x_0),\ \sum_{i\in I}\gamma_i=1,\ \gamma_i\in(0,1).$$

The strategy of the 1-st player is $U_{K_1}=(U_1,U_2)$, 2-nd – $U_{K_2}=(U_3,U_4)$, 3-rd – Z.

Remark 1. In order not to complicate symbols, during investigation of the game Γ we shall keep to the ones accepted earlier, i.e., U_{K_1}, U_{K_2} and etc, while bearing in mind that now 3-person game rather then coalitional one is considered.

Definition 3. Situation $(U^*_{K_1},U^*_{K_2},Z^*)\in\bigcup_{K_1}\times\bigcup_{K_2}\times\mathrm{Z}$ we call maximal by Pareto-Slater in the game Γ if for all initial positions $(t_0,x_0)\in[0,\vartheta]\times R^n$ and strategies $U_{K_1}\in\bigcup_{K_1}$, $U_{K_2}\in\bigcup_{K_2}$, $Z\in\mathrm{Z}$ the system of inequalities

$$\Phi_i(U_{K_1},U_{K_2},Z^*,t_0,x_0)$$
$$\ge \Phi_i(U^*_{K_1},U^*_{K_2},Z^*,t_0,x_0), \quad i=1,2,$$

is incompatible and at least one of them is strict, and the inequality

$$\Phi_3(U^*_{K_1},U^*_{K_2},Z,t_0,x_0) \le \Phi_3(U^*_{K_1},U^*_{K_2},Z,t_0,x_0)$$

is held.

Theorem 1. Let the following assumptions be made:

1) $[(D_{11}>0)\vee(D_{12}>0)]\wedge[(D_{21}>0)\vee(D_{22}>0)]$,
2) $[(D_{31}<0)\vee(D_{32}<0)]\wedge[(D_{41}<0)\vee(D_{42}<0)]$,
3) $[(D_{13}<0)\vee(D_{14}<0)]\wedge[(D_{23}<0)\vee(D_{24}<0)]$,
4) $[(D_{33}>0)\vee(D_{34}>0)]\wedge[(D_{43}>0)\vee(D_{44}>0)]$.

Equilibrium situation by Pareto-Slater in the game Γ with $\|x_0\|\ne 0$ generates the coalitional guaranteed equilibrium

$$(U^*_{K_1},U^*_{K_2},Z^*,\{F_i(U^*_{K_1},U^*_{K_2},Z^*,t_0,x_0)\}), \quad i\in I,$$

in the game (1)-(3).

5 Sufficient optimality conditions for the linear-quadratic game

Let the prize function of i-th player be defined as

$$F_i(U,Z,t_0,x_0)=x^T C_i x$$
$$+\int_{t_0}^{\vartheta}\left[\sum_{k\in I}u_k^T[t]D_{ik}u_k[t]+z^T[t]L_i z[t]\right]dt,$$

where all matrices C_i, D_{ij}, L_i are constant and symmetric, $u_k[t]=\Theta_k(t)x(t)=u_k(t,x(t))$, $k\in I$, and $z[t]=P(t)x(t)=z(t,x(t))$ are the realizations of strategy and uncertainty, correspondingly.

Let us introduce the following matrices:

$$D_1^1=D_{11}+D_{21},\quad D_2^1=D_{12}+D_{22},$$
$$D_3^1=D_{13}+D_{23},\quad D_4^1=D_{14}+D_{24},$$
$$D_1^2=D_{31}+D_{41},\quad D_2^2=D_{32}+D_{42},$$
$$D_3^2=D_{33}+D_{43},\quad D_4^2=D_{34}+D_{44},$$
$$D_1(\gamma)=-(\gamma_1 D_{11}+\gamma_2 D_{21}+\gamma_3 D_{31}+\gamma_4 D_{41}),$$
$$D_2(\gamma)=-(\gamma_1 D_{12}+\gamma_2 D_{22}+\gamma_3 D_{32}+\gamma_4 D_{42}),$$
$$D_3(\gamma)=-(\gamma_1 D_{13}+\gamma_2 D_{23}+\gamma_3 D_{33}+\gamma_4 D_{43}),$$
$$D_4(\gamma)=-(\gamma_1 D_{14}+\gamma_2 D_{24}+\gamma_3 D_{34}+\gamma_4 D_{44}).$$
$$L^1=L_1+L_2,\quad L^2=L_3+L_4,\quad L(\gamma)=-\sum_{i\in I}\gamma_i L_i,$$
$$C^1=C_1+C_2,\quad C^2=C_3+C_4,\quad C(\gamma)=-\sum_{i\in I}\gamma_i C_i$$

and functions

$$W_1=(t,x,u_1,u_2,u_3,u_4,z,V_1)$$
$$=\frac{\partial V_1}{\partial t}+\left(\frac{\partial V_1}{\partial x}\right)^T\left(A(t)x+\sum_{i\in I}B_i(t)u_i+z\right)$$
$$+\sum_{i\in I}u_i^T D_i^1 u_i+z^T L^1 z,$$
$$W_2=(t,x,u_1,u_2,u_3,u_4,z,V_2)$$
$$=\frac{\partial V_2}{\partial t}+\left(\frac{\partial V_2}{\partial x}\right)^T\left(A(t)x+\sum_{i\in I}B_i(t)u_i+z\right),$$
$$W_3=(t,x,u_1,u_2,u_3,u_4,z,V_3)$$
$$=\frac{\partial V_3}{\partial t}+\left(\frac{\partial V_3}{\partial x}\right)^T\left(A(t)x+\sum_{i\in I}B_i(t)u_i+z\right)$$
$$+\sum_{i\in I}u_i^T D_i(\gamma)u_i+z^T L(\gamma)z,$$

where $V_i=V_i(t,x)$ are Lyapunov-Bellman functions.

Theorem 2. Let there exist functions $u_i^*(t,x), i\in I$, $z^*(t,x)$, continuously differentiable functions $V_i(t,x)$, $j=1,2,3$, and numbers $\gamma_i\in(0,1)$, $i\in I$, so that:

1) $D_1^1<0,\ D_2^1<0,\ D_3^2<0,\ D_4^2<0$;
2) $L(\gamma)<0$;
3) for all $(t,x)\in[0,\vartheta]\times R^n$ the equalities

$$W_1(t,x,u_1^*(t,x),u_2^*(t,x),u_3^*(t,x),$$
$$u_4^*(t,x),z^*(t,x),V_1(t,x))$$
$$=\max_{u_1,u_2}W_1(t,x,u_1(t,x),u_2(t,x),u_3^*(t,x),$$
$$u_4^*(t,x),z^*(t,x),V_1(t,x))=0,$$
$$W_2(t,x,u_1^*(t,x),u_2^*(t,x),u_3^*(t,x),$$
$$u_4^*(t,x),z^*(t,x),V_2(t,x))$$
$$=\max_{u_3,u_4}W_2(t,x,u_1^*(t,x),u_2^*(t,x),u_3(t,x),$$
$$u_4(t,x),z^*(t,x),V_2(t,x))=0,$$
$$W_3(t,x,u_1^*(t,x),u_2^*(t,x),u_3^*(t,x),$$
$$u_4^*(t,x),z^*(t,x),V_3(t,x))$$
$$=\max_{z}W_3(t,x,u_1^*(t,x),u_2^*(t,x),u_3^*(t,x),$$
$$u_4^*(t,x),z(t,x),V_3(t,x))=0$$

are fulfilled;

4) for all $x\in R^n$ the equalities

$$V_i(\vartheta,x)=x^T C_i x,\ i=1,2,\ V_3(\vartheta,x)=x^T C(\gamma)x$$

are fulfilled;

5) the system of matrix equations of Riccathy type

$$\dot Q_1(t)+A^T(t)Q_1(t)+Q_1(t)A(t)$$
$$-Q_1(t)\left(\sum_{i\in K_1}B_i(t)(D_i^1)^{-1}B_i^T(t)Q_1(t)\right.$$
$$+\sum_{i\in K_2}B_i(t)(D_i^2)^{-1}B_i^T(t)Q_2(t)$$
$$+Q_2(t)\left[\sum_{i\in K_2}B_i(t)(D_i^2)^{-1}B_i^T(t)Q_2(t)\right]^T$$
$$\left.+L^{-1}(\gamma)Q_3(t)+Q_3(t)\left[L^{-1}(\gamma)\right]^T\right)$$
$$+Q_2(t)\sum_{i\in K_2}B_i(t)(D_i^2)^{-1}D_i^1(D_i^2)^{-1}B_i^T(t)Q_2(t)$$

$$+Q_3(t)L^{-1}(\gamma)L^1L^{-1}(\gamma)Q_3(t)=0\,,\ Q_1(\vartheta)=C^1\,,$$

$$\dot{Q}_2(t)+A^T(t)Q_2(t)+Q_2(t)A(t)$$

$$-Q_2(t)\Bigg(\sum_{i\in K_1}B_i(t)(D_i^1)^{-1}B_i^T(t)Q_1(t)$$

$$+Q_1(t)\left[\sum_{i\in K_1}B_i(t)(D_i^1)^{-1}B_i^T(t)\right]^T$$

$$+\sum_{i\in K_2}B_i(t)(D_i^2)^{-1}B_i^T(t)Q_2(t)$$

$$+L^{-1}(\gamma)Q_3(t)+Q_3(t)\left[L^{-1}(\gamma)\right]^T\Bigg)$$

$$+Q_1(t)\sum_{i\in K_1}B_i(t)(D_i^1)^{-1}D_i^2(D_i^1)^{-1}B_i^T(t)Q_1(t)$$

$$+Q_3(t)L^{-1}(\gamma)L^2L^{-1}(\gamma)Q_3(t)=0\,,\ Q_2(\vartheta)=C^2\,,$$

$$\dot{Q}_3(t)+A^T(t)Q_3(t)+Q_3(t)A(t)$$

$$-Q_3(t)\Bigg(\sum_{i\in K_1}B_i(t)(D_i^1)^{-1}B_i^T(t)Q_1(t)$$

$$+Q_1(t)\left[\sum_{i\in K_1}B_i(t)(D_i^1)^{-1}B_i^T(t)\right]^T$$

$$+\sum_{i\in K_2}B_i(t)(D_i^2)^{-1}B_i^T(t)Q_2(t)$$

$$+Q_2(t)\left[\sum_{i\in K_2}B_i(t)(D_i^2)^{-1}B_i^T(t)\right]^T+L^{-1}(\gamma)Q_3(t)\Bigg)$$

$$+Q_1(t)\sum_{i\in K_1}B_i(t)(D_i^1)^{-1}D_i(\gamma)(D_i^1)^{-1}B_i^T(t)Q_1(t)$$

$$+Q_2(t)\sum_{i\in K_2}B_i(t)(D_i^2)^{-1}D_i(\gamma)(D_i^2)^{-1}B_i^T(t)Q_2(t)=0\,,$$

$$Q_3(\vartheta)=C(\gamma)\,,$$

has solution $Q_i(t)$, $i=1,2,3$, which can be extended onto $[0,\vartheta]$.

Then $(U_1^*,U_2^*,U_3^*,U_4^*,Z^*,F^*)$ is the coalitional guaranteed equilibrium in the game (1)-(3),

$$U_i^*\div u_i^*(t,x)=\Theta_i(t)x$$

$$=(D_i^1)^{-1}B_i^T(t)Q_i(t)x,\ i\in K_1\,,$$

$$U_j^*\div u_j^*(t,x)=\Theta_j(t)x$$

$$=(D_j^2)^{-1}B_j^T(t)Q_j(t)x,\ j\in K_2\,,$$

$$Z^*\div z^*(t,x)=P(t)x=-L^{-1}(\gamma)Q_3(t)x\,,$$

and the summary prizes of the players inside each coalition are

$$\Phi_i(U_1^*,U_2^*,U_3^*,U_4^*,Z^*)=V_i(t_0,x_0),\ i=1,2,3\,,$$

where functions Φ_i are defined earlier.

Remark 2. For the individual prizes of players (guarantees) to be found it is necessary to make appropriate functions W_i, $i=1,2,3,4$, to substitute in them the solutions indicated in the theorem 2, to equate them to zero and to integrate.

6 Conclusion

This paper has provided an approach to solving the network problems by the applicability of coalitional game theory. The principle of threats and couterthreats simulates potential conflict of interests of agents modeled as coalitions of players. In a network wherein informational exchange and conflicts are possible, either through a central controller or among agents themselves, the concept of coordinated equilibrium arises. We have suggested the coalitional guaranteed equilibrium. The results confirm the apparent utility of this equilibrium for solving the network problems: consideration the coalitions, rather than individual players, allows simulating groups of interacting agents in the network. The results show also that coalitional guaranteed equilibrium is preferably when compared to non-cooperative schemes. Sufficient conditions of optimality obtained in the paper allow reaching a consensus between coalitions.

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