OPTIMAL FEEDBACK CONTROL OF TRAVELING WAVE IN A PIECEWISE LINEAR FHN MODEL

Masashi Takeuchi
Dept. of Electrical and Info. Syst.
Osaka Prefecture University
Japan

Keiji Konishi
Dept. of Electrical and Info. Syst.
Osaka Prefecture University
Japan
konishi@eis.osakafu-u.ac.jp

Naoyuki Hara
Dept. of Electrical and Info. Syst.
Osaka Prefecture University
Japan

Abstract

Our previous study [Konishi, Takeuchi, Shimizu, Chaos 2011] proposed a simple systematic design procedure of a periodic impulsive force and a time-continuous feedback force to eliminate traveling waves in a piecewise linear FitzHugh–Nagumo (FHN) model. As our previous study used only the integral control method in *classical* control theory, it is not easy to specify its system performance. The present paper introduces an optimal control method in *modern* control theory, which can specify its system performance. Furthermore, we show that the designed force is valid not only for the piecewise linear function but also for a class of smooth nonlinear FHN models.

Key words

FitzHugh–Nagumo model, traveling wave, optimal feedback control, piecewise linear model.

1 Introduction

Excitable media, such as cardiac tissue and the Belousov–Zhabotinsky reaction, have received considerable attention in the field of nonlinear science (Mikhailov and Showalter, 2006). It is known that the spatial waves and spatiotemporal chaos in cardiac tissue induce a major health problem, since irregular activation, such as ventricular tachycardia and ventricular fibrillation, decreases the ability of the heart to pump blood. One of current treatments for the irregular activation is to apply a *high*-voltage electric shock to a patient's chest to eliminate it. However, this shock often causes physical and mental strain to the patient. Therefore, a practical use of *low*-voltage electric shock is anticipated (Sinha and Sridhar, 2008; Takagi et al., 2004). On the other hand, the Belousov–Zhabotinsky (BZ) reaction have attracted increasing interest (Mikhailov and Showalter, 2006). It was reported that a feedback light-intensity control can stabilize and track unstable propagating waves in the photosensitive BZ reaction (Mihaliuk et al., 2002; Sakurai et al., 2002).

Some researchers have proposed various feedback control methods for eliminating spatiotemporal behavior in excitable media (Sinha and Sridhar, 2008). Yuan, Chen, and Yang showed that an external force injected into the resting regions to counter propagation waves can eliminate the propagation waves (Yuan et al., 2007). The global feedback control proposed by Yoneshima, Konishi, and Kokame is to apply a force uniformly to every region of the medium only when the area of active region is at local minimal values (Yoneshima et al., 2008). Guo *et al.* provided the local feedback control which identifies the spiral tip areas and makes them unexcitable. This method can experimentally eliminate spiral turbulence (Guo et al., 2010). Sakaguchi and Nakamura eliminate breathing spiral waves in the Aliev–Panfilov model by using the delayed feedback control (Sakaguchi and Nakamura, 2010).

From a viewpoint of practical applications, it would be desirable to know a systematic design of external forces for elimination of spatiotemporal behavior because the systematic design does not require trial-and-error testing. However, most studies on the elimination of spatiotemporal behavior were investigated only by numerical simulations (Sinha and Sridhar, 2008). A systematic design procedure of the single impulsive nonfeedback force was provided by Osipov and Collins (Osipov and Collins, 1999). Although a periodic impulsive force and a time-continuous feedback force have the potential to achieve a low-amplitude elimination, their procedure cannot be used for these forces. Our previous study proposed a simple systematic design procedure for such forces (Konishi et al., 2011). This study focused on a one-dimensional FitzHugh–Nagumo (FHN) model with a piecewise linear function (Ohta and Kiyose, 1996; Ohta et al., 1997; Koga, 1993; Rinzel and Keller, 1973; Tonnelier, 2003b; Tonnelier, 2003a), and obtained simple and analytical results. The proposed procedure is useful for designing nonfeedback and feedback control systems. However, in our previous study, the following problems, which

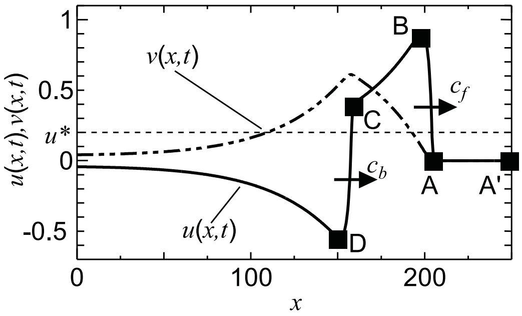

Figure 1. Spatial distribution of traveling wave

are important subjects from a practical viewpoint, remain unsolved: (i) it is impossible to specify the system performance, such as the transient time for elimination and the amplitude of the external force, in designing the feedback controller; (ii) it is unclear whether the procedure can be used for FHN model with smooth nonlinear functions.

The present paper shows that problems (i) and (ii) can be solved by using *modern* control theory and by employing a smooth nonlinear function, respectively. As our previous study (Konishi et al., 2011) used only the integral control method in *classical* control theory, problem (i) cannot be systematically solved. The present paper introduces an optimal feedback control method in *modern* control theory to systematically solve problem (i). Furthermore, for problem (ii), the force designed by our procedure for the piecewise linear FHN model is applied to a smooth nonlinear FHN model: we see that the designed force is valid not only for the piecewise linear model but also for a class of smooth nonlinear models.

2 Piecewise linear FitzHugh-Nagumo model

Now consider the one-dimensional piecewise linear FHN model (Ohta and Kiyose, 1996; Ohta et al., 1997; Koga, 1993; Rinzel and Keller, 1973):

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = f[u(x,t)] - v(x,t) + D\dfrac{\partial^2 u(x,t)}{\partial x^2} \\ \dfrac{\partial v(x,t)}{\partial t} = \varepsilon\{u(x,t) - \gamma v(x,t)\} + e(t) \end{cases}, \tag{1}$$

$$f[u] = H[u - u^*] - u, \tag{2}$$

where $u(x,t)$ and $v(x,t)$ are fast and slow variables, respectively. x denotes position and t is continuous time. The diffusion coefficient for the fast variable is denoted by $D > 0$. Here $0 < \varepsilon \ll 1$ and $\gamma \in (0, u^*/(1-u^*))$ are the parameters. $u^* \in (0, 1/2)$ is the threshold of $f[u]$. H represents the step function.

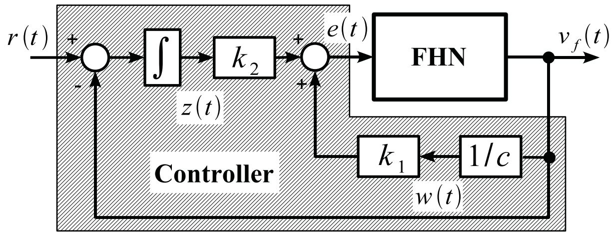

Figure 2. Block diagram of a state feedback with integral control

The weak external force $e(t)$ is applied with spatial uniformity to the slow dynamics. Let us assume that one traveling wave propagates through a one-dimensional space $x \in (-\infty, +\infty)$. Figure 1 sketches the spatial distribution of the traveling wave

3 Feedback control

This section derives a linear time-invariant system (Konishi et al., 2011) and designs an optimal feedback controller on the basis of the linear quadratic regulator.

3.1 Linear time-invariant system

This subsection reviews our previous work (Konishi et al., 2011). The velocity of the traveling wave at wave front (curve AB in Fig. 1), c_f, is given by

$$c_f = \{1 - 2(u^* + v_f)\}\sqrt{\frac{D}{(u^* + v_f)(1 - u^* - v_f)}}. \tag{3}$$

The wave front velocity c_f depends on $v(x,t) = v_f$ in front of the wave (A–A' region in Fig. 1). $u(x,t)$ and $v(x,t)$ are with spatial uniformity in the A–A' region. As the external force $e(t)$ is applied uniform spatially to the whole medium, $u(x,t)$ and $v(x,t)$ maintain their spatial uniformity in this region. Here they can be considered as the variables $u_f(t)$ and $v_f(t)$ in this region. In addition, the wave front velocity satisfying Eq. (3) can also be a time variable $c_f(t)$. Since $u(x,t)$ and $v(x,t)$ in this region have their spatial uniformity, we can neglect the diffusion term in Eq. (1). As $u_f(t) < u^*$, the nonlinear function is simplified to $f[u] = -u$. The parameter ε is assumed to be a sufficiently small positive value; thus, we ignore the fast mode and focus only on the slow dynamics. In consequence, the force $e(t)$ and the variable $v_f(t)$ are approximately given by the linear time-invariant (LTI) system,

$$\begin{cases} \dot{w}(t) = aw(t) + be(t) \\ v_f(t) = cw(t) \end{cases}, \tag{4}$$

where $w(t)$ describes the slow dynamics and the system parameters (a, b, c) are

$$a := \eta^{(+)},\ b := -\frac{1}{\eta^{(+)} - \eta^{(-)}},\ c = -1 - \eta^{(+)},$$
$$\eta^{(\pm)} := \{-(1+\varepsilon\gamma) \pm \sqrt{(1-\varepsilon\gamma)^2 - 4\varepsilon}\}/2.$$

3.2 Controller

This paper proposes a state feedback with integral control (Kuo and Golnaraghi, 2003), as shown in Fig. 2, to eliminate the traveling waves. The external force $e(t)$ is given by

$$\begin{cases} e(t) = k_1 w(t) + k_2 z(t) \\ \dot{z}(t) = r(t) - v_f(t) \end{cases}, \tag{5}$$

where $z(t)$ is the additional variable and $k_{1,2}$ are the feedback gains we have to design. Our main goal is to stop the traveling wave front (i.e., $\lim_{t\to+\infty} c_f(t) = 0$); thus, according to Eq. (3), the controller must track $v_f(t)$ to $1/2 - u^*$. In order to achieve this goal, the reference signal $r(t)$ should be a step input with amplitude $1/2 - u^*$. The reason we use the additional variable $z(t)$ is that the integral unit is required to track without steady-state error.

Let us design the gains $k_{1,2}$ by using the linear quadratic regulator (Zak, 2002). Combining LTI system (4) and controller (5), we have

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}e(t) + \begin{bmatrix}0 \ 1\end{bmatrix}^T r(t) \\ v_f(t) = \boldsymbol{c}\boldsymbol{x}(t) \end{cases}, \tag{6}$$

$$e(t) = \boldsymbol{k}\boldsymbol{x}(t), \tag{7}$$

where $\boldsymbol{x}(t) := \begin{bmatrix}w(t) \ z(t)\end{bmatrix}^T$ is the system variable. The system matrices ($\boldsymbol{A}$, $\boldsymbol{b}$, $\boldsymbol{c}$) and the feedback gain vector $\boldsymbol{k}$ are written as

$$\boldsymbol{A} := \begin{bmatrix} a & 0 \\ -c & 0 \end{bmatrix},\ \boldsymbol{b} := \begin{bmatrix} b \\ 0 \end{bmatrix},\ \boldsymbol{c} := \begin{bmatrix} c \\ 0 \end{bmatrix}^T,\ \boldsymbol{k} := \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}^T. \tag{8}$$

Since it is obvious that $(\boldsymbol{A}, \boldsymbol{b})$ is controllable (i.e., $\det\begin{bmatrix}\boldsymbol{b}\ \boldsymbol{A}\boldsymbol{b}\end{bmatrix} = -b^2 c \neq 0$), the feedback gain $\boldsymbol{k}$ can be designed by simple procedures. In order to obtain an optimal system performance, this paper employs the linear quadratic regulator: controller (7) is designed such that the performance index,

$$J = \int_0^\infty \left\{\bar{\boldsymbol{x}}(t)^T \boldsymbol{Q}\bar{\boldsymbol{x}}(t) + \mu\bar{e}(t)^2\right\} \mathrm{d}t, \tag{9}$$

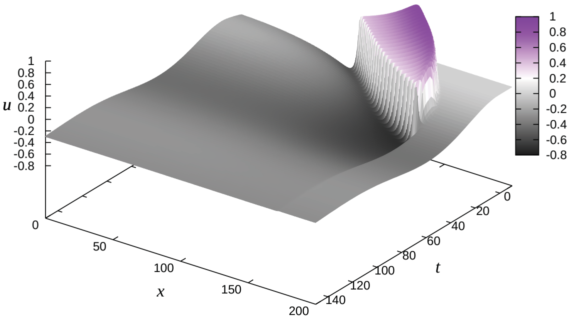

Figure 3. Spatial distribution of the traveling wave with integral control ($k_1 = 0, k_2 = 0.003$).

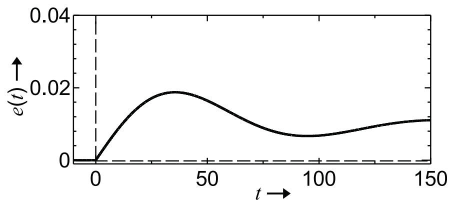

Figure 4. External force with integral control ($k_1 = 0, k_2 = 0.003$).

$$\bar{\boldsymbol{x}}(t) := \boldsymbol{x}(t) - \boldsymbol{x}_\infty,\ \bar{e}(t) := e(t) - e_\infty,$$

is minimized, where $\boldsymbol{x}_\infty := \lim_{t\to\infty} \boldsymbol{x}(t)$ and $e_\infty := \lim_{t\to\infty} e(t)$. Here $\boldsymbol{Q} > 0$ and $\mu > 0$ are the weights we can arbitrarily choose. The feedback gain is given by

$$\boldsymbol{k} = -\frac{1}{\mu}\boldsymbol{b}^T\boldsymbol{P}, \tag{10}$$

where $\boldsymbol{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} > 0$ satisfies the algebraic Riccati equation,

$$\boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} + \boldsymbol{Q} - \frac{1}{\mu}\boldsymbol{P}\boldsymbol{b}\boldsymbol{b}^T\boldsymbol{P} = \boldsymbol{0}. \tag{11}$$

Now let us design the feedback gain $\boldsymbol{k}$ according to the above procedure. For simplicity, the weight $\boldsymbol{Q}$ is fixed at $\boldsymbol{Q} = \mathrm{diag}\{q_1, q_2\}$. Substituting Eq. (8) into Eq. (11), we have

$$p_{12} = \sqrt{\mu q_2 / b^2},$$
$$p_{11} = \left\{a\mu + \sqrt{a^2\mu^2 - b^2\mu(2cp_{12} - q_1)}\right\}/b^2,$$
$$p_{22} = p_{12}(a - b^2 p_{11}/\mu)/c.$$

Here, we obtain the optimal gain (10),

$$k_1 = -\frac{1}{\mu}bp_{11},\ k_2 = -\frac{1}{\mu}bp_{12}. \tag{12}$$

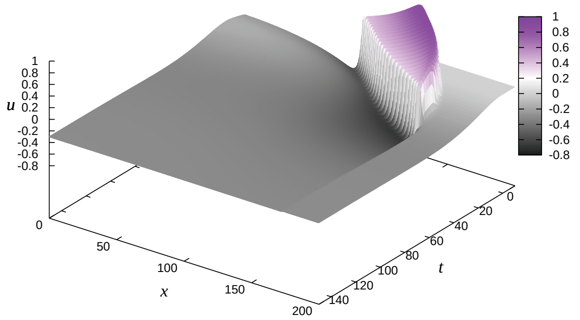

Figure 5. Spatial distribution of the traveling wave with optimal control (case (I): $k_1 = 0.0598, k_2 = 0.0041$).

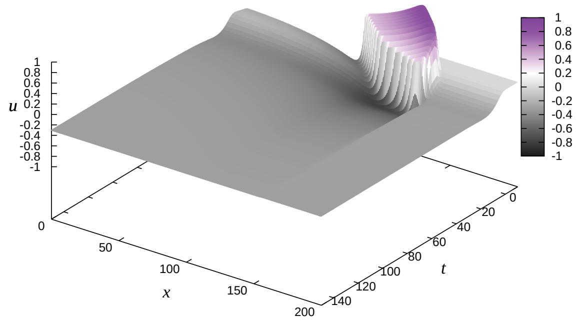

Figure 7. Spatial distribution of the traveling wave with optimal control (case (II): $k_1 = 0.3134, k_2 = 0.0632$).

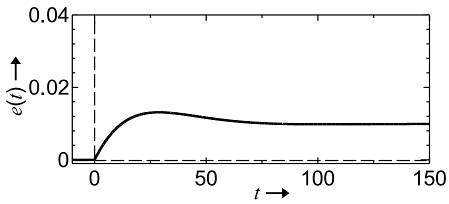

Figure 6. External force with optimal control (case (I): $k_1 = 0.0598, k_2 = 0.0041$).

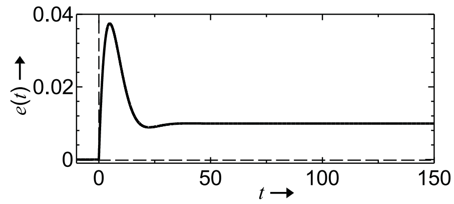

Figure 8. External force with optimal control (case (II): $k_1 = 0.3134, k_2 = 0.0632$).

As a consequence, we have optimal feedback controller (5) with gain (12).

3.3 Numerical examples

Throughout this paper, we assume that the parameters of FHN model (1) are known and fixed at

$$u^* = 0.2,\ D = 1.0,\ \varepsilon = 0.03,\ \gamma = 0.1. \qquad (13)$$

Let us review the numerical result of the integral control, which corresponds to a particular case of controller (5) (i.e., $k_1 = 0$ and $k_2 > 0$), discussed in our previous work (Konishi et al., 2011). Figures 3 and 4 show the spatial distribution of the traveling wave and the external force just before and after the control start time ($t = 0$). For $t > 0$, the wave front AB slows down and the wave back CD maintains its velocity, and then the wave back catches up with the front. Eventually, the traveling wave disappears at $t \approx 34.5$.

Now we design the optimal controller proposed in the preceding section. Consider the two specifications of system performance: (I) preference for low peak force and (II) preference for rapid elimination of traveling waves.

For case (I), the weights in index (9) are set to $q_1 = q_2 = 1.0$, $\mu = 6 \times 10^4$. From these weights and parameters (13), optimal gain (12) can be obtained: $k_1 = 0.0598$ and $k_2 = 0.0041$. The spatial distribution of the traveling wave and the external force are shown in Figs. 5 and 6. It can be seen that, compared with the results of integral control shown in Figs. 3 and 4, the peak force becomes low.

For case (II), the weights are set to $q_1 = q_2 = 1.0$, $\mu = 250$. Optimal gain (12) can be obtained: $k_1 = 0.3134$ and $k_2 = 0.0632$. The spatial distribution of the traveling wave and the external force are shown in Figs. 7 and 8. We see that the traveling wave rapidly disappears. From the two cases, it can be confirmed that the optimal controller works well on numerical simulations.

4 FHN model with smooth nonlinear functions

This section investigates whether the force designed for the piecewise linear FHN model can be valid for smooth nonlinear FHN models. Let us introduce the smooth nonlinear function shown in Fig. 9,

$$f[u] = -u + 0.5 + 0.5\tanh(\alpha(u - u^*)). \qquad (14)$$

It should be noted that this function converges on piecewise linear function (2) as $\alpha \to +\infty$. We have ob-

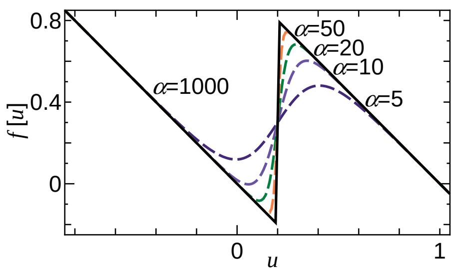

Figure 9. Smooth nonlinear function $f[u]$ ($u^* = 0.2$)

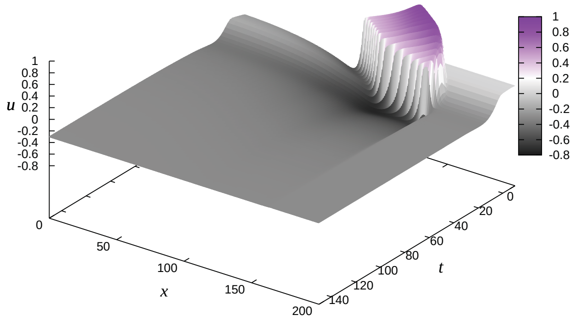

Figure 10. Spatial distribution of the traveling wave in optimal controlled FHN model with function (14) ($\alpha = 10$, $k_1 = 0.3134$, $k_2 = 0.0632$).

served that the traveling wave can propagate in FHN model with function (14) for $\alpha \geq 10$; however, it cannot propagate for $\alpha < 10$. We have numerically confirmed that the integral controller ($k_1 = 0, k_2 = 0.003$) and the optimal controller ($k_1 = 0.3134$, $k_2 = 0.0632$) designed for the piecewise linear FHN model are valid for the smooth nonlinear FHN model for $\alpha \geq 10$. Figure 10 shows the optimal control of traveling wave in FHN model with function (14) ($\alpha = 10$). It can be seen that the designed controller works well even for FHN model with smooth function (14).

5 Conclusion

The present paper provided a systematic procedure for designing an optimal feedback controller to eliminate the traveling wave in the piecewise liner FHN model. The designed controller achieves our goal: the rapid disappearance of wave and the low peak force. Furthermore, it was shown that the controller designed for the piecewise linear FHN model is valid for a class of smooth nonlinear FHN models.

Acknowledgments

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