Estimates of control times for some classes of control systems of neutral type

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Abstract

It is shown that the control time is the adequate characteristic of the state of chaos for the dynamical control system of neutral type.

1 Introduction

For considered below classes of dynamical control systems, we use controls of two kinds, and namely, basic controls and local controls. We assume the following for the classes of dynamical control systems. First, the dynamical system, that corresponds to each basic control, has the complex behavior with abundance of periodic and everywhere dense trajectories. Second, any dynamical control system is locally controlled along every basic trajectories, i.e. along trajectories corresponding to basic controls.

By first assumption, any dynamical control system is globally ε-controlled, and namely, for every pair of points there exists a trajectory of the basic dynamical system that goes through an ε-neighborhood of these points. By second assumption, there exists a controllable trajectory goes through these points. In this case, any dynamical control system is globally controlled provided it is locally controlled.

In connection with above, the main problem is find estimates of control times for dynamical control systems of the considered classes. In particular, this problem was considered in [1, 2]. Also, this problem was solved in [3, 4, 5, 6] where the upper and lower bounds of control times were obtained for some classes of dynamical control systems. In particular, the bounds of control times were obtained for hyperbolic dynamical control systems. It was shown that the control time is the adequate characteristic of the state of chaos for the dynamical control system.

In this paper, an analogous problem is investigated for neutral dynamical control systems. In [4], it was shown that the control time T satisfies to the following estimates

$$\frac{\underline{C}}{\varepsilon^{\frac{d}{d+1}}} \leq T \leq \frac{\overline{C}}{\varepsilon}, \tag{1}$$

where d is dimension of the state space, ε is the radius of a neighborhood of the target point, The values $\underline{C}, \overline{C}$ depend on parameters of the state space and of the dynamical control system.

Estimates (1) were obtained without regard to the state of chaos of the dynamical control system. Below, we investigate the impact of the state of chaos on upper and lower bounds of control times.

2 Description of dynamical control systems

Let $X = \{x\}$, $\dim X = d$ be a Riemannian manifold and $U = \{u\} \subset R^1$ be a neighborhood of null control $u^0 = 0$. By dist denote a metric on X, by Vol denote a measure of volume on X, and by vol denote a normalized measure of volume on X, i.e. for any measurable set $A \subset X$ it is fulfilled $\mathrm{vol}(A) = \frac{1}{V}\,\mathrm{Vol}(A)$ where $V = \mathrm{Vol}(X)$.

Consider a local family of maps $f(.,u) : X \to X$, u/inU, where the set X is the state space, the set U is the control space.

A map f is called a neutral map iff $\mathrm{dist}(x_*, x_{**}) = \mathrm{dist}(f(x_*), f(x_{**}))$ for any $x_*, x_{**} \in X$.

By $f_0(.) = f(., u^0)$ denote the basic map. We shall assume that the map f_0 is some neutral map and that the minimal f_0-invariant set is the state space X. If f_0 is the neutral map then the probabilistic f_0-invariant measure is vol.

We assume that $f(.,u) = f_0(.) + uf_1(.)$. Consider the dynamical control system in the form

$$x_{n+1} = f_0(x_n) + uf_1(x_n), \quad n = 0, 1, 2, \ldots, \tag{2}$$

where the map $f_1 : X \to X$ is such that dynamical control system (2) is local controllable. Denote

$$\vec{U} = U^d = U \times \cdots \times U, \quad \vec{u} = (u_1, \ldots, u_d) \in \vec{U},$$
$$F(x, \vec{u}) = f(x, u_1) \circ \cdots \circ f(x, u_d) \tag{3}$$

and consider the dynamical control system

$$x_{n+1} = F(x_n, \vec{u}), \quad n = 0, 1, 2, \ldots. \tag{4}$$

Form the sequence of the accessibility sets from a initial point x_0 under local controls $\vec{u} \in \vec{U}$.

$$M_0 = \{x_0\}, M_1 = F(x_0, \vec{U}), \ldots, M_{n+1} = F(M_n, \vec{U}),$$
$$n = 0, 1, 2, \ldots. \tag{5}$$

The sequence of relative volumes of the accessibility sets from x_0 is as follows

$$v_0 = 0, v_1 = \mathrm{vol}(M_1), \ldots, v_n = \mathrm{vol}(M_n), \ldots. \quad (6)$$

3 Dynamical control systems on torus with the basic maps generated by shifts

We consider some classes of dynamical control systems on torus of dimension d. Torus $\mathbb{T}^d$ is the factorization $\mathbb{R}^d/\mathbb{Z}^d$. Let x be coordinates in $\mathbb{R}^d$. Further, we shall assume that the map f_0 depends on parameter a which corresponds to some class of dynamical control systems. We shall assume also that the basic map is the shift, i.e. $f_0(x, a) = x + a$.

Let the dynamical control system be given by the equation

$$x_{n+1} = x_n + a + u f_1(x_n), \quad \mathrm{mod}\, 1, \quad n = 0, 1, 2, \ldots, \quad (7)$$

i.e. we consider the fractional parts of the right part of equation (2). The value of parameter a is the constant (independent of time) d-dimensional vector. Further, we assume that the vector a depends on some small parameter ε. We shall consider a few kinds of dependence on ε, i.e. $a = a(\varepsilon)$, where $a(0) = 0$. Let $a(\varepsilon) = a_0\varepsilon^\alpha$ where $0 \le \alpha \le 1$, $a_0 \ge 0$.

3.1 The basic map generated by zero-shift

Consider the case $a_0 = 0$. The control systems takes the form

$$x_{n+1} = x_n + u f_1(x_n), \quad n = 0, 1, 2, \ldots,$$

Sequence (5) of accessibility sets M_n is the sequence of embedded balls with center at the point x_0 and radius $n\varepsilon$, (see fig. 1).

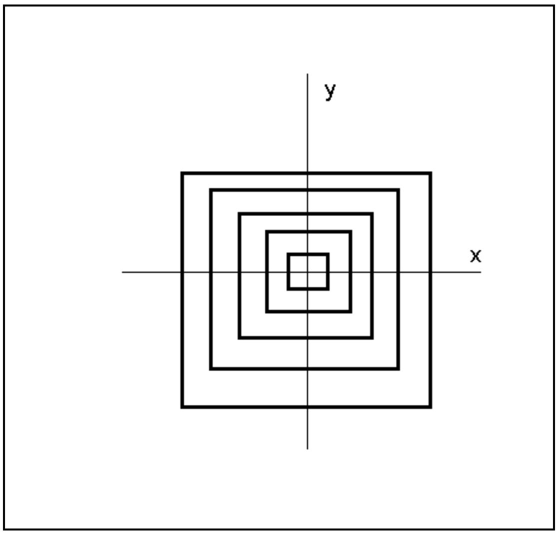

Figure 1: $a_0 = 0$

Hence from (6), volume $v_n = \mathrm{vol}(M_n) = (n\varepsilon)^d$ (in the cubic metric). Since $(n\varepsilon)^d = V$ then we get that the control time $n_1(\varepsilon) = \frac{R}{\varepsilon}$, $R = V^{1/d}$ is the right part of (1).

3.2 The basic map generated by nonzero shifts

We shall consider for simplicity the case $d = 2$. Torus $\mathbb{T}^2$ is the factorization $\mathbb{R}^2/\mathbb{Z}^2$. Let (x, y) be coordinates on plane $\mathbb{R}^2$.

3.2.1 The case $\alpha = 1$

First, consider the case $0 < a_0 \le 1$. Sequence (5) of accessibility sets M_n is the sequence of embedded balls with center at the point $na_0\varepsilon$ and with radius $n\varepsilon$. Volume $v_n = \mathrm{vol}(M_n) = (n\varepsilon)^2$ (in cubic metric). Since $(n\varepsilon)^d = V$ then we get that the control time $n_1(\varepsilon) = \frac{R}{\varepsilon}$, i.e. it is the right part of (1).

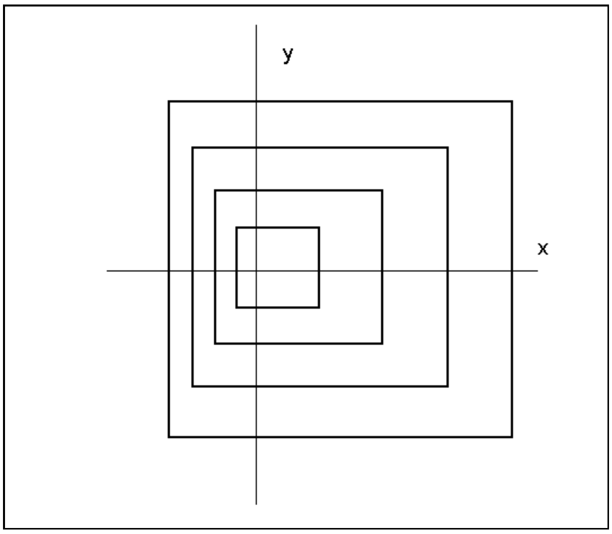

Figure 2: $\alpha = 1, a_0 = \frac{1}{2}$.

Second, consider the case $a_0 > 1$. Sequence (5) of accessibility sets M_n lies as it is shown in fig. 3. The boundary of set $\bigcup_{n\ge2} M_n$ is approximated with deficiency by the line $y = \frac{1}{a_0 - 1}x + \zeta(\varepsilon)$ and with surplus by the line $y = \frac{1}{a_0-1}x + \eta(\varepsilon)$, where $\zeta(\varepsilon) \le \eta(\varepsilon)$. Since $\zeta(\varepsilon) \to 0$, $\eta(\varepsilon) \to 0$, $\frac{\zeta(\varepsilon)}{\eta(\varepsilon)} \to 1$ as $\varepsilon \to 0$ then obtained below asymptotic estimates of the control time as $\varepsilon \to 0$ independent of kinds the functions $\zeta(\varepsilon), \eta(\varepsilon)$. Therefore, we shall assume that these functions are the zero-functions. Thus, we assume that $y = \frac{1}{a_0-1}x$.

Further, we shall get upper bounds of control times. The set $\bigcup_{n\ge2} M_n$ is appoximated with deficiency by the triangle with the base $y = 2n\varepsilon$ (diameter of M_n) and

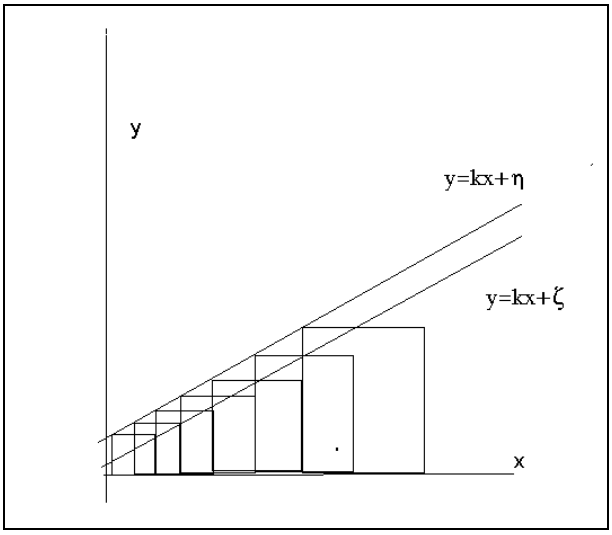

Figure 3: $\alpha = 1, a_0 > 1$.

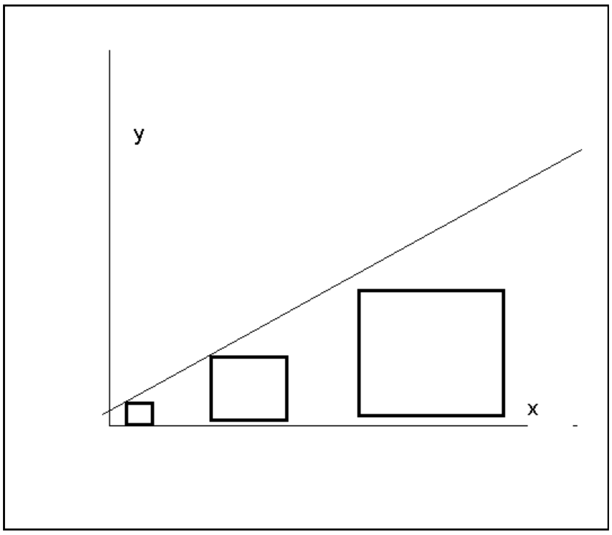

Figure 4: $\alpha = 1, a_0 = \frac{1}{2}$.

with the height $x = n\varepsilon(a_0 - 1)$ (see fig. 3). Therefore, the volume $\mathrm{vol}(\bigcup_{n\geq 2} M_n)$ is $\underline{V}_n = (n\varepsilon)^2(a_0 - 1)$. From $\underline{V}_n = V$ we get that the estimate of the control time is

$$\overline{n}_1(\varepsilon) = \frac{1}{\varepsilon}\frac{\sqrt{V}}{a_0 - 1},$$

i.e. it is the right part of (1).

The lower bounds can be obtained analogously from the equality $\overline{V}_n + v_n = V$.

3.2.2 The case of values α close to unity

Let $\gamma = 1 - \alpha$ be a small parameter. In this case, $x = n\varepsilon^{1-\gamma}$ is the center and $y = 2n\varepsilon$ is the radius of the accessibility set M_n, $y = \varepsilon^{\gamma}x$ is the equation of the line bounding the accessibility sets. Analogously as in the case $\alpha = 1$, the union of accessibility sets $\bigcup_{n\geq 2} M_n$ can be approximated with deficiency by the triangle with the base $2n\varepsilon$ (diameter of M_n) and with the height $n\varepsilon^{1-\gamma}$. (see fig. 4). Therefore, the volume $\underline{V}_n = \mathrm{vol}(\bigcup_{n\geq 2} M_n)$ is equal to $\underline{V}_n = (n\varepsilon)(n\varepsilon^{1-\gamma}) = n^2\varepsilon^{2-\gamma}$. From $\underline{V}_n = V$, we get that the lower bound of the control time is equal to

$$n_1(\varepsilon) = \frac{\sqrt{V}}{\varepsilon^{1-\gamma/2}}.$$

From the equation $n\varepsilon = a_0\varepsilon^{1-\gamma}$. i.e. from the equation $r_n(\varepsilon) = a(\varepsilon)$, we get the value n when the sets M_n are self-intersecting. Thus, $n_0(\varepsilon) = \frac{a_0}{\varepsilon^{\gamma}}$. The inequality $n_0 < n_1$ is true when $0 \leq \gamma < \frac{2}{3}$, i.e. when $\frac{1}{3} < \alpha \leq 1$, since the inequality $n_0 < n_1$ is equivalent to the inequality

$$\frac{a_0}{\varepsilon^{\gamma}} < \frac{\sqrt{V}}{\varepsilon^{1-\gamma/2}} \Leftrightarrow \varepsilon^{1-\frac{3}{2}\gamma} < \frac{\sqrt{V}}{a_0}.$$

Notice that if $\gamma \to 1$ then the lower bound and the upper bound of control time are close.

3.2.3 The case of values α close to zero: $0 \leq \alpha < 1/3$

We shall assume that the shift $a(\varepsilon) = \varepsilon^{\alpha}$ for sufficient small α. Further, we shall find boundaries for values of parameter α. The diameter of the ball M_n is equal to $2n\varepsilon$. Therefore, the value n_0 of numbers n, when the sets M_n are self-intersecting, can be calculated by formula

$$n_0(\varepsilon) = \frac{a(\varepsilon)}{2\varepsilon} = \frac{\varepsilon^{\alpha}}{2\varepsilon} = \frac{1}{2\varepsilon^{1-\alpha}}.$$

For n steps where $n \leq n_0$, the total volume $V_n = v_1 + \cdots + v_n$ is equal to (in special metric)

$$V_n = \varepsilon^2(1^2 + 2^2 + \cdots + n^2) \sim \varepsilon^2\frac{n^3}{3}.$$

We get the estimate of control time from equation $V_n = V$, i.e. from equation $\frac{\varepsilon^2 n^3}{3} = R^2$. Hence,

$$n_1(\varepsilon) = \left(3R^2\right)^{\frac{1}{3}}\left(\frac{1}{\varepsilon}\right)^{\frac{2}{3}}.$$

For small values of parameter ε it is true the inequality $n_1(\varepsilon) < n_0(\varepsilon)$ that is equivalent to the inequalities $\frac{1}{\varepsilon^{2/3}} < \frac{1}{2\varepsilon^{1-\alpha}}$, i.e., $\varepsilon^{\frac{1}{3}-\alpha} < 2$ for $0 \leq \alpha < \frac{1}{3}$. Therefore for $0 \leq \alpha < \frac{1}{3}$, the value of total volume will exceed the value the volume of state space before than the balls M_n will be self-intersecting.

3.3 The basic system has deterministic shifts (deterministic intermixing)

We shall assume that the values $a_n, n = 1, 2, \ldots$ of parameter a are assigned so that sets (3.1) are not intersected. Then the control system takes the form

$$x_{n+1} = x_n + a_{n+1} + uf_1(x_n), \mathrm{mod} 1, \quad n = 0, 1, 2, \ldots, \tag{8}$$

The volume $v_n = \mathrm{vol}(M-n)$ can be calculated by formula $v_n = (n\varepsilon)^d$ and the total volume

$$V_n = v_1 + \cdots + v_n = \varepsilon^d(1^d + \cdots + n^d) \sim \varepsilon^d \frac{n^{d+1}}{d+1}.$$

From the equation $V_n = 1$, we define the value of parameter n which is the control time. Thus, the control time is the left part of inequalities (1).

3.4 The basic system has random shifts (stochastic intermixing)

We shall assume that the value $a_n, n-1, 2.\dots$ of parameter a are assigned corresponding to the distribution $p(.)$. We shall assume also that $p(.)$ is density of uniform distribution and values x_0, a_n, $n = 1, \dots$ are independent (white noise). Then the control system takes the form (8). Further, we shall get the estimate for control time $T(x_0)$ in the mean on initial conditions.

Define the sequence of the accessibility sets from initial point x_0 for the sifts $a_1, \dots, a_n, \dots$. We call the point a_n as the center of the set M_n. Notice that the terms of sequence (6) independent of the terms of the sequence $a_0, a_1, \dots, a_n, \dots$ where $a_0 = x_0$. Denote $\vec{a}_n = (a_0, a_1, \dots, a_n)$ and

$$\begin{gathered}\mathcal{A}_0(\vec{a}_0) = M_0(\vec{a}_0), \quad \mathcal{B}_0(\vec{a}_0) = X \setminus M_0(\vec{a}_0),\\ \mathcal{A}_1(\vec{a}_1) = M_1(\vec{a}_1), \quad \mathcal{B}_1(\vec{a}_1) = X \setminus M_1(\vec{a}_1),\\ A_{n+1}(\vec{a}_{n+1}) = M_{n+1}(\vec{a}_{n+1}) \cap \mathcal{A}_n(\vec{a}_n),\\ B_{n+1}(\vec{a}_{n+1}) = M_{n+1}(\vec{a}_{n+1}) \setminus \mathcal{A}_{n+1}(\vec{a}_{n+1}).\end{gathered}$$

Then

$$M_{n+1}(\vec{a}_{n+1}) = A_{n+1}(\vec{a}_{n+1}) + B_{n+1}(\vec{a}_{n+1}). \tag{9}$$

Define

$$\begin{aligned}\mathcal{A}_{n+1}(\vec{a}_{n+1}) &= \mathcal{A}_n(\vec{a}_n) + B_{n+1}(\vec{a}_{n+1}),\\ \mathcal{B}_{n+1}(\vec{a}_{n+1}) &= \mathcal{B}_n(\vec{a}_n) \setminus B_{n+1}(\vec{a}_{n+1}).\end{aligned} \tag{10}$$

Hence,

$$\mathcal{B}_{n+1}(\vec{a}_{n+1}) = X \setminus \mathcal{A}_n(\vec{a}_n).$$

For relative volumes, it is fulfilled the following equations

$$\begin{aligned}\mathrm{vol}(\mathcal{A}_{n+1}(\vec{a}_{n+1})) &= \mathrm{vol}(\mathcal{A}_n(\vec{a}_n)) + \mathrm{vol}(B_{n+1}(\vec{a}_{n+1})),\\ \mathrm{vol}(\mathcal{B}_{n+1}(\vec{a}_{n+1})) &= \mathrm{vol}(\mathcal{B}_n(\vec{a}_n)) - \mathrm{vol}(B_{n+1}(\vec{a}_{n+1})).\end{aligned}$$

Define the mean values of relative volumes by formulas

$$\alpha_n = \int_X \mathrm{vol}(A_n(\vec{a}_n))dp_n(\vec{a}_n),$$

$$\beta_n = \int_X \mathrm{vol}(B_n(\vec{a}_n))dp_n(\vec{a}_n), \tag{11}$$

$$b_n = \int_X \mathrm{vol}(\mathcal{B}_n(\vec{a}_n))dp_n(\vec{a}_n), \tag{12}$$

where $p_n(\vec{a}_n)$ is the joint distribution of the values $(a_0, \dots, a_n) = \vec{a}_n$.

Since for any n, the invariant measure of shift $x + a_n$ is vol and $p(.) = 1$, then it is true the following equation

$$\frac{\beta_{n+1}}{b_n} = \frac{v_{n+1}}{v}, \tag{13}$$

where $v = \mathrm{vol}(X) = 1$. By formulas (10) – (13), it is follows that it is true the following equations

$$b_{n+1} = b_n - \beta_{n+1} = b_n - \beta_n v_{n+1} = b_n(1 - v_{n+1}), \quad b_0 = 1.$$

Obviously,

$$v_n = \frac{a}{V}(n\varepsilon)^d,$$

where the number a depends on Riemannian metric dist of manifold X. Therefore

$$b_n = (1 - v_n)\dots(1 - v_1)b_0.$$

Hence,

$$\ln b_n = \ln(1 - v_n) + \cdots + \ln(1 - v_1).$$

Further, provided the value v_n is small, we get that

$$\ln b_n \sim -(v_n + \cdots + v_1) = -\frac{a}{V}\varepsilon^d(1^d + \cdots + n^d) \sim -\frac{a}{V}\varepsilon^d\frac{n^{d+1}}{d+1}. \tag{14}$$

It follows from below formula (18). Hence,

$$b_n \sim \exp\left(-\frac{a}{V}\varepsilon^d\frac{n^{d+1}}{d+1}\right). \tag{15}$$

Let δ be any small number. Find a number n such that $b_n < \delta$. By formula (15), this inequality is equivalent to the following inequality

$$-\frac{a}{V}\varepsilon^d\frac{n^{d+1}}{d+1} < \ln\delta. \tag{16}$$

Further, we shall assume that the number δ depends on ε, i.e., $\delta = \delta(\varepsilon)$. Inequality (16) is equivalent to the following inequality

$$n_1(\varepsilon) := \left(\frac{a(d+1)}{V}\right)^{\frac{1}{d+1}}\left(\frac{1}{\varepsilon^d}\ln\frac{1}{\delta(\varepsilon)}\right)^{\frac{1}{d+1}} < n. \tag{17}$$

The left part of (17) give the estimate of mean control time. In the typical situation $\delta(\varepsilon) = a\varepsilon^d$. Notice that it is true the following formula

$$r_n(\varepsilon) := n\varepsilon \sim \varepsilon^{\frac{1}{d+1}}\ln\frac{1}{\delta(\varepsilon)}, \tag{18}$$

where $r_n(\varepsilon)$ is radius of the set M_n having the volume v_n. Formula (18) substantiates use formula (3.4).

4 Conclusion

The cases considered in previous sections show that the estimates of control times depend on the degree of chaos of the basic dynamical system. The more are the values of the shifts , the more are the degrees of chaos. The more is randomness, the less is the time of control. Thus, this situation for neutral systems on torus is analogous to the situation for hyperbolic systems on torus.

References

[1] Ott E, Grebodi C., Yorke J. Controlling chaos. // Physical review letters. — 1990. v. 64, No 11 — pp. 1195– 1199.

[2] Yang L., Liu Z., Zheng Y."Midle" periodic orbit and its application to chaos control. // International journal of bifurcation and chaos. – 2001. v. 12, No 8, pp. 1869 – 1876.

[3] Khryashchev S.M. Control times in systems of chaotic behavior. I. Its Estimation.// Autom. Remote Control. – 2004, v. **64** (No 10), pp.1566–1579.

[4] Khryashchev S.M. Control times in systems of chaotic behavior. II. Its Estimation.// Autom. Remote Control. – 2004. v. **64** (No 11), pp.1782–1792.

[5] Khryashchev S.M. Estimation of transport times for nonstationary systems generated by quasi-linear hyperbolic maps. // Electronic journal "Differential equations and processes of control", http://www.neva.ru/journal/ – 2004. No 4, pp. 77 — 115.

[6] Khryashchev S.M. Application of statistic methods for estimation of control times for determinate dynamical systems.// Electronic journal "Difererential equations and processes of control", http://www.neva.ru/journal/ – 2006. No 1, pp. 1-35.

[7] Khryashchev S.M. Estimation of transport times for chaotic dynamical control systems. Proceedings of International Conference Physics and Control (PhysCon-05) – 2005. pp. 28 – 33.

[8] Newhouse Sh. Entropy and Volume // Ergodic Theory and Dynamical Systems. — 1988. v. 8* (Conley Memoria Issue) — p. 283 – 300.

[9] Yomdin Y. Volume growth and entropy // Israel J.math. – 1987. v.57, pp.285 – 300.

[10] Katok A., Hasselblatt B. Introduction to the modern theory of dynamical systems. 1999.