CYBERNETICS AND PHYSICS, VOL. 6, NO. 4, 2017 , 222–230

# ALGORITHMS FOR THE INVESTIGATION OF NONLINEAR SYSTEMS WITH FIRST INTEGRALS

**Valentin Irtegov**
Matrosov Institute for System Dynamics
and Control Theory SB RAS
Russia
irteg@icc.ru

**Larisa Burlakova**
Matrosov Institute for System Dynamics
and Control Theory SB RAS
Russia
burlakova.larisa@gmail.com

**Abstract**

The paper presents the results of qualitative analysis [Poincare A., 1947] of conservative systems. The modified Routh-Lyapunov technique is used as a tool for their study. Special attention is paid to the algorithms of finding and the analysis of invariant manifolds on which the elements of algebra of problem's first integrals assume a stationary value.

**Key words**

Dynamic systems, invariant manifolds, first integrals.

## 1 Introduction

We present an approach to the analysis of dynamic systems having smooth first integrals. Our approach is based on the Routh-Lyapunov method for the analysis of such systems and computer algebra methods.

Using the Gröbner bases technique, we find stationary sets of differential equations of the systems, i.e. the sets of any finite dimension, for which necessary conditions of an extremum of problem's first integrals are satisfied. Zero dimensional sets are known as stationary solutions, while positive dimensional sets are called invariant manifolds (IMs). Further, we investigate properties of these sets (stability in the sense of Lyapunov, bifurcations and etc.).

When bifurcations of the stationary sets are analyzed, the Gröbner bases method is also applied for finding the sets passing through bifurcation points. To investigate the stability of the stationary sets, we use the *Mathematica* package STABILITY [Banshchikov, Burlakova, Irtegov, and Titorenko, 2011] developed by the authors together with their colleagues. The algorithms of the package are based on the Lyapunov stability theorems for linear approximation and the 2nd Lyapunov method.

In this paper, our approach is demonstrated by the study of two problems: dynamic systems described by Euler's equations with first integrals, and the problem of motion of a rigid body in double force field. Similar problems arise, for example, in space dynamics [Sarychev and Gutnik , 2015], quantum mechanics [Adler, Marikhin, and Shabat, 2012], [Smirnov, 2008].

## 2 The Family of Euler's Equations

The differential equations of the problem can be written as [Borisov, Mamaev, and Sokolov, 2001]:

$$\left.\begin{array}{l}
\dot{s}_1=\alpha(r_1 s_2-\alpha r_2 r_3)-(\beta r_3-s_2)(\beta r_2+s_3),\\
\dot{s}_2=\beta\left(\beta r_1 r_3-r_2 s_1\right)+(\alpha r_3-s_1)\left(\alpha r_1+s_3\right),\\
\dot{s}_3=(\beta r_1-\alpha r_2)s_3,\\
\dot{r}_1=r_2(\alpha r_1+\beta r_2+2s_3)-r_3 s_2\\
\quad -x\left((\alpha^2+\beta^2)r_3 s_2+\beta s_3^2\right),\\
\dot{r}_2=r_3 s_1-r_1(\alpha r_1+\beta r_2+2s_3)\\
\quad +x\left((\alpha^2+\beta^2)r_3 s_1+\alpha s_3^2\right),\\
\dot{r}_3=r_1 s_2-r_2 s_1+x(\beta s_1-\alpha s_2)s_3.
\end{array}\right\} \quad (1)$$

Here $r_i, s_i$ are the phase variables, $\alpha, \beta$, are some constants, $x$ is the parameter of the family.

Equations (1) can be interpreted as the Kirchhoff equations for the motion of a rigid body in ideal fluid for $x = 0$, as the Poincaré-Zhukowskii equations for a rigid body with an ellipsoidal cavity filled with a liquid for $x = 1$, and as the Euler equations on the Lie algebras $so(4)$ and $so(3,1)$ for $x > 0$ and $x < 0$, respectively.

Equations (1) have the following first integrals:

$$\left.\begin{array}{l}
2H=s_1^2+s_2^2+2(\alpha r_1+\beta r_2)s_3+2s_3^2\\
-(\alpha^2+\beta^2)r_3^2=2h,\\
V_1=r_1 s_1+r_2 s_2+r_3 s_3=c_1,\\
V_2=r_1^2+r_2^2+r_3^2+x(s_1^2+s_2^2+s_3^2)=c_2,\\
V_3=x(\beta s_1-\alpha s_2)^2 s_3^2+(r_1 s_1+r_2 s_2)\big((\alpha^2\\
+\beta^2)(r_1 s_1+r_2 s_2)+2(\alpha s_1+\beta s_2)s_3\big)\\
+s_3^2\big(s_1^2+s_2^2+(\alpha r_1+\beta r_2+s_3)^2\big)=c_3.
\end{array}\right\} \quad (2)$$

The problem is to find the stationary sets (both zero and non-zero dimension) for equations (1) and to investigate their stability.

### 2.1 Finding Stationary Sets

The method of obtaining the stationary sets for equations (1), which is used in this work, reduces this problem to solving a system of polynomial algebraic equations with parameters. In order to find the desired solutions, we construct a linear combination of first integrals (2)

$$2K = 2\lambda_0 H - 2\lambda_1 V_1 - \lambda_2 V_2 - \lambda_3 V_3 \quad (\lambda_i = \mathrm{const}) \tag{3}$$

and write down the conditions of stationarity for the integral $K$ with respect to the phase variables $r_i, s_i$:

$$\left.\begin{aligned}
&\partial K/\partial s_1 = s_1\lambda_0 - r_1\lambda_1 - x s_1\lambda_2 - \\
&\big((\alpha^2+\beta^2)\, r_1\,(r_1 s_1 + r_2 s_2) \\
&+ (\alpha r_2 s_2 + r_1\,(2\alpha s_1 + \beta s_2))\, s_3 + \\
&\big((1+x\beta^2)\, s_1 - x\alpha\beta s_2\big)\, s_3^2\big)\,\lambda_3 = 0, \\
&\partial K/\partial s_2 = s_2\lambda_0 - r_2\lambda_1 - x s_2\lambda_2 - \\
&\big((\alpha^2+\beta^2)\, r_2\,(r_1 s_1 + r_2 s_2) \\
&+ (\beta r_1 s_1 + r_2\,(\alpha s_1 + 2\beta s_2))\, s_3 + \\
&(s_2 + x\alpha\,(-\beta s_1 + \alpha s_2))\, s_3^2\big)\,\lambda_3 = 0, \\
&\partial K/\partial s_3 = (\alpha r_1 + \beta r_2 + 2 s_3)\,\lambda_0 - r_3\lambda_1 - \\
&x s_3\lambda_2 - ((\alpha s_1 + \beta s_2)\,(r_1 s_1 + r_2 s_2) + \\
&\big((\alpha r_1 + \beta r_2)^2 + (1+x\beta^2)\, s_1^2 - 2x\alpha\beta s_1 s_2 \\
&+ (1+x\alpha^2)\, s_2^2\big)\, s_3 + \\
&3\,(\alpha r_1 + \beta r_2)\, s_3^2 + 2 s_3^3\big)\,\lambda_3 = 0, \\
&\partial K/\partial r_1 = \alpha s_3\lambda_0 - s_1\lambda_1 - r_1\lambda_2 - \\
&\big((\alpha^2+\beta^2)\, s_1\,(r_1 s_1 + r_2 s_2) + s_1\,(\alpha s_1 \\
&+\beta s_2)\, s_3 + \alpha\,(\alpha r_1 + \beta r_2)\, s_3^2 + \alpha s_3^3\big)\,\lambda_3 = 0, \\
&\partial K/\partial r_2 = \beta s_3\lambda_0 - s_2\lambda_1 - r_2\lambda_2 - \\
&\big((\alpha^2+\beta^2)\, s_2\,(r_1 s_1 + r_2 s_2) + s_2\,(\alpha s_1 \\
&+\beta s_2) s_3 + \beta\,(\alpha r_1 + \beta r_2)\, s_3^2 + \beta s_3^3\big)\,\lambda_3 = 0, \\
&\partial K/\partial r_3 = (\alpha^2+\beta^2)\, r_3\lambda_0 + s_3\lambda_1 + r_3\lambda_2 = 0
\end{aligned}\right\} \tag{4}$$

These equations allow one to determine both the stationary solutions and the IMs for equations (1).

### 2.2 Solving Stationary Equations with Respect to Phase Variables

For equations (4), we find both general solutions (existing without any restrictions on the parameters of the problem) and particular solutions (existing under some conditions on the parameters). For this purpose, we construct a Gröbner basis for system (4) with respect to the phase variables. After a factorization the basis has the form:

$$\left.\begin{aligned}
&\big(a_1 s_1 + a_2 s_2 + a_3 s_3\big)\big(a_4 + a_5 s_1^2 + a_6 s_2^2 \\
&+ a_7 s_1 s_3 + a_8 s_2 s_3 + a_9 s_3^2\big) = 0, \\
&s_3\big(a_{10} s_1 + a_{11} s_3\big)\big(a_{12} + a_{13} s_1^2 + a_{14} s_2^2 \\
&+ a_{15} s_1 s_3 + a_{16} s_2 s_3 + a_{17} s_3^2\big) = 0, \\
&s_3\, f_1(s_1, s_2, s_3) = 0, \qquad f_2(s_1, s_2, s_3) = 0, \\
&f_3(s_1, s_2, s_3) = 0, \qquad s_3\, f_4(s_1, s_2, s_3) = 0, \\
&f_5(s_1, s_2, s_3) = 0, \qquad s_3\, f_6(s_1, s_2, s_3) = 0, \\
&a_{18} r_2 + f_7(s_1, s_2, s_3) = 0, \\
&a_{19} r_1 + f_8(s_1, s_2, s_3) = 0, \\
&a_{20} r_3 + a_{21} s_3 = 0.
\end{aligned}\right\} \tag{5}$$

Here $f_i$ are the polynomials of the 4th-6th degrees, $a_j$ are the polynomials of $\lambda_0, \lambda_1, \lambda_2, \lambda_3, x, \alpha, \beta$.

System (5) is decomposed into several subsystems. We have computed a lexicographic Gröbner basis for each of the subsystems. Below, some of these bases are represented.

$$\left.\begin{aligned}
&(\alpha^2+\beta^2)^2\lambda_3\chi\, r_2^4 - 2\alpha^2\chi\,\kappa\, r_2^2 \\
&+\alpha^4\kappa\big(\lambda_1^2 + \lambda_2\kappa\big) = 0, \\
&\alpha r_1 + \beta r_2 = 0,\ r_3 = 0, s_3 = 0, \\
&\alpha^3\lambda_1\kappa\, s_1 - \alpha^2\beta\lambda_2\kappa\, r_2 + \beta\chi\, r_2^3 = 0, \\
&\alpha^2\lambda_1\,\kappa s_2 + \alpha^2\lambda_2\kappa\, r_2 - \chi\, r_2^3 = 0, \\
&\text{where } \kappa = \lambda_0 - x\lambda_2,\ \chi = (\alpha^2+\beta^2)^2\lambda_2\lambda_3.
\end{aligned}\right\} \tag{6}$$

$$\left.\begin{aligned}
&b_{30} s_3^4 s_1^4 + (b_{31} s_3^2 + b_{35}) s_3^3 s_1^3 + (b_{32} s_3^4 + b_{22} s_3^2 \\
&+ b_{10}) s_3^2 s_1^2 + (b_{33} s_3^6 + b_{21} s_3^4 + b_9 s_3^2 + b_{16}) s_3 s_1 \\
&+ b_{34} s_3^8 + b_{20} s_3^6 + b_5 s_3^4 + b_7 s_3^2 + b_{14} = 0, \\
&(b_{28} s_3^2 + b_4) s_3 s_2 + b_{37} s_3^3 s_1^3 + (b_{38} s_3^2 + b_{41}) s_3^2 s_1^2 \\
&+ (b_{39} s_3^4 + b_{19} s_3^2 + b_{12}) s_3 s_1 + b_{40} s_3^6 \\
&+ b_{23} s_3^4 + b_1 s_3^2 + b_{17} = 0, \\
&b_{45} s_3 r_2 + b_{42} s_3^2 s_1^2 + s_3(b_{43} s_3^2 + b_{46}) s_1 \\
&+ b_{44} s_3^4 + b_{29} s_3^2 + b_{13} = 0, \\
&s_3(b_{27} s_3^2 + b_3) r_1 + b_{36} s_3^3 s_1^3 \\
&+ (b_{25} s_3^2 + b_{18}) s_3^2 s_1^2 + (b_{24} s_3^4 + b_2 s_3^2 \\
&+ b_{11}) s_3 s_1 + b_{26} s_3^6 + b_6 s_3^4 + b_8 s_3^2 + b_{15} = 0, \\
&b_{47} r_3 + b_{48} s_3 = 0,
\end{aligned}\right\} \tag{7}$$

where $b_i$ are polynomials of $\lambda_j, x, \alpha, \beta$.

We can obtain the information on dimension of the solutions and find the solutions directly from the above bases.

System (6) has the finite number of solutions: 4 general solutions. Below, some of them are represented.

$$\left.\begin{aligned}
&r_1 = \pm\tfrac{\beta\sqrt{(\rho+\lambda_1\sqrt{d})/\lambda_3}}{\alpha^2+\beta^2}, \\
&r_2 = \mp\tfrac{\alpha\sqrt{(\rho+\lambda_1\sqrt{d})/\lambda_3}}{\alpha^2+\beta^2},\ r_3 = 0, \\
&s_1 = \mp\tfrac{\beta\sqrt{(\rho+\lambda_1\sqrt{d})/(d\,\lambda_3)}}{(\alpha^2+\beta^2)}, \\
&s_2 = \pm\tfrac{\alpha\sqrt{(\rho+\lambda_1\sqrt{d})/(d\,\lambda_3)}}{(\alpha^2+\beta^2)},\ s_3 = 0; \\
&\text{where } \rho = \lambda_0 - x\lambda_2,\ d = -\rho/\lambda_2.
\end{aligned}\right\} \tag{8}$$

These are the families of stationary solutions of equations (1), which are parameterized by $\lambda_i$.

System (7) has infinitely many solutions. For finding the general solutions of equations (7), it is necessary to solve an equation of 4th degree. In this case, the solutions will be bulky. Here, we restrict ourselves the particular solutions of this system, which have been obtained for $\lambda_1 = 0$. Below, some of these solutions are represented.

$$\left.\begin{array}{l} r_1 = (\beta\sigma + \alpha\varrho)\varrho/((\alpha^2+\beta^2)\lambda_3 s_3), \\ r_2 = -(\alpha\sigma - \beta\varrho)\varrho/((\alpha^2+\beta^2)\lambda_3 s_3), \\ r_3 = 0, \\ s_1 = \pm(\alpha\sigma - \beta\varrho)\sqrt{\lambda_2}/((\alpha^2+\beta^2)\lambda_3 s_3), \\ s_2 = \pm(\beta\sigma + \alpha\varrho)\sqrt{\lambda_2}/((\alpha^2+\beta^2)\lambda_3 s_3), \end{array}\right\} \quad (9)$$

where $\varrho = \sqrt{\lambda_0 - \lambda_2 x - \lambda_3 s_3^2}$, $\sigma = \sqrt{(\alpha^2+\beta^2)\,x\lambda_3 s_3^2 - \varrho^2}$. Solutions (9) are the families of one-dimensional IMs of equations (1).

Having eliminated the problem's parameters and the parameters $\lambda_i$ from the expressions of stationary solutions (8), we obtain four equations

$$r_1 + s_1 = 0,\; r_2 + s_2 = 0,\; r_3 = 0,\; s_3 = 0,$$

which determine the IM of system (1). The vector field on this IM can be written as:

$$\dot r_1 = r_2(\alpha r_1 + \; r_2), \dot r_2 = -r_1(\alpha r_1 + \beta r_2).$$

The latter system obviously has the first integral:

$$W = r_1^2 + r_2^2.$$

Since the 2nd variation of the above integral is sign definite, then the equilibrium position $r_1 = r_2 = 0$ of the system is stable.

### 2.3 Solving Stationary Equations with Respect to Some Part of Phase Variables and Parameters

Let us consider another technique for finding the solutions of equations (4). Using this technique, it is possible to obtain IMs together with the first integrals of differential equations on these IMs [Irtegov and Titorenko, 2009]. The latter allows us to set the problem for finding and the analysis of the stationary sets of these differential equations. Following this technique, we have computed a lexicographical Gröbner basis for equations (4) with respect to the variables $r_3, s_2, \lambda_1, \lambda_2, \lambda_0$. As a result, we have obtained a system which is decomposed into two subsystems:

$$\left.\begin{array}{l} \lambda_0 - \lambda_3\,(\alpha r_1 + \beta r_2 + s_3)\,s_3 = 0,\; \lambda_2 = 0, \\ \alpha\lambda_1 + (\alpha^2+\beta^2)\lambda_3\,(\alpha r_1 s_1 + \beta r_2 s_1 \\ +s_1 s_3) = 0, \\ \alpha s_2 - \beta s_1 = 0,\; \alpha r_3 - s_1 = 0. \end{array}\right\} \quad (10)$$

$$\left.\begin{array}{l} b_{12}\lambda_0^2 + b_2\lambda_0 + b_1 = 0, \\ b_{15}\lambda_2 + b_7\lambda_0 + b_4 = 0, \\ b_{11}\lambda_1 + b_6\lambda_0 + b_8 = 0, \\ b_{10}s_2 + b_{14}\lambda_0 + b_5 = 0, \\ b_9 r_3 + b_{13}\lambda_0 + b_3 = 0, \end{array}\right\} \quad (11)$$

where $b_i$ are polynomials of $s_1,\, s_3,\, r_1,\, r_2,\, \lambda_3, x, \alpha, \beta$.

It is easy to see that system (10) has one solution, and system (11) has two solutions.

The latter two expressions of (10) determine the IM of equations (1).

The differential equations of vector field on this IM are given by:

$$\left.\begin{array}{l} \dot s_1 = (\beta r_1 - \alpha r_2)s_1,\; \dot s_3 = (\beta r_1 - \alpha r_2)s_3, \\ \dot r_1 = \alpha r_1 r_2 + 2r_2 s_3 - \beta s_1^2(\beta^2 x + 1)/\alpha^2 \\ +\beta(r_2^2 - (s_1^2 + s_3^2)x), \\ \dot r_2 = -\alpha r_1^2 - r_1(\beta r_2 + 2s_3) + \alpha(s_1^2 + s_3^2)x \\ +s_1^2(\beta^2 x + 1)/\alpha. \end{array}\right\} \quad (12)$$

The first three expressions of (10) are the first integrals of equations (12).

The general solutions of system (11) are bulky, here we represent the particular solutions obtained when $s_3 = 0$:

$$r_3 = \frac{\sigma\, s_1}{r_2(\alpha r_2 - \beta r_1)},\; s_2 = -\frac{r_1 s_1}{r_2},\; s_3 = 0,$$
$$\lambda_0 = 0,\; \lambda_1 = 0,\; \lambda_2 = 0;$$
$$r_3 = 0,\; s_2 = \frac{r_2 s_1}{r_1},\; s_3 = 0,\; \lambda_0 = \frac{\lambda_3 \sigma\, s_1^2}{r_1^2},$$
$$\lambda_1 = \frac{\lambda_3 \sigma s_1(s_1^2 - (\alpha^2+\beta^2)(r_1^2 - s_1^2 x))}{r_1(r_1^2 - s_1^2 x)},$$
$$\lambda_2 = -\frac{\lambda_3 \sigma\, s_1^4}{r_1^4 - r_1^2 s_1^2 x},\; (\text{where } \sigma = r_1^2 + r_2^2).$$

The first three expressions of each of the above solutions define the IMs of equations (1), and the latter three expressions of each of the solutions are the first integrals of differential equations on these IMs.

### 2.4 Parametric Analysis of Stationary Sets

Using the Gröbner bases technique, we have found a series of the solutions of equations (4) under some conditions imposed on the parameters.

For the case $\lambda_0 = x\,\lambda_2,\; \lambda_1 = 0,\; x = -1/(\alpha^2+\beta^2)$, the solution

$$r_1 = -\frac{\alpha s_3}{\alpha^2+\beta^2},\; r_2 = -\frac{\beta s_3}{\alpha^2+\beta^2},\; r_3 = 0 \quad (13)$$

has been obtained. It represents the 3-dimensional IM of equations (1).

For the case $\lambda_0 = 0,\; \lambda_1 = 0,\; \lambda_2 = 0$, the solution

$$\beta r_2 + \alpha r_1 + s_3 = 0,\; \beta s_1 - \alpha s_2 = 0$$

has been found. It represents the 4-dimensional IM of equations (1).

It is easily verified that the above solutions pass through the zero solution. The elements of the families of one-dimensional IMs (9) also pass through the zero solution when $\lambda_0 = \lambda_1 = \lambda_2 = 0$. So, the zero solution is a bifurcation point.

**2.4.1 Stability of Stationary Sets** Let us investigate the stability of both the zero solution and the IM passing through this solution by the Routh-Lyapunov method [Lyapunov, 1954]. In simple cases, the problem is reduced to verifying the sign-definiteness conditions for the 2nd variation of integral $K$ (3) obtained in the neighbourhood of the solution under study.

The 2nd variation of the integral $K$ in the neighbourhood of the zero solution can be written as:

$$\left.\begin{array}{l} 2\,\delta^2 K = -\lambda_2 y_1^2 - \lambda_2 y_2^2 - ((\alpha^2+\beta^2)\lambda_0 \\ +\lambda_2) y_3^2 - 2\lambda_1 y_1 y_4 + (\lambda_0 - \lambda_2 x) y_4^2 \\ -2\lambda_1 y_2 y_5 + (\lambda_0 - \lambda_2 x) y_5^2 + 2\alpha\lambda_0 y_1 y_6 \\ +2\beta\lambda_0 y_2 y_6 - 2\lambda_1 y_3 y_6 + (2\lambda_0 - \lambda_2 x) y_6^2. \end{array}\right\} \quad (14)$$

Here $y_i$ are the deviations of the perturbed solution from the unperturbed one.

Using Sylvester's criterion, we can write down the conditions for the positive definiteness of the quadratic form $\delta^2 K$ as

$$\left.\begin{array}{l} \lambda_2 < 0,\ D_1 < 0,\ (\alpha^2+\beta^2)\lambda_0^2(\lambda_0 - x\lambda_2) \\ +D_1(2\lambda_0 - x\lambda_2) < 0, \\ (D_1 + D_2\lambda_0)(\lambda_1^2 + D_2(\lambda_0 - x\lambda_2)) > 0, \end{array}\right\} \quad (15)$$

where $D_1 = \lambda_1^2 + \lambda_2(\lambda_0 - x\lambda_2),\ D_2 = (\alpha^2+\beta^2)\lambda_0 + \lambda_2$.

Inequalities (15) are compatible under the following constraints imposed on the parameters $\lambda_i, \alpha, \beta, x$:

$$\left.\begin{array}{l} \alpha \neq 0 \text{ and } \beta \neq 0 \text{ and } \lambda_2 < 0 \text{ and } \Big( (\lambda_0 > 0 \\ \text{and} \lambda_0 + \frac{\lambda_2}{\alpha^2+\beta^2} < 0 \text{ and } x > \frac{\lambda_1^2 + D_2\lambda_2}{D_2\lambda_2}) \\ \text{or } \big(\lambda_0 \leq 0 \text{ and } x > \frac{\lambda_1^2 + (\lambda_2 + D_2)\lambda_0}{\lambda_2^2}\big)\Big). \end{array}\right\} \quad (16)$$

Conditions (16) are sufficient for the stability of the zero solution.

Further, let us investigate the stability of IM (13).

The variation of the integral $\tilde{K} = 2\lambda_0 H - \lambda_2 V_2 - \lambda_3 V_3$ in the neighbourhood of this IM is:

$$2\Delta\tilde{K} = -\lambda_2 y_2^2 - \lambda_2 y_3^2 - \lambda_3(\alpha y_2 + \beta y_3)^2 s_3^2$$
$$-(\alpha^2+\beta^2)\lambda_3(s_1 y_2 + s_2 y_3)^2.$$

Here $y_1 = r_1 + \alpha s_3/(\alpha^2+\beta^2)$, $y_2 = r_2 + \beta s_3/(\alpha^2+\beta^2)$, $y_3 = r_3$ are the deviations of the perturbed solution from the unperturbed one.

Next, we introduce the following variables $z_1 = (\alpha y_2 + \beta y_3)s_3$, $z_2 = s_1 y_2 + s_2 y_3$. In the variables $y_2, y_3, z_1, z_2$, the $\Delta\tilde{K}$ has the form: $2\Delta\tilde{K} = -\lambda_2\big(y_2^2 + y_3^2\big) - \lambda_3\big(z_1^2 + (\alpha^2+\beta^2)z_2^2\big)$.

The latter quadratic form is sign definite with respect to the variables $y_2, y_3, z_1, z_2$ when the following conditions $\alpha^2 + \beta^2 \neq 0$ and $\lambda_2 > 0, \lambda_3 > 0$ (or $\lambda_2 < 0, \lambda_3 < 0$) hold. Hence, these conditions are sufficient for the stability of IM (13) with respect to the variables $y_2, y_3$.

### 2.5 Euler's Equations at x = 0

Let us consider the problem of motion of a rigid body in ideal fluid in case [Sokolov, 2001]. The differential equations of motion

$$\left.\begin{array}{l} \dot{r_1} = (\alpha r_1 + \beta r_2 + 2s_3)r_2 - r_3 s_2, \\ \dot{r_2} = -(\alpha r_1 + \beta r_2 + 2s_3)r_1 - r_3 s_1, \\ \dot{r_3} = r_1 s_2 - r_2 s_1, \\ \dot{s_1} = -(\beta s_3 + (\alpha^2+\beta^2)r_2)r_3 \\ +(\alpha r_1 + \beta r_2 + s_3)s_2, \\ \dot{s_2} = (\alpha s_3 + (\alpha^2+\beta^2)r_1)r_3 \\ -(\alpha r_1 + \beta r_2 + s_3)s_1, \\ \dot{s_3} = (\beta r_1 - \alpha r_2)s_3 \end{array}\right\} \quad (17)$$

admit the following first integrals:

$$\left.\begin{array}{l} 2H = (s_1^2 + s_2^2 + 2s_3^2) + 2(\alpha r_1 + \beta r_2)s_3 \\ -(\alpha^2+\beta^2)r_3^2 = 2h, \\ V_1 = s_1 r_1 + s_2 r_2 + s_3 r_3 = c_1, \\ 2V_2 = r_1^2 + r_2^2 + r_3^2 = c_2, \\ 2V_3 = (r_1 s_1 + r_2 s_2)((\alpha^2+\beta^2)(r_1 s_1 + r_2 s_2) \\ +2(\alpha s_1 + \beta s_2)s_3) \\ +s_3^2(s_1^2 + s_2^2 + (\alpha r_1 + \beta r_2 + s_3)^2) = 2c_3. \end{array}\right\} \quad (18)$$

For obtaining the stationary solutions and the IMs of system (17), we construct a linear combination of problem's first integrals (18):

$$K = \lambda_0 H - \lambda_1 V_1 - \lambda_2 V_2 - \lambda_3 V_3. \quad (19)$$

The necessary conditions for integral $K$ (19) to have an extremum with respect to the variables $s_1, s_2, s_3, r_1, r_2, r_3$

$$\left.\begin{array}{l} \partial K/\partial s_1 = 0, \partial K/\partial s_2 = 0, \partial K/\partial s_3 = 0, \\ \partial K/\partial r_1 = 0, \partial K/\partial r_2 = 0, \partial K/\partial r_3 = 0. \end{array}\right\} \quad (20)$$

define the families of stationary solutions and the families of IMs for differential equations (17).

The Gröbner basis technique is applied for finding solutions of system (20). We have constructed a Gröbner basis for this system with respect to $\lambda_0, \lambda_1, \lambda_2, r_3, s_3$. As a result, we have obtained the system:

$$\left.\begin{array}{l} \lambda_2(pz^2\lambda_2 + q^2w^2\lambda_3) = 0, \quad -q^2w\lambda_1 \\ -z((\beta r_1 + \alpha r_2)s_1^2 - 2(\alpha r_1 - \beta r_2)s_1 s_2 \\ -(\beta r_1 + \alpha r_2)s_2^2)\lambda_2 - Gq^2w^2\lambda_3 = 0, \\ -pq^2\lambda_0 - (\beta^2 r_1^4 - 2\alpha\beta r_1^3 r_2 + r_2^2(\alpha^2 r_2^2 \\ +s_1^2) - 2r_1(\alpha\beta r_2^3 + r_2 s_1 s_2) + r_1^2(Gr_2^2 \\ +s_2^2))\lambda_2 = 0, \quad -yz\lambda_2 - q^2 w s_3\lambda_3 = 0, \\ -pz(\alpha r_1 + \beta r_2)\lambda_2 + qw^2(Gr_3 - \alpha s_1 \\ -\beta s_2)\lambda_3 = 0. \end{array}\right\} \quad (21)$$

Here the following denotations are used:

$$q = \beta s_1 - \alpha s_2,\ p = r_1^2 + r_2^2,\ w = r_1 s_1 + r_2 s_2,$$
$$y = r_1 s_2 - r_2 s_1, z = \beta r_1 - \alpha r_2,\ G = \alpha^2 + \beta^2.$$

Let us consider one family of the solutions of system (21):

$$\left.\begin{array}{l} s_3 = xy/pz,\ r_3 = y/z,\ \lambda_2 = -q^2 w^2 \lambda_3/pz^2, \\ \lambda_1 = -\left(w\left(-pq^2 + Gy^2 + Gpz^2\right)\lambda_3\right)/pz^2, \\ \lambda_0 = w^2\left(y^2 + pz^2\right)\lambda_3/p^2 z^2, \end{array}\right\} \quad (22)$$

where $\lambda_3$ is the family parameter.

The expressions for $r_3, s_3$ (22) define the IM of differential equations (17). The vector field on this IM has the form:

$$\left.\begin{array}{l} \dot{r}_1 = r_2\left(\frac{2wy}{pz} + \alpha r_1 + \beta r_2\right) - \frac{ys_2}{z}, \\ \dot{r}_2 = -r_1\left(\frac{2wy}{pz} + \alpha r_1 + \beta r_2\right) + \frac{ys_1}{z}, \\ \dot{s}_1 = \frac{-y(wy\beta + Gpzr_2) + z(wy + pz(\alpha r_1 + \beta r_2))s_2}{pz^2}, \\ \dot{s}_2 = \frac{y(wy\alpha + Gpzr_1) - z(wy + pz(\alpha r_1 + \beta r_2))s_1}{pz^2}. \end{array}\right\} \quad (23)$$

The expressions for $\lambda_0, \lambda_1, \lambda_2$ (22) are the first integrals of equations (23).

One can show that the following integrals of initial differential equations (17)

$$\tilde{\lambda}_0 = \frac{(V_1(HV_1 \pm M\lambda_3))}{(V_1^2 - 4GV_2^2)},$$
$$\tilde{\lambda}_1 = \frac{(2GHV_1V_2^2 \pm (V_1^2 - 2GV_2^2)M\lambda_3)}{(V_2(V_1^2 - 4GV_2^2))},$$
$$\tilde{\lambda}_2 = \frac{(V_1^2(4GHV_2^2 \pm V_1M\lambda_3))}{(V_1^2 - 4GV_2^2)},$$

correspond to the integrals $\lambda_0, \lambda_1, \lambda_2$. Here $M = \sqrt{(V_1^2(H^2 - 2V_3) + 8GV_2^2V_3)}$.

These nonlinear combinations of the fist integrals of the initial system can be used to analyze it by the technique applied above.

## 3 A Rigid Body under the Influence of Two Force Fields

The rotation of a rigid body around a fixed point in uniform gravitational and magnetic force fields is considered. The distribution of mass in the body corresponds to the Kowalewski integrable case.

The equations of motion of the body in the coordinate system rigidly attached to the body can be written as:

$$\left.\begin{array}{l} 2\dot{p} = b\delta_3 + q\,r, \quad \dot{\gamma}_1 = \gamma_2 r - \gamma_3 q, \\ 2\dot{q} = x_0\gamma_3 - p\,r,\ \dot{\gamma}_2 = \gamma_3 p - \gamma_1 r, \\ \dot{r} = -b\delta_1 - x_0\gamma_2,\ \dot{\gamma}_3 = \gamma_1 q - \gamma_2 p, \\ \dot{\delta}_1 = \delta_2 r - \delta_3 q,\ \dot{\delta}_2 = \delta_3 p - \delta_1 r, \\ \dot{\delta}_3 = \delta_1 q - \delta_2 p. \end{array}\right\} \quad (24)$$

Here $p, q, r$ are the projections of the angular velocity vector onto the axes related to the body, $\gamma_1, \gamma_2, \gamma_3$ are the direction cosines of the upward vertical, $\delta_1, \delta_2, \delta_3$ are the direction cosines of the constant magnetic moment vector, the parameters $x_0, b$ are proportional to the coordinate of the mass center of the body and the coordinate of the constant magnetic moment vector, respectively.

The equations admit the following first integrals:

$$\left.\begin{array}{l} 2H = 2(p^2 + q^2) + r^2 + 2(x_0\gamma_1 - b\,\delta_2) = 2h, \\ V_1 = (p^2 - q^2 - x_0\gamma_1 - b\,\delta_2)^2 + (2p\,q - x_0\gamma_2 \\ +b\,\delta_1)^2 = c_1,\ V_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1, \\ V_3 = \delta_1^2 + \delta_2^2 + \delta_3^2 = 1, \\ V_4 = \gamma_1\delta_1 + \gamma_2\delta_2 + \gamma_3\delta_3 = c_2. \end{array}\right\} \quad (25)$$

When $b = 0$, the system under consideration corresponds to the Kowalewski integrable case.

On the invariant manifold of codimension 2

$$p^2 - q^2 - x_0\gamma_1 - b\,\delta_2 = 0,\ 2pq - x_0\gamma_2 + b\,\delta_1 = 0 \quad (26)$$

system (24) has an additional cubic integral [Bogoyavlenskii, 1984] and is completely Liouville integrable. Further, we study the above differential equations written on IM (26):

$$\left.\begin{array}{l} 2\dot{p} = q\,r + b\,\delta_3, \qquad \dot{\delta}_1 = r\,\delta_2 - q\,\delta_3, \\ 2\dot{q} = x_0\gamma_3 - p\,r, \qquad \dot{\delta}_2 = \delta_3 p - \delta_1 r, \\ \dot{r} = -2(p\,q + b\,\delta_1),\ \ \dot{\delta}_3 = \delta_1 q - \delta_2 p, \\ x_0\dot{\gamma}_3 = -((p^2 + q^2)\,q + b\,(p\,\delta_1 + q\,\delta_2)). \end{array}\right\} \quad (27)$$

The first integrals of equations (27) are given by

$$\left.\begin{array}{l} 2\tilde{H} = 4p^2 + r^2 - 4b\,\delta_2 = 2\tilde{h}, \\ \tilde{V}_2 = \gamma_3^2 + \frac{(2p\,q + b\,\delta_1)^2}{x_0^2} + \frac{(q^2 - p^2 + b\delta_2)^2}{x_0^2} = 1, \\ V_3 = \delta_1^2 + \delta_2^2 + \delta_3^2 = 1, \\ \tilde{V}_4 = \frac{2p\,q\,\delta_2 + (p^2 - q^2)\,\delta_1}{x_0} + \gamma_3\,\delta_3 = \tilde{c}_2, \\ 2V_5 = (p^2 + q^2)\,r - 2x_0 p\,\gamma_3 + 2bq\,\delta_3 = m. \end{array}\right\} \quad (28)$$

Within the framework of the study of the phase space of system (27), we state the problem to find IMs of this system for their simplest classification and to investigate their stability.

### 3.1 Finding Invariant Manifolds

Likewise as above, we construct a linear combination of first integrals (28)

$$2K = \lambda_0\tilde{H} - \lambda_1\tilde{V}_2 - \lambda_2 V_3 - 2\lambda_3\tilde{V}_4 - \lambda_4 V_5 \quad (29)$$

and write down the necessary conditions for the integral $K$ to have an extremum with respect to the phase variables $p, q, r, \gamma_3, \delta_1, \delta_2, \delta_3$:

$$\left.\begin{array}{l}\partial K/\partial p = 4\lambda_0 p - \frac{2\lambda_1[(p^2+q^2)\,p+b(q\,\delta_1-p\,\delta_2)]}{x_0^2}\\ \quad -\frac{2\lambda_3(p\,\delta_1+q\,\delta_2)}{x_0} + \lambda_4\,(x_0\gamma_3 - p\,r) = 0,\\ \partial K/\partial q = -\frac{2\lambda_1\,[(p^2+q^2)\,q+b\,(p\,\delta_1+q\,\delta_2)]}{x_0^2}\\ \quad +\frac{2\lambda_3\,(q\,\delta_1-p\,\delta_2)}{x_0} - \lambda_4\,(q\,r + b\,\delta_3) = 0,\\ \partial K/\partial r = 2\lambda_0\,r - \lambda_4\,(p^2+q^2) = 0,\\ \partial K/\partial\gamma_3 = -\lambda_1\gamma_3 - \lambda_3\,\delta_3 + \lambda_4 x_0 p = 0,\\ \partial K/\partial\delta_1 = -\frac{\lambda_1 b\,(2p\,q+b\,\delta_1)}{x_0^2} - \lambda_2\,\delta_1\\ \quad -\frac{\lambda_3\,(p^2-q^2)}{x_0} = 0,\\ \partial K/\partial\delta_2 = -2b\,\lambda_0 - \lambda_2\,\delta_2 - \frac{\lambda_1 b\,(q^2-p^2+b\,\delta_2)}{x_0^2}\\ \quad -\frac{2\lambda_3 p\,q}{x_0} = 0,\\ \partial K/\partial\delta_3 = -\lambda_2\,\delta_3 - \lambda_3\,\gamma_3 - \lambda_4\,bq = 0.\end{array}\right\} \quad (30)$$

We shall find the solutions of stationary equations (30) with two procedures. The 1st procedure is based on solving these equations with respect to some part of the phase variables and the family parameters of the integral $K$. This technique was already used in the given work.

The 2nd procedure finds new IMs by eliminating the family parameters from the known solutions of the stationary equations. Both techniques provide a possibility to reveal embedded in one another IMs.

### 3.2 Applying First Procedure

We find the IMs of various dimension for equations (27). Since first integrals correspond to IMs of codimension 1, let us begin with IMs of codimension 2. To this end, we take, e.g., $\delta_1, \delta_2, \lambda_1, \lambda_2, \lambda_0, \lambda_4$ as unknowns and construct a Gröbner basis with respect to the lexicographic ordering $\delta_1 > \delta_2 > \lambda_1 > \lambda_2 > \lambda_0 > \lambda_4$ for the polynomials of system (30). As a result, we have the following system:

$$\begin{array}{l}\lambda_4\,g_1(p,q,r,\gamma_3,\lambda_3,\lambda_4) = 0, g_2(p,q,r,\lambda_0,\lambda_4) = 0,\\ g_3(q,\gamma_3,\delta_3,\lambda_2,\lambda_3,\lambda_4) = 0,\\ g_4(p,\gamma_3,\delta_3,\lambda_1,\lambda_3,\lambda_4) = 0,\\ g_5(p,q,r,\gamma_3,\delta_2,\delta_3,\lambda_3,\lambda_4) = 0,\\ g_6(p,q,r,\gamma_3,\delta_1,\delta_3,\lambda_3,\lambda_4) = 0,\end{array}$$

where $g_j (j = 1,\ldots,6)$ are the polynomials of the basis. The resulting system is bulky, therefore it is not represented explicitly here.

The system can be decomposed into two subsystems given below.

The subsystem 1:

$$\left.\begin{array}{l}\lambda_4 b\,x_0(\varrho - 2(p^2+q^2)pq) - \lambda_3(x_0\gamma_3(2p(p^2\\ +q^2) + x_0\gamma_3 r) + b\,(b\,\delta_3 r\\ -2q\,(p^2+q^2))\,\delta_3) = 0,\\ 2\lambda_0\,b\,x_0\,(\varrho - 2(p^2+q^2)\,p\,q])\,r - \lambda_3\,(p^2\\ \quad +q^2)\,(x_0\,\gamma_3\,(2p\,(p^2+q^2) + x_0\gamma_3\,r)\\ \quad +b\,(b\,\delta_3\,r - 2q\,(p^2+q^2))\,\delta_3) = 0,\\ \lambda_2\,x_0\,(2(p^2+q^2)\,p\,q - \varrho) + \lambda_3\,b\,(2(p^2\\ \quad +q^2)q^2 - \varrho_2) = 0,\\ \lambda_1\,b\,(2(p^2+q^2)\,p\,q - \varrho) + \lambda_3\,x_0\,(2(p^2\\ \quad +q^2)p^2 + \varrho_2) = 0,\end{array}\right\} \quad (31)$$

$$\left.\begin{array}{l}2b\,(p^2+q^2)\,r\,\delta_2 + b\,(b\,r\,\delta_3 r + q(r^2\\ \quad -2(p^2+q^2)))\,\delta_3 - (p\,r - x_0\,\gamma_3)\\ \quad \times(2p\,(p^2+q^2) + x_0\,\gamma_3\,r) = 0,\\ -2b\,(p^2+q^2)\,\delta_1 - p\,[2q\,(p^2+q^2) + b\,\delta_3\,r]\\ \quad -x_0\gamma_3\,q\,r = 0.\end{array}\right\} \quad (32)$$

The subsystem 2:

$$\left.\begin{array}{l}\lambda_4 = 0,\ \lambda_0 = 0,\ -(\lambda_2\delta_3 + \lambda_3\gamma_3) = 0,\\ \quad -(\lambda_1\gamma_3 + \lambda_3\delta_3) = 0,\end{array}\right\} \quad (33)$$

$$\left.\begin{array}{l}(x_0^2\gamma_3^2 + b^2\,\delta_3^2)\,\delta_2 - (2x_0\gamma_3 p\,q\\ \quad +b\,(p^2-q^2)\,\delta_3)\,\delta_3 = 0,\\ (x_0^2\gamma_3^2 + b^2\,\delta_3^2)\,\delta_1 + (2b\,\delta_3 p\,q - x_0\,\gamma_3(p^2\\ \quad -q^2))\,\delta_3 = 0.\end{array}\right\} \quad (34)$$

Here $\varrho = (b\,\delta_3 p - x_0\gamma_3\,q)\,r$, $\varrho_2 = (b\,\delta_3 q + x_0\gamma_3 p)\,r$.

Let us analyze the subsystem 1.

It can be easy verified by IM definition that equations (32) define the IM of codimension 2 for differential equations (27).

The equations of vector field on IM (32) are given by:

$$\left.\begin{array}{l}2\dot{p} = q\,r + b\,\delta_3,\ 2\dot{q} = x_0\gamma_3 - p\,r,\ \dot{r} = \frac{\varrho_2}{p^2+q^2},\\ \dot{\gamma}_3 = \frac{b\,[b q\,r\,\delta_3 - (p^2+q^2)(2q^2-r^2)]\,\delta_3}{2x_0\,(p^2+q^2)\,r} + \frac{p\,\gamma_3\,q}{r}\\ \quad +\frac{(x_0^2\gamma_3^2 - 2(p^2+q^2)^2)\,q}{2x_0\,(p^2+q^2)},\\ \dot{\delta}_3 = \frac{[b\,r\,\delta_3 - 2(p^2+q^2)\,q]\,p\,\delta_3}{2\,(p^2+q^2)\,r} - \frac{1}{2b}\\ \quad +\frac{x_0\gamma_3\,p\,(2p\,(p^2+q^2)+x_0\gamma_3\,r)}{2b\,(p^2+q^2)\,r}.\end{array}\right\} \quad (35)$$

From (31), we find the values for $\lambda_0, \lambda_1, \lambda_2, \lambda_4$ which are the first integrals of equations (35).

In a similar manner, we have established that equations (34) also define the IM of codimension 2 for differential equations (27), and the values of $\lambda_1, \lambda_2$ found from the two latter expressions of (33) are the first integrals for the equations of vector field on this IM. Obviously, these integrals are dependent. We have also found the families of IMs of codimension 3, 4 and 5.

Let us consider the latter. In order to obtain this family, we take $\delta_1, \delta_2, \delta_3, \gamma_3, r, \lambda_0$ as unknowns and construct a Gröbner basis with respect to the lexicographic ordering $\delta_1 > \delta_2 > \delta_3 > \gamma_3 > r > \lambda_0$ for the polynomials of system (30). A result will be the following system:

$$\lambda_0(4\lambda_1\lambda_2 - 4\lambda_3^2) + \lambda_4^2\alpha_1 = 0, \quad (36)$$

$$\left.\begin{array}{l} \lambda_4\,\alpha_1\,r - 2\alpha_2\,(p^2+q^2) = 0, \\ -\alpha_2\,\gamma_3 - \lambda_4\,(\lambda_3\,b\,q + \lambda_2 x_0 p) = 0, \\ \alpha_2\,\delta_3 - \lambda_4\,(\lambda_1\,b\,q + \lambda_3 x_0\,p) = 0, \\ 2\alpha_1\,\alpha_2\,\delta_2 - 2\lambda_1\,\alpha_2\,b\,(p^2-q^2) \\ +4\lambda_3\,\alpha_2\,x_0 p\,q + \lambda_4^2\,\alpha_1\,b\,x_0^2 = 0, \\ -\alpha_1\,\delta_1 - 2\lambda_1 b\,p\,q - \lambda_3 x_0\,(p^2-q^2) = 0, \end{array}\right\} \quad (37)$$

where $\alpha_1 = \lambda_1 b^2 + \lambda_2 x_0^2,\ \alpha_2 = \lambda_3^2 - \lambda_1\lambda_2$.

Equations (37) define the family of IMs of codimension 5 for differential equations (27). The parameters of the family are $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. This family possesses an extremal property: the integral $K$ (29) takes a stationary value on the elements of the family when $\lambda_0 = -\lambda_4^2\alpha_1/(4\alpha_2)$ (this value is found from equation (36)).

Obviously the solutions found by the described technique are related. Indeed, on substituting expressions (37) (resolved with respect to $\delta_1, \delta_2, \delta_3, \gamma_3, r$) into equations (34), the latter equations become identities. Hence, one can conclude that the elements of IMs family (37) are submanifolds of IM (32).

Thus, the procedure presented above allows one to find the embedded in one another IMs families. In the case considered, the latter is caused by the technique applied. In general case, this technique enables us to classify IMs on the basis of their embedding and degree of their degeneration.

The IMs families found for the differential equations written on IM (26) can be "lifted up" as invariant into the phase space of system (24). To this end, it is sufficient to add the equations of IM (26) to the equations of the IMs families.

### 3.3 Applying 2nd Procedure

Let us eliminate the parameter $\lambda_4$ from equations (37) with the aid of one of the equations, e.g., the first. The value of $\lambda_4$ found from this equation is:

$$\lambda_4 = -2\alpha_2\,(p^2+q^2)(\alpha_1\,r)^{-1}. \quad (38)$$

Next, construct a lexicographic Gröbner basis with respect to the lexicographic ordering $\delta_1 > \delta_2 > \delta_3 > \gamma_3$ for the polynomials of a resulting system (after eliminating $\lambda_4$ from equations (37)). The system obtained

$$\left.\begin{array}{l} \alpha_1\,\gamma_3\,r + 2(p^2+q^2)(\lambda_3 bq + \lambda_2 x_0 p) = 0, \\ \alpha_1\,r\,\delta_3 - 2(p^2+q^2)(\lambda_1 bq + \lambda_3 x_0 p) = 0, \\ \alpha_1^2 r^2\,\delta_2 + \alpha_1\,[\lambda_1 b\,(q^2-p^2) + 2\lambda_3\,x_0\,p\,q] \\ \quad\times r^2 - 2\alpha_2(p^2+q^2)^2 = 0, \\ -\alpha_1\,\delta_1 - 2b\lambda_1 p\,q + \lambda_3 x_0(q^2-p^2) = 0 \end{array}\right\} \quad (39)$$

defines the IMs family of codimension 4 for the initial differential equations, which is parameterized by $\lambda_1, \lambda_2, \lambda_3$.

Expression (38) is the first integral for the equations of vector field on the elements of IMs family (39). The latter is verified by IM definition.

The elements of IMs family (37) are submanifolds of the IMs family found. This can be verified by direct substitution of expressions (37) (resolved with respect to $\delta_1, \delta_2, \delta_3, \gamma_3, r$) into equations (39).

The above example shows that the presented procedure also provides a possibility to find embedded in one another IMs families by eliminating the family parameters from the equations of known IMs families. In this case, the resulting IMs family includes the initial one.

## 4 Conclusion

In the given work, nonlinear systems which are described by differential equations with polynomial first integrals were considered. The algorithms for the study of extremal properties of the first integrals of such systems have been proposed. With the aid of these algorithms, new invariant manifolds have been obtained for both Euler's equations on Lie algebras and the equations of motion of a rigid body under the influence of two force fields, and their properties have been investigated.

In this paper, we restricted our study to the linear combinations of the basic integrals only. For the exhaustive analysis of the problems on the base of the proposed approach, it is necessary to investigate in detail the properties of the algebra of the first integrals of these problems.

**Acknowledgements**

This research was supported financially by the Russian Foundation for Basic Research (16-07-00201a) and the Council on Grants of the President of the Russian Federation for State Support to Leading Scientific Schools of the Russian Federation (NSh-8081.2016.9).

## References

Adler, V. E., Marikhin, V. G., and Shabat, A. B. (2012). Quantum Tops as Examples of Commuting Differential Operators. *Theoret. and Math. Phys.,* **172**(3), pp. 1187–1205.

Banshchikov, A. V., Burlakova, L. A., Irtegov, V. D., and Titorenko, T. N.(2011). The software package for selecting and investigation the stability of stationary sets of mechanical systems. Certificate of state registration of the program on a computer, number 2011615235, on July 5. (In Russian)

Bogoyavlenskii, O. I.(1984). Two integrable cases of a rigid body dynamics in a force field. *USSR Acad. Sci. Doklady,* **275**(6), pp. 1359–1363.

Borisov, A. V., Mamaev, I. S., and Sokolov, V. V. (2001). A new integrable case on so(4), *Dokl. Phys.* **46**(12), pp. 888–889.

Irtegov, V. D. and Titorenko, T. N.(2009). The invariant manifolds of systems with first integrals.*J. Appl. Math. Mech.*, **73**(4), pp. 379–384

Lyapunov, A. M. (1954). *On Permanent Helical Motions of a Rigid Body in Fluid. Collected Works, vol. 1*. USSR Acad. Sci., Moscow–Leningrad. (In Russian)

Poincare, A. (1947) *On curves defined by differential equations*. OGIS. Moscow–Leningrad. (In Russian)

Sarychev, V. A. and Gutnik, S. A. ( 2015). Dynamics of a Satellite Subject to Gravitational and Aerodynamic Torques. Investigation of Equilibrium Positions. *Cosmic Research.*, **53**(6), pp. 449–457.

Smirnov, A. V. (2008). Systems of sl(2, C) tops as two-particle systems. *Theoretical and Mathematical Physics*, **157**(1), pp. 1370–1382 .

Sokolov, V. V. (2001). A new integrable case for the Kirchhoff equations. *Theoretical and Math. Phys.* **129** (1), pp. 1335–1340