ON BIFURCATIONS IN CONSERVATIVE SYSTEMS

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Abstract

In the present paper, the problem of analysis related to bifurcations in the neighborhood of peculiar families of stationary sets is considered.

Key words

bifurcation, invariant manifolds, stability.

Introduction

The procedure of finding and qualitative analysis of invariant manifolds (IMs), which attribute stationary values to the elements of algebra of the problem's first integrals, is presently one of perspective and widely used approaches to investigation of conservative systems with a sufficiently large number of first integrals. We call such invariant manifolds the invariant manifolds of steady motions (IMSMs). This approach is suitable to investigate the stability of the indicated class of IMSMs on the basis of Lyapunov's 2nd method. Comparison of the approach to other methods of analysis of such systems can be found, for example, in [1]. The foundations of the technique may be traced to the works by Routh [2] and Lyapunov [3], in which the procedures of finding stationary solutions and investigation of their stability have been proposed and grounded. In the present paper, we consider the problem of bifurcation in the neighborhood of "peculiar" families of stationary sets. The stationary sets, which attribute stationary values to the several first integrals of the problem will be called peculiar.

In the cases, when the equations of stationarity for the family of first integrals, which are used for obtaining stationary solutions and IMSMs, are nonlinear and bulky, their solutions may be efficiently obtained with the application of computer algebra methods. In particular, the algorithms, which are based on the Gröbner basis method [5], are efficient in many cases.

1 Peculiar Stationary Sets

Now consider the problem of obtaining IMs and investigation of their bifurcations and stability for one system having a large number of first integrals. We shall use the Routh–Lyapunov technique.

In the capacity of a typical problem we consider the Lagrange top in a central field of forces. It is known [6], the differential equations of a body's motion in Euler's form are:

$$\begin{aligned}
&A\dot{p} = (A-C)qr_0 + z_0\gamma_2 - \mu(A-C)\gamma_2\gamma_3,\\
&A\dot{q} = (C-A)r_0p - z_0\gamma_1 - \mu(C-A)\gamma_3\gamma_1,\\
&C\dot{r} = 0,\ \ \dot{\gamma_1} = r_0\gamma_2 - q\gamma_3,\\
&\dot{\gamma_2} = p\gamma_3 - r_0\gamma_1, \dot{\gamma_3} = q\gamma_1 - p\gamma_2. \qquad (1)
\end{aligned}$$

where A, C are the body's main inertia moments; z_0 is the coordinate of the mass center; μ is the parameter characterizing the gravitation force; p, q, r are projections of the body's angular rate onto the axes bound up with the body; $\gamma_1, \gamma_2, \gamma_3$ are the directional cosines of the "vertical" in the axes bound up with the body. Above Euler's equations assume also other interpretations, which include mechanical ones [7] as well.

It is well known that system (1) has the following first integrals:

$$\begin{aligned}
&2H = Ap^2 + Aq^2 + 2z_0\gamma_3 + \mu(A\gamma_1^2 + A\gamma_2^2 +\\
&C\gamma_3^2) = 2h,\\
&V_1 = Ap\gamma_1 + Aq\gamma_2 + Cr_0\gamma_3 = m,\\
&V_3 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = c,\ \ r = r_0 = const.
\end{aligned}$$

Let us construct a complete linear bundle of these integrals

$$\begin{aligned}
&2K_0 = Ap^2 + Aq^2 + 2z_0\gamma_3 + \mu(A\gamma_1^2 + A\gamma_2^2 + C\gamma_3^2) -\\
&2\lambda_1(Ap\gamma_1 + Aq\gamma_2 + Cr_0\gamma_3) + \lambda_3(\gamma_1^2 + \gamma_2^2 + \gamma_3^2),\\
&\lambda_i = const,\ \ i = 1, 3. \qquad (2)
\end{aligned}$$

While following the Routh–Lyapunov technique for finding the stationary sets of equations (1), we write down the stationary conditions for this family of first integrals:

$$\frac{\partial K_0}{\partial p} = A(p - \lambda_1\gamma_1) = 0,\ \frac{\partial K_0}{\partial q} = A(q - \lambda_1\gamma_2) = 0,$$
$$\frac{\partial K_0}{\partial \gamma_1} = -\lambda_1 Ap + (\mu A + \lambda_3)\gamma_1 = 0,$$
$$\frac{\partial K_0}{\partial \gamma_2} = -\lambda_1 Aq + (\mu A + \lambda_3)\gamma_2 = 0,$$
$$\frac{\partial K_0}{\partial \gamma_3} = z_0 - \lambda_1 Cr_0 + (\mu C + \lambda_3)\gamma_3 = 0. \quad (3)$$

In accordance with Routh–Lyapunov's theorem, it is possible to state that the family of solutions

$$p = q = \gamma_1 = \gamma_2 = 0,\ \ \gamma_3 = 1,\ \ r_0 = const \quad (4)$$

for the system (1) is a family of stationary solutions of the initial system of differential equations with parameter r_0. The elements of the obtained family of the body's permanent rotations are peculiar. This may be concluded from equations (3). From the equations it follows that the obtained solutions are realized as stationary ones when the condition

$$\mu C + \lambda_3 + z_0 - \lambda_1 Cr_0 = 0 \quad (5)$$

holds. Consequently, these attribute a stationary value to the one-parameter family of first integrals with parameter λ_1. Therefore, it is possible to put the following two parameters r_0 and λ_1 in correspondence to the family of peculiar stationary solutions (4).

Now let us investigate stability of the obtained motions. To this end, it is sufficient to write out the conditions of sign-definiteness for the second variation of integral K_0 in the neighborhood of the scrutinized motion when the variations of integrals V_1 and V_3 are equated to zero. The desired condition is

$$\lambda_1 Cr_0 - \lambda_1^2 A > z_0 + \mu(C - A).$$

The above condition of stability, which is a condition of sign-definiteness for the family of integrals of equations of perturbed motion, contains parameter λ_1, because the permanent rotation under scrutiny is peculiar. By choosing λ_1 such that the left-hand side of the inequality is maximum (to this end, it is necessary to put $\lambda_1 = Cr_0(2A)^{-1}$), we obtain the following condition of stability of permanent rotations belonging to the family (4) [6]:

$$C^2 r_0^2 > 4A[z_0 + \mu(C - A)].$$

The latter is both the necessary and sufficient condition of stability with the precision up to the boundary.

Now let us find the invariant manifolds of steady motions (IMSMs) for the system (1). Compute the determinant of the linear system (3):

$$detJ = A^2[\mu A + \lambda_3 - \lambda_1^2 A]^2(\mu C + \lambda_3). \quad (6)$$

After removing λ_3 from the latter expression with the use of relation (5), the expression has the form:

$$[\mu(C - A) - z_0 + \lambda_1 Cr_0 - \lambda_1^2 A]^2(\lambda_1 Cr_0 - z_0). (7)$$

Consider now the situation, when determinant (6) turns zero under the following condition imposed on the second multiplier:

$$\mu A + \lambda_3 - \lambda_1^2 A = 0.$$

When the value of λ_3 is defined by the latter equation, the family of first integrals (2) writes:

$$2\tilde{K}_0 = Ap^2 + Aq^2 + 2z_0\gamma_3 + \mu(A\gamma_1^2 + A\gamma_2^2 + C\gamma_3^2)$$
$$-2\lambda_1(Ap\gamma_1 + Aq\gamma_2 +$$
$$Cr_0\gamma_3) + A(\lambda_1^2 - \mu)(\gamma_1^2 + \gamma_2^2 + \gamma_3^2).$$

The two-parameter family of IMSMs

$$p - \lambda_1\gamma_1 = 0,\ \ q - \lambda_1\gamma_2 = 0,$$
$$z_0 - \lambda_1 Cr_0 + (\mu(C - A) + \lambda_1^2 A)\gamma_3 = 0 \quad (8)$$

attributes a stationary value to the above bundle of first integrals. This fact is almost obvious since stationarity conditions for $\tilde{K}_0$ with respect to the problem's variables may be written in the following form:

$$\frac{\partial \tilde{K}_0}{\partial p} = A(p - \lambda_1\gamma_1) = 0,\ \frac{\partial \tilde{K}_0}{\partial q} = A(q - \lambda_1\gamma_2) = 0,$$
$$\frac{\partial \tilde{K}_0}{\partial \gamma_1} = -\lambda_1 A(p - \lambda_1\gamma_1) = 0,$$
$$\frac{\partial \tilde{K}_0}{\partial \gamma_2} = -\lambda_1 A(q - \lambda_1\gamma_2) = 0,$$
$$\frac{\partial \tilde{K}_0}{\partial \gamma_3} = z_0 - \lambda_1 Cr_0 + (\mu(C - A) + \lambda_1^2 A)\gamma_3 = 0 \quad (9)$$

and the matrix of coefficients for the unknowns $(p - \lambda_1\gamma_1)$, $(q - \lambda_1\gamma_2)$, $(z_0 - \lambda_1 Cr_0 + (\mu(C - A) + \lambda_1^2 A)\gamma_3)$ in equations (9) has the rank of three. The latter guarantees the invariance of the family of manifolds (8) for the initial system of differential equations [4]. Note that when we give the above mechanical interpretation to the variables of the differential equations

(1) we must consider the first integral's constant (the Kasimir function) $\gamma_1^2+\gamma_2^2+\gamma_3^2=c$ to be equal to one, and take account of this restriction both in the discussion of properties of the system's phase space and in operating with the invariant manifolds in this phase space.

From the kinematic viewpoint, the family of IMs represents a family of our body's regular precessions. In this case, λ_1 is the angular rate of precession, and r_0 is the rate of proper rotation.

When $\lambda_1=0$, the equations of stationarity for $\tilde{K}_0$ have another family of solutions (IMs):

$$p=q=0,\ \ \gamma_3=\frac{z_0}{\mu(A-C)},\ \ r_0=const.,\quad (10)$$

which – under the chosen interpretation – corresponds to the family of the body's permanent rotations with the axis of symmetry inclined with respect to the "vertical". In particular, if the value of the family's parameter r_0 is zero then this is the equilibrium position having the axis inclined with respect to the "vertical" axis.

The vector field on the elements of the obtained family of IMSMs (8) is defined by differential equations (1) of the initial problem. These equations write:

$$\begin{aligned}&A\dot{p}=q\,\Omega,\ \ A\dot{q}=-p\,\Omega,\ \ \dot{\gamma_3}=0,\\&\Omega=(A-C)r_0+\frac{z_0}{\lambda_1}+\frac{\mu(A-C)(z_0-\lambda_1Cr_0)}{\lambda_1[\mu(C-A)+\lambda_1^2A]}.\end{aligned}$$

Within the frames of the approach discussed, stability of the elements of the obtained family of IMSMs may be investigated on the basis of Lyapunov's 2nd method.

To this end, we introduce the deviations from the general element (an arbitrary fixed λ_1) of the family of IMSMs (8):

$$\begin{aligned}&y_1=p-\lambda_1\gamma_1,\ \ y_2=q-\lambda_1\gamma_2,\\&y_3=z_0-\lambda_1Cr_0+(\mu(C-A)+\lambda_1^2A)\gamma_3.\end{aligned}$$

Hence the expression for the first integral $\tilde{K}_0$ represented in terms of deviations for the equations of the perturbed motion writes:

$$2\Delta\tilde{K}_0=Ay_1^2+Ay_2^2+[\mu(C-A)+\lambda_1^2A]^{-1}y_3^2.$$

Applying Zubov's theorem [8] on stability of IMs, we can easily obtain sufficient conditions of stability for the elements of the family of regular precessions. These conditions represented as conditions of signdefiniteness for the quadratic form $\Delta\tilde{K}_0$ write:

$$\mu(C-A)+\lambda_1^2A>0.\quad (11)$$

For the family of permanent rotations (10) the expression of integral $\tilde{K}_0$ for the equations of perturbed motion is

$$2\Delta\tilde{K}_0=Ay_1^2+Ay_2^2+\mu(C-A)y_3^2,$$

where y_1,y_2,y_3 are deviations of p,q,γ_3 in the perturbed motion. The quadratic form $\Delta\tilde{K}_0$ is signdefinite for $C>A$. When this condition holds, elements of the family of permanent rotations (10) are stable with respect to the variables p,q,γ_3 and r_0.

When removing γ_1,γ_2 from the problem's integrals with the use of equations (8), we have:

$$\begin{aligned}&2\tilde{H}=A(p^2+q^2)=2\tilde{h}-2z_0\gamma_3^0+\mu(A-C)\gamma_3^{02}\\&-Cr_0^2-\mu A,\ \ \tilde{V}_1=A(p^2+q^2)=\lambda_1(m-Cr_0\gamma_3^0),\\&\tilde{V}_3=p^2+q^2=\lambda_1^2(1-\gamma_3^{02}).\end{aligned}$$

From the latter we can conclude, in particular, that differential equations on the manifold (8) have the first integral:

$$W=p^2+q^2=const.$$

and the constants of first integrals of the initial problem are related by the following conditions:

$$\begin{aligned}&\lambda_1(m-Cr_0\gamma_3^0)=\lambda_1^2A(1-\gamma_3^{02}),\\&2\tilde{h}-2z_0\gamma_3^0-\mu(C-A)\gamma_3^{02}-Cr_0^2-\mu A=\\&\lambda_1^2A(1-\gamma_3^{02}).\end{aligned}\quad (12)$$

Note also that the following resonance relations take place between integrals on IMSMs (8):

$$2\tilde{H}=\tilde{V}_1,\ \tilde{V}_1=A\tilde{V}_3.$$

If γ_3^0 is removed from the first relation of (12) with the use of equations of IMSMs (8), then we obtain

$$\begin{aligned}&-\lambda_1^5A^3+\lambda_1^4A^2m-2\lambda_1^3A^2\mu(C-A)-\lambda_1^2A[z_0Cr_0\\&-2\mu(C-A)m]-\lambda_1[\mu^2A(C-A)^2-Az_0^2\\&+C^2r_0^2\mu(C-A)]+\mu(C-A)[Cr_0z_0\\&+\mu(C-A)m]=0.\end{aligned}\quad (13)$$

Therefore, for fixed r_0 and m, there may be up to five regular precessions (when equation (13) has five real roots) on the integral manifold $V_1=m$.

2 On bifurcations stationary sets

The most interesting situations of branching invariant manifolds arise when we deal with peculiar stationary sets. In the problem under scrutiny we have obtained above the one-parameter family of peculiar steady permanent rotations (4) and the two-parameter family of regular precessions (8). Consider now the problem of branching these families having different dimensions. Let for a fixed $r_o = r_0^*$ a concrete motion belong to the family of peculiar permanent rotations. Introduce the following definition.

Definition. *Let us speak that a subfamily of the family of precessions (8) adjoins to the above peculiar permanent rotation ($r_o = r_0^*$) if for some fixed value of parameter r_0^* there exists such an interval of variations of parameter λ_1 and such λ_1^*, which belongs to this interval, that when the values of the parameters are r_0^*, λ_1^* then values of all the phase variables coincide for the families (4) and (8).*

Following this definition, values of parameter λ_1, which correspond to the case of adjoining of the precessions to peculiar permanent rotations, may be determined from the equation

$$\lambda_1^2 A - \lambda_1 C r_0 + z_0 + \mu(C - A) = 0. \qquad (14)$$

When solving this equation, we can find the two desired values:

$$\lambda_{1_{(1,2)}} = \frac{1}{2A}(Cr_0 \pm \sqrt{C^2 r_0^2 - 4A[z_0 + \mu(C - A)]}).$$

Therefore, two families of regular precessions adjoin to each peculiar permanent rotation, whose angular rate r_0 satisfies the condition

$$C^2 r_0^2 - 4A[z_0 + \mu(C - A)] > 0.$$

The account of the fact that the latter inequality is both the necessary and sufficient (without the boundary) condition of stability for permanent rotations allows one to conclude that we have proved the following

Proposition: *Not less than two families of regular precessions adjoin to each stable permanent rotation.*

Now consider the cases when the number of such adjoining families is larger. Note that the constant of the integral of areas at the moment of adjoining these precessions to permanent rotations must have the value $V_1 = m = Cr_0$ for $\gamma_3 = 1$. Having substituted this value into (13), we obtain the factorized relation:

$$[\lambda_1^2 A - \lambda_1 C r_0 + z_0 + \mu(C - A)][\lambda_1^3 A^2 + \lambda_1 A[\mu(C - A) - z_0] - \mu(C - A)Cr_0] = 0. \qquad (15)$$

The first multiplier here coincides with (14). Now we have to find out the condition, when the equation obtained has multiple roots. For this purpose we write down the resultant for the multipliers of equation (15). It can readily be verified that the resultant, being equated to zero, writes:

$$\Delta = \mu(A - C)C^4 r_0^4 - C^2 r_0^2 A[z_0^2 + 4\mu z_0(A - C) - 4\mu^2(A - C)^2] + 4A^2 z_0^2[z_0 - \mu(A - C)] = 0. (16)$$

When solving the latter equation with respect to $C^2 r_0^2$, we obtain the two roots:

$$(C^2 r_0^2)_1 = 4A[z_0 + \mu(C - A)],$$
$$(C^2 r_0^2)_2 = \frac{A z_0^2}{\mu(A - C)}.$$

Hence, there are three regular precessions that adjoin to the peculiar permanent rotation, which lies on the boundary of stability. The following value of λ_1 corresponds to these precessions:

$$\tilde{\lambda}_1 = \sqrt{[z_0 + \mu(C - A)]A^{-1}}.$$

The third of the coinciding roots here is the root of the cubic multiplier in (15).

It can readily be verified, the second solution of (16) corresponds to the family of peculiar regular precessions, whose equations write:

$$p = \pm\sqrt{\mu(A - C)A^{-1}}\gamma_1, \; q = \pm\sqrt{\mu(A - C)A^{-1}}\gamma_2, \; \gamma_3 = const.. \qquad (17)$$

These precessions are realized under fixed values of

$$(C^2 r_0^2)_2 = A z_0^2[\mu(A - C)]^{-1},$$
$$\lambda_1 = \sqrt{\mu(A - C)A^{-1}} \qquad (18)$$

and under an arbitrary constant $\gamma_3 = const$. Here γ_3 plays the role of the family's parameter. Note, the family of precessions (17) is degenerate among precessions (8) in the sense that both of the multipliers in the expression of the determinant (6) turn zero on these precessions.

If the value of the angular rate r_0, which corresponds to the latter family of peculiar regular precessions, is substituted into (14), then, by solving the the equation obtained, it is possible to find the following two values for λ_1:

$$(\lambda_1)_1 = \sqrt{[\mu(A - C)]A^{-1}},$$
$$(\lambda_1)_2 = [z_0 - \mu(A - C)][\mu A(A - C)]^{-1/2}. (19)$$

Therefore, in case when the value of the angular rate is defined by the first formula in (18), there are three precessions which adjoin to the corresponding peculiar permanent rotation. Furthermore, two of them coincide (are multiple).

Consider now the problem of stability of the precessions, which adjoin to permanent rotations.

Having substituted the values of roots of equation (15) into the conditions of stability for the precessions (11), we obtain the following inequality:

$$C^2 r_0^2 - 2Az_0 > \pm C r_0 \sqrt{C^2 r_0^2 - 4A[z_0 + \mu(C-A)]}.$$

Having squared the latter inequality and performing an elementary transformation, we find out that, when the inequalities

$$4A[z_0 + \mu(C-A)] < C^2 r_0^2 < \frac{Az_0^2}{\mu(A-C)}$$

hold, both precessions, which adjoin to the permanent rotation having an angular rate r_0, are stable.

Now we consider the problem of stability of precessions, which correspond to the right boundary in the latter system of inequalities. The values of parameters r_0 and λ_1 for the two coinciding kinds of precessions, which are included into this family, are defined by (18). Obtaining the sufficient conditions of stability is reduced here to finding the conditions of signdefiniteness for the 2nd variation of integral $\tilde{K}_0$ in the neighborhood of the precessions. This variation writes:

$$2\Delta\tilde{K}_0 = A(y_1^2 + y_2^2).$$

So, these precessions are stable with respect to some part of the variables. The third family of precessions, which are adjoining to (4) for the value of parameter r_0 indicated above, corresponds to $(\lambda_1)_1 < (\lambda_1)_2$ (19), and so, these precessions are stable.

As it follows from the form of roots of equation (14), one of the roots is growing, and the other is decreasing side by side with the increase of r_0. Hence, under the value of the parameter

$$C^2 r_0^2 > Az_0^2[\mu(A-C)]^{-1},$$

the two subfamilies of precessions shall again adjoin to peculiar permanent rotations (4). Furthermore, only precessions of one of these subfamilies shall be stable.

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