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MULTIESTIMATES FOR LINEAR-GAUSSIAN SYSTEMS UNDER COMMUNICATION CONSTRAINTS

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Abstract
A state estimation problem is an important part of the more general control problem under incomplete information. In many cases control strategies are built on the base of various algorithms of state estimation. In this work, estimation problems for linear systems are considered under mixed disturbances. The determined disturbances are constrained by convex and compact sets, and the random ones are standard Wiener processes. Random information sets (multiestimates) are defined. In the absence of random components the multiestimates coincide with information sets from the theory of guaranteed estimation. The structure of multiestimates is considered: they are the sum of a random vector and a determinate set, which depend on parameters. In turn, the given set of parameters unequivocally defines the conditional and unconditional probability of inclusion of the multiestimate in a covering set. Special cases are considered, and the question on the form of the covering set is discussed. A modified problem is considered under communication constraints, in which the limited capacity of the digital data link and random errors in a communication channel are taken into consideration. Relations between the accuracy of restoration of multiestimate's parameters, the length of a transferred word, and a transmission frequency are received in the noiseless case. A number of results is illustrated on an example.

Key words
Multiestimates, communication channel, estimation, control, filtering.

1 Introduction

There are many control problems under incomplete information in which state estimation algorithms are used. A motion correction problem is one of them [Kurzhanski, 1977]. To solve this problem it is necessary at some instant to find a set of state vectors compatible with measurements. After that on the rest of time interval, a control is chosen to minimize a terminal cost uniformly for all the initial conditions from the found set. Hence, algorithms for set estimates are necessary and useful for control. For linear multistage systems with the mixed uncertainty including both determined and random disturbances, in [Katz and Kurzhanski, 1975] estimates in the form of sets were offered. However these estimates were not reduced to information sets from [Kurzhanski, 1977; Kurzhanski and Vályi, 1996] in the absence of random disturbances. In this connection, in [Anan'ev, 2000] the concept of random information set for systems with discrete time and with the mixed uncertainty is entered. The entered sets are already reduced to earlier known in the absence of random disturbances, but generally demand the further processing, as they depend on random and not observable parameters. In work [Anan'ev, 2007], a generalization of the random information sets named for brevity multiestimates is offered for multistage stochastic inclusions. An inclusion of the multiestimates in a covering set taking into account the communication constraints is considered in [Anan'ev, 2008]. The received results in many respects lean on [Savkin and Petersen, 2003; Huber, 1981; Matheron, 1975]. Various aspects of estimation theory with uncertain disturbances are examined in [Schweppe, 1973; Chernousko, 1994]. In this work, linear continuous time systems are considered mostly in a case when the determined disturbances are subordinated to geometrical restrictions and the random disturbances are Wiener processes. At the same time, it makes sense to give in the beginning the general statement of a problem and to plan the scheme of its decision. A realization of this scheme for rather simple case is offered below.

Let $(\Omega, \mathcal{F})$ be a standard measurable space, and P be a probability on $\mathcal{F}$. Consider two functions $x(d, \omega)$ and $y(d, \omega)$ mapping $D \times \Omega$ in U and S respectively, where D, U, S are separable metrizable spaces. The function $x(d, \omega)$ not observable in experiment is supposed continuous in d and measurable in ω. The observable function $y(d, \omega)$ has analogous properties. Let's enter the random closed set of compatible parameters:

$\mathcal{D}(s,\omega) = y_\omega^{-1}(\{s\})$. We substitute it in the function x: $\mathcal{X}(s,\omega) = x(\mathcal{D}(s,\omega),\omega)$. The received set is named the **random information set** or the **multiestimate** of the value x with respect to observation $y = s$.

Problem. *For $\varepsilon > 0$ it is necessary to find a set $\hat{\mathcal{X}}(s) \subset U$ so that $p(d,s,\{\omega : \mathcal{X}(s,\omega) \subset \hat{\mathcal{X}}(s)\}) \geq 1-\varepsilon$ for all $d \in D$, or, reducing search, only for all $d \in D_s = \cup_\omega \mathcal{D}(s,\omega)$. Here $y(d,\omega) = s$, and $p(d,s,A) = E(I_A \mid y(d,\cdot) = s)$ is a regular conditional probability* [Shiryayev, 1991].

In a case when functions x, y do not depend on parameter d, the Problem is reduced to construction of a confidence set with conditional probability of inclusion not smaller $1-\varepsilon$. If functions x, y depend only on parameter d, the Problem coincides with one of construction of the information set in the theory of guaranteed estimation [Kurzhanski, 1977; Kurzhanski and Vályi, 1996]. In Problem, not all aspects and the difficulties meeting in specific cases are reflected far. In particular, for evolutionary systems it is rather important to find the recurrent procedure of construction of covering sets. Besides, the Problem can be complicated due to the presence of communication constraints. In this case, the information to the Center of Processing Information (CPI) may arrive during the discrete instants by the words consisting of integers and having the limited length. The communication channel for simplicity is often supposed noiseless and not having delay. The coding device in a communication channel is exploited for an information transfer about parameters of the multiestimate and of the signal. In CPI the information is decoded and utilized for the approximated restoration of a covering set. Other schemes are used for channels with noise. One of them we suggest at the end of the paper.

2 Linear-Gaussian Continuous Systems

Consider a linear system in Ito's form:

$$\left.\begin{array}{l} dx = (A(t)x + v(t))dt + d\xi(t),\ t \in [0,T], \\ dy = (C(t)x + w(t))dt + d\eta(t),\ y_0 = 0, \end{array}\right\} \quad (1)$$

where $x(t) \in R^n$ is the unmeasured vector, $y(t) \in R^m$ is the measured one. The initial state x_0 has Gaussian distribution $x_0 \sim N(\bar{x}_0,\gamma_0)$ with mean value $\bar{x}_0$ and the covariance γ_0. Uncertain parameters in (1) are restricted by the convex and compact constraints $\bar{x}_0 \in \bar{X}_0$, $v(t) \in V$, $w(t) \in W$. The random processes $\xi(t)$, $\eta(t)$ in (1) are supposed to be Wiener ones with zero means and $\mathrm{cov}(d\xi,d\xi) = Q(t)dt$, $\mathrm{cov}(d\eta,d\eta) = R(t)dt$. The processes $\xi(t)$, $\eta(t)$ are independent and do not depend on the initial vector x_0.

Recall that for any matrix C the pseudoinverse matrix C^- is defined by conditions $CC^-C = C$, $C^- = C'L = MC'$, where L, M are matrices of the suitable size, the symbol $'$ means the transposition, [Liptser and Shiryayev, 2000]. Let $\ker C = \{x : Cx = 0\}$, $\mathrm{im}\, C = \{y : y = Cx,\ \exists x\}$, then the following equality $\{x : Cx \in W\} = C^-(W \cap \mathrm{im}\, C) \oplus \ker C$ takes place.

We introduce purely random processes defined by

$$\left.\begin{array}{l} dx^0 = A(t)x^0dt + d\xi(t),\ x_0^0 \sim N(0,\gamma_0), \\ dy^0 = C(t)x^0dt + d\eta(t),\ y_0^0 = 0,\ t \in [0,T], \end{array}\right\} \quad (2)$$

and determinate functions for which

$$\left.\begin{array}{ll} \dot{x}^1 = A(t)x^1 + v(t), & x_0^1 = \bar{x}_0, \\ \dot{y}^1 = C(t)x^1 + w(t), & y_0^1 = 0, \quad t \in [0,T]. \end{array}\right\} \quad (3)$$

It is clear that $x(t) = x^0(t) + x^1(t)$, $y(t) = y^0(t) + y^1(t)$. Using the function $y^{1t}(\cdot) = \{y^1(s) : s \in [0,t]\}$ we construct the determinate information set, which is denoted by $\mathcal{X}^1(t,y)$, [Kurzhanski, 1977]. To this end for given $\Delta > 0$ and for any $t \in [0,T]$, consider the finite sequence

$$y^{1t}(\cdot\Delta) = \{y^1(\Delta),\ldots,y^1(j_t^\Delta\Delta),y^1(t)\}$$

consisting of $j_t^\Delta + 1$ elements, where $j_t^\Delta = \lfloor t/\Delta \rfloor$. Here $\lfloor\cdot\rfloor$ denotes the integer part of a real number. Let $X(t,s)$ be the fundamental matrix of the first equation in (1).

Let's put $(j_t^\Delta + 1)\Delta = t$ by definition and introduce the designations:

$C_i^\Delta = \int_{(i-1)\Delta}^{i\Delta} C(s)X(s,(i-1)\Delta)ds$, $X_i^\Delta = X(i\Delta,(i-1)\Delta)$, $\mathcal{W}_i^\Delta = \{\int_{(i-1)\Delta}^{i\Delta} w(s)ds : w(s) \in W\} \subset R^n$, $\mathcal{V}_i^\Delta = \Big\{\Big[\int_{(i-1)\Delta}^{i\Delta} X(i\Delta,s)v(s)ds;$ $\int_{(i-1)\Delta}^{i\Delta}\int_s^{i\Delta} C(t)X(t,s)dtv(s)ds\Big] : v(s) \in V\Big\} \subset R^{n+m}$, $\tilde{C}_i^\Delta = [C_i^\Delta,F]$, $\tilde{X}_i^\Delta = [X_i^\Delta,D]$, where $F = [O_{m\times n},I_m]$, $D = [I_n,O_{n\times m}]$, I_n is a unity matrix, $O_{m\times n}$ is a zero matrix. Using this notation we can write the multistage system

$$\left.\begin{array}{l} x_i^1 = X_i^\Delta x_{i-1}^1 + Dv,\ v \in \mathcal{V}_i^\Delta,\ y^1(i\Delta) = \\ y^1((i-1)\Delta) + C_i^\Delta x_{i-1}^1 + Fv + w,\ w \in \mathcal{W}_i^\Delta, \\ x_0^1 \in \bar{X}_0,\ y^1(0) = 0,\ i = 1,\ldots,j_t^\Delta + 1. \end{array}\right\} \quad (4)$$

From now on, we use the recurrent sets defined by the equations:

$$\left.\begin{array}{l} \bar{\mathcal{X}}_i^\Delta = \left(\mathcal{X}_{i-1}^\Delta \times \mathcal{V}_i^\Delta\right) \bigcap \left(\tilde{C}_i^{\Delta -}((y_i^1 - \mathcal{W}_i^\Delta) \bigcap \right. \\ \left. \mathrm{im}\tilde{C}_i^\Delta) \oplus \ker\tilde{C}_i^\Delta\right),\ y_i^1 = y^1(i\Delta) - y^1((i-1)\Delta), \\ \mathcal{X}_i^\Delta = \tilde{X}_i^\Delta\bar{\mathcal{X}}_i^\Delta,\ \mathcal{X}_0^\Delta = \bar{X}_0,\ i = 1,\ldots,j_t^\Delta + 1. \end{array}\right\} \quad (5)$$

Theorem 1. *The following convergence takes place in Hausdorf metric uniformly with respect to $t \in [0,T]$:* $\lim_{\Delta\to 0}\mathcal{X}_{j_t^\Delta+1}^\Delta = \mathcal{X}^1(t,y)$.

Theorem 1 follows from the compactness of the initial set and the weak compactness of disturbances.

Definition. The sets

$$\mathcal{X}(t,y) = \mathcal{X}^1(t,y) + x^0(t), \quad t \in [0,T], \quad (6)$$

are called the random information sets or the multiestimates.

The sets $\mathcal{X}^1(t,y)$ and (6) coincide in the absence of random components. Owing to the law of averages and Tchebychev's inequality [Shiryayev, 1991] we have

$$\sum_{k=1}^N y(t)_{(k)}/N \to Ey(t) = y^1(t) \quad \text{if} \quad N \to \infty,$$

where $y(t)_{(k)}$ are realizations of measured vectors. Unfortunately, repetition of experiences and ways of specification of multiestimates on the basis of given relation, as a rule, are impossible. Therefore in the present work other methods connected with construction of covering set $\mathcal{Z}(t)$, depending on the measured signal $y^t(\cdot)$ and providing inclusion $\mathcal{X}(t,y) \subset \mathcal{Z}(t)$ with given conditional probability irrespective of realization of uncertain parameters $\{\bar{x}_0, v(\cdot), w(\cdot)\}$, are offered.

3 Constructing of covering set

In system (1), the sum $y(t) = y^0(t) + y^1(t)$ is measured, and it is obviously impossible to find out the summand $y^1(t)$, on which the determinate set in multiestimate (6) is under construction. Let's act as follows. Let the symbol $\mathcal{Y}^{1t}$ mean the set of all signals in system (3) on the segment $[0, t]$. It is convex compact set in the space of all continuous vector functions. Fixing $y^{1t} \in \mathcal{Y}^{1t}$ and marking out the random element $y^{0t} = y(\cdot) - y^{1t}$, we can find the conditional distribution $\mathrm{Law}(x^0(t) \mid y^{0t}, y^{1t}) = N(m^0(t), \gamma(t))$ for the vector $x^0(t)$ from (2). According to Kalman's theory [Liptser and Shiryayev, 2000; Shiryayev, 1991] the parameters $m^0(t)$, $\gamma(t)$ satisfy the equations

$$\left.\begin{array}{l} dm^0 = A(t)m^0dt + K(t)(dy^0 - C(t)m^0dt), \\ m^0(0) = 0; \quad K(t) = \gamma(t)C'(t)R^{-1}(t), \\ \dot{\gamma} = A(t)\gamma + \gamma A'(t) + Q(t) - K(t)C(t)\gamma, \\ \gamma(0) = \gamma_0. \end{array}\right\} \quad (7)$$

Let $k(t)$ be a vector satisfying the first equation in (7), where the signal y^0 is replaced by y, and $\alpha(t)$ be a vector satisfying to the similar equation, where the signal y^0 is replaced by $-y^1$. We have the obvious equality $m^0(t) = k(t) + \alpha(t)$ that follows from the relation $y^0(t) = y(t) - y^1(t)$. We form the set

$$\mathcal{Z}(t) = k(t) + \bigcup_{y^{1t} \in \mathcal{Y}^{1t}} (\alpha(t) + \mathcal{X}^1(t, y)) \quad (8)$$

and use it as a covering one for the multiestimate $\mathcal{X}(t, y)$ from (6). Now our goal is to count up the conditional probability $P(\mathcal{X}(t, y) \subset \mathcal{Z}(t) \mid y^{0t}, y^{1t})$. In force (6), (8) we have the equality of events

$$\{\mathcal{X}(t, y) \subset Z(t)\} = \{x^0(t) - m^0(t) \in \mathcal{D}(t)\}, \quad (9)$$

where

$$\left.\begin{array}{l} \mathcal{D}(t) = \Big\{x : x + \alpha(t) + \mathcal{X}^1(t, y) \\ \subset \bigcup_{y^{1t} \in \mathcal{Y}^{1t}} \big(\alpha(t) + \mathcal{X}^1(t, y)\big)\Big\}. \end{array}\right\} \quad (10)$$

The set $\mathcal{D}(t)$ depends on the concrete realization y^{1t} and contains zero at any such realization. Thus, the required conditional probability subject to (9), (10) is equal to

$$P(\mathcal{X}(t, y) \subset \mathcal{Z}(t) \mid y^{0t}, y^{1t}) = \int_{\mathcal{D}(t)} f(x, \gamma(t))dx, \quad (11)$$

where $f(x, \gamma)$ is the density of a gaussian distribution with zero mean and the covariance γ. As the distribution can be singular, we give an expression of the type (11), which is true and for the singular case. Consider the matrix representation $\gamma = T\Lambda^{1/2}(\Lambda^{1/2})'T'$, where T is the orthogonal matrix, $\Lambda^{1/2} = [\mathrm{diag}\{\lambda_1^{1/2}, \dots, \lambda_k^{1/2}\}; O_{(n-k)\times k}]$. Here $k = \mathrm{rank}\,\gamma$, λ_i are eigenvalues that distinct from zero of the matrix γ. Then equality (11) is transformed to the following:

$$\left.\begin{array}{l} P(\mathcal{X}(t, y) \subset \mathcal{Z}(t) \mid y^{0t}, y^{1t}) \\ = \int_{(\Lambda^{1/2})^- T'\mathcal{D}(t)} f(x, I_k)dx, \end{array}\right\} \quad (11)'$$

where I_k is the unity $k \times k$ matrix.

The value y^{1t} is actually realized together with y^t and is unknown. Therefore, it is possible to guarantee the inclusion $\mathcal{X}(t, y) \subset \mathcal{Z}(t)$ with conditional probability not smaller than the minimum of value (11) ((11)$'$) over all $y^{1t} \in \mathcal{Y}^{1t}$. In order to provide the high probability of covering of the multiestimates, consider the concentration ellipsoid $\mathcal{E}_t(l) = \{x \in \mathrm{im}\,\gamma(t) : x'\gamma(t)^- x \le l^2\}$ such that $P(x^0(t) - m^0(t) \in \mathcal{E}_t(l)) = \int_{\|x\| \le l} f(x, I_k)dx \ge 1 - \varepsilon$, $k = \mathrm{rank}\,\gamma(t)$. Then we have the equality of the events

$$\{\mathcal{X}(t, y) \subset \mathcal{Z}(t) + \mathcal{E}_t(l)\} = \{x^0(t) - m^0(t) \in \tilde{\mathcal{D}}(t)\}, \quad (12)$$

where

$$\left.\begin{array}{l} \tilde{\mathcal{D}}(t) = \Big\{x : x + \alpha(t) + \mathcal{X}^1(t, y) \\ \subset \bigcup_{y^{1t} \in \mathcal{Y}^{1t}} \big(\alpha(t) + \mathcal{X}^1(t, y)\big) + \mathcal{E}_t(l)\Big\}. \end{array}\right\} \quad (13)$$

Whereas $\mathcal{E}_t(l) \subset \tilde{\mathcal{D}}(t)$, the inequality

$$P(\mathcal{X}(t, y) \subset \mathcal{Z}(t) + \mathcal{E}_t(l) \mid y^{0t}, y^{1t}) \ge 1 - \varepsilon \quad (14)$$

is valid for any signal y^{1t}.

4 Evolutionary equations for covering set with the account of confidence sets

In this section, we try to reduce the exhaustive search of uncertain signals y^{1t} under construction of covering set for a multiestimate and simultaneously to construct the evolutionary covering set. Note that $y^0(t) \sim N(0, J(t))$, where matrices $J(t)$ are defined by the evolutionary relations

$$\dot{J} = C(t)L + L'C'(t) + R(t), \quad L(0) = O_m;$$
$$\dot{L} = A(t)L + PC'(t), \quad L(0) = O_{n\times m};$$
$$\dot{P} = A(t)P + PA'(t) + Q(t), \quad P(0) = \gamma_0.$$

For any $t \in [0, T]$ let us choose the concentration ellipsoid $\mathbf{Y}_t(l) = \{x \in \mathrm{im}\, J(t) : x'J(t)^- x \le l^2\}$ such that $P(y^0(t) \in \mathbf{Y}_t(l)) = \int_{\|x\| \le l} f(x, I_k)dx > 0.99$, $k = \mathrm{rank}\, J(t)$. Then it is possible to assume practically that $y^1(t) \in y(t) - \mathbf{Y}_t(l)$, i.e. the confidence set $y(t) - \mathbf{Y}_t(l)$ covers the signal $y^1(t)$. Therefore in (8), (10), and (13), we can consider the narrower set

$$\left.\begin{array}{l} \bar{\mathcal{Y}}^{1t} = \{y^{1t} \in \mathcal{Y}^{1t} : y^1(s) \in y(s) - \mathbf{Y}_s(l), \\ s \in [0, t]\}. \end{array}\right\} \quad (15)$$

In case of small matrices Q, R and γ_0 in (1) inclusion (15) essentially reduces the search of the determined signals under construction of covering set. For simplification in case of small random parameters it is possible a little to expand inclusion (15):

$$y^1(s) \in y(s) - \mathbf{Y}_s(l), \; s \in [0, t]. \quad (15)'$$

With the account of restrictions (15) consider an evolutionary construction of such covering set, which in the absence of random parameters, coincides with $\mathcal{X}^1(t, y)$. Let us pass to the approximating discrete scheme (4), (5). Purely random system (2) in discrete scheme can be written as

$$x_i^0 = X_i^\Delta x_{i-1}^0 + \xi_i^\Delta, \quad \mathrm{cov}(\xi_i^\Delta, \xi_i^\Delta) = Q_i^\Delta$$
$$= \int_{(i-1)\Delta}^{i\Delta} X(i\Delta, s)Q(s)X'(i\Delta, s)ds;$$
$$y_i^0 = C_i^\Delta x_{i-1}^0 + \eta_i^\Delta, \quad \mathrm{cov}(\eta_i^\Delta, \eta_i^\Delta) = R_i^\Delta$$
$$= \int_{(i-1)\Delta}^{i\Delta} \Big(\Big(\int_s^{i\Delta} C(t)X(t, s)dt\Big)Q(s)$$
$$\Big(\int_s^{i\Delta} C(t)X(t, s)dt\Big)' + R(s)\Big)ds, \quad \mathrm{cov}(\xi_i^\Delta, \eta_i^\Delta)$$

$= S_i^\Delta = \int_{(i-1)\Delta}^{i\Delta} X(i\Delta, s)Q(s)\Big(\int_s^{i\Delta} C(t)$
$X(t,s)dt\Big)' ds.$

Note that $y_i^0 \sim N(0, J_i)$, where matrices J_i defined by the recurrent relations

$$\left.\begin{array}{l} J_i = C_i^\Delta P_{i-1} C_i^{\Delta'} + R_i^\Delta, \quad P_i \\ = X_i^\Delta P_{i-1} X_i^{\Delta'} + Q_i^\Delta, \quad P_0 = \gamma_0, \quad i \geq 1. \end{array}\right\} \quad (16)$$

Choosing the concentration ellipsoid $\mathbf{Y}_i^\Delta(l) = \{x \in \mathrm{im} J_i : x' J_i^- x \leq l^2\}$ such that $\int_{\|x\| \leq l} f(x, I_k)dx = P(y_i^0 \in \mathbf{Y}_i^\Delta(l)) > 0.99,\ k = \mathrm{rank}\, J_i$, we can assume that the inclusion $y_i^{1\Delta} \in y_i - \mathbf{Y}_i^\Delta(l)$ is practically valid.

Introduce the sets

$$\left.\begin{array}{l} \tilde{\mathcal{X}}_i^\Delta = \tilde{X}_i^\Delta(\tilde{\mathcal{X}}_{i-1}^\Delta \times \mathcal{V}_i^\Delta), \quad \tilde{\mathcal{X}}_0^\Delta = \bar{X}_0, \\ \mathcal{Y}_i^\Delta = \tilde{C}_i^\Delta(\tilde{\mathcal{X}}_{i-1}^\Delta \times \mathcal{V}_i^\Delta) + \mathcal{W}_i^\Delta, \\ \mathbf{Y}_i^\Delta(l, y) = \mathcal{Y}_i^\Delta \bigcap \big(y_i - \mathbf{Y}_i^\Delta(l)\big). \end{array}\right\} \quad (17)$$

The set $\mathbf{Y}_i^\Delta(l, y)$ represents the family of all signals y_i^1 that can be realized in system (4) accordance with the confidence set $y_i - \mathbf{Y}_i^\Delta(l)$. In the absence of random parameters we obtain the singleton $\mathbf{Y}_i^\Delta(l, y) = \{y_i^1\}$.

Using Kalman's filter we get

$$\left.\begin{array}{l} m_i^0 = X_i^\Delta m_{i-1}^0 + K_{i-1}(y_i^0 - C_i^\Delta m_{i-1}^0), \\ m_0^0 = 0;\ K_{i-1} = (S_i^\Delta + X_i^\Delta \gamma_{i-1} C_i^{\Delta'})(R_i^\Delta \\ + C_i^\Delta \gamma_{i-1} C_i^{\Delta'})^-, \quad \gamma_i = X_i^\Delta \gamma_{i-1} X_i^{\Delta'} + Q_i^\Delta \\ - K_{i-1}(S_i^\Delta + X_i^\Delta \gamma_{i-1} C_i^{\Delta'})', \quad i \geq 1, \end{array}\right\} \quad (18)$$

where $\mathrm{Law}(x_i^0 \mid y^{0i}, y^{1i}) = N(m_i^0, \gamma_i)$. Here $y^{1i} = \{y_1^1, \dots, y_i^1\}$ and y^{0i} is defined similarly.

As in (8), let k_i be a vector satisfying the first equation in (18), where the signal y^0 is replaced by y, and α_i be a vector satisfying to the similar equation, where the signal y^0 is replaced by $-y^{1\Delta}$. Then we can form the set

$$\mathcal{Z}_i = k_i + \bigcup\nolimits_{y^1 \in \mathbf{Y}^\Delta(l,y)} (\alpha_i + \mathcal{X}_i^\Delta) \quad (19)$$

and use it as a covering one for the discrete multiestimate $\mathcal{X}_i^\Delta(y) = x_i^0 + \mathcal{X}_i^\Delta$. Note that $\mathcal{Z}_i = \mathcal{X}_i^\Delta$ if the random parameters are absent.

Unfortunately, covering sets (19) are not recurrent as a rule. Now we a little expand sets (19) and make their recurrent ones. We examine the recurrent system of sets:

$$\left.\begin{array}{l} \mathcal{A}_i = (X_i^\Delta - K_{i-1} C_i^\Delta)\mathcal{A}_{i-1} - K_{i-1}\mathbf{Y}_i^\Delta(l, y), \\ \mathcal{A}_0 = \{0\},\ \tilde{\mathcal{X}}_i^\Delta = \Big(\hat{\mathcal{X}}_{i-1}^\Delta \times \mathcal{V}_i^\Delta\Big) \bigcap \\ \Big\{z \in R^{2n+m} : \tilde{C}_i^\Delta z \in \mathbf{Y}_i^\Delta(l, y) - \mathcal{W}_i^\Delta\Big\},\ \hat{\mathcal{X}}_i^\Delta \\ = \tilde{X}_i^\Delta \tilde{\mathcal{X}}_i^\Delta,\ \hat{\mathcal{X}}_0^\Delta = \bar{X}_0,\ i = 1, \dots, j_t^\Delta + 1. \end{array}\right\} \quad (20)$$

Theorem 2. *Let $\mathcal{A}_i$, $\tilde{\mathcal{X}}_i^\Delta$, $\hat{\mathcal{X}}_i^\Delta$ be the sets defined by* (20). *Then with the account of restrictions* (17) *we receive $\bigcup_{y^1 \in \mathbf{Y}^\Delta(l,y)} (\alpha_i + \mathcal{X}_i^\Delta) \subset \mathcal{A}_i + \hat{\mathcal{X}}_i^\Delta,\ i \geq 1$. The set $\hat{\mathcal{Z}}_i = k_i + \mathcal{A}_i + \hat{\mathcal{X}}_i^\Delta$ covers the multiestimate $\mathcal{X}_i^\Delta(y)$ with conditional probability*

$$P(\mathcal{X}_i^\Delta(y) \subset \hat{\mathcal{Z}}_i \mid y^{0i}, y^{1i}) = \int_{\mathcal{B}_i} f(x, \gamma_i)dx, \quad (21)$$

where the function f is the same as in (11), *and the set $\mathcal{B}_i$ is of the form*

$$\mathcal{B}_i = \Big\{x : x + \alpha_i + \mathcal{X}_i^\Delta \subset \mathcal{A}_i + \hat{\mathcal{X}}_i^\Delta\Big\}. \quad (22)$$

The minimum of relation (21) *over all $y^1 \in \mathbf{Y}^\Delta(l, y)$ is the guaranteed result of inclusion's probability. Note that $\hat{\mathcal{Z}}_i = \mathcal{X}_i^\Delta$ if the random parameters are absent.*

This theorem is established by induction. The same as in (12) – (14), it is possible to add the ellipsoid $\mathcal{E}_i(l)$ to set $\hat{\mathcal{Z}}_i$ in order to obtain the guaranteed result of type (14).

Note that for any given signal y^{1t} in (3) the $k_{j_t^\Delta+1}$ converges almost surely to the $k(t)$ when $\Delta \to 0$. Thus, taking into consideration Theorem 1 we come to

Theorem 3. *Suppose $d(\cdot,\cdot)$ is the Hausdorf metric for compact sets. Then $d(\mathcal{Z}_{j_t^\Delta+1}, \mathcal{Z}(t)$ converges almost surely to zero when $\Delta \to 0$. Here $\mathcal{Z}(t)$ is defined by* (8), *where the set $\mathcal{Y}^{1t}$ is replaced by* (15). *The same convergence takes place for the multiestimates $\mathcal{X}_{j_t^\Delta+1}^\Delta(y)$ and $\mathcal{X}(t, y)$.*

5 Example

Consider the system of Ito's equations:

$dx = d\xi, \quad t \in [0, T], \quad \mathrm{Law}\,(x_0) = N(\bar{x}_0, \gamma_0),$
$dy = d\eta + (x + w)dt, \quad y_0 = 0, \quad \|w(t)\| \leq \nu,$
$x, y \in R^2, \quad \|\bar{x}_0\| \leq \mu.$

Here $w(\cdot)$ is an uncertain function; $\xi(t)$, $\eta(t)$ are independent Wiener processes with $\mathrm{cov}(d\xi, d\xi) = Qdt$, $\mathrm{cov}(d\eta, d\eta) = Rdt$. Let's divide the segment $[0, T]$ into n_1 equal parts and set $\Delta = T/n_1$, $t_i = \Delta i$, $i = 1, \dots, n_1$. Integrating the system by steps with the help of Cauchy's formula we get

$x_i = x_{i-1} + \xi_i, \quad \xi_i = \xi(t_i) - \xi(t_{i-1}), \quad y_i$
$= \Delta x_{i-1} + \eta_i + w_i, \quad \|w_i\| \leq \nu\Delta, \quad \mathrm{cov}\,(\eta_i, \eta_i)$
$= R\Delta + Q\Delta^3/3, \quad \mathrm{cov}\,(\xi_i, \eta_i) = Q\Delta^2/2,$
$\mathrm{cov}\,(\xi_i, \xi_i) = Q\Delta; \quad i = 1, \dots, n_1.$

Here $C^\Delta = \Delta I_2$, $V = \{0\}$, $\mathcal{W}^\Delta = \{w : \|w\| \leq \nu\Delta\}$, $\bar{X}_0 = \{x : \|x\| \leq \mu\}$. The determinate component is constant: $x_i^1 \equiv \bar{x}_0$, and the determinate sets are of the form $\mathcal{X}_i^\Delta = \bar{X}_0 \cap \cap_{i=1}^t (y_i^1 - W)/\Delta$. Let us write system (18):

$m_i^0 = m_{i-1}^0 + K_{i-1}(y_i^0 - \Delta m_{i-1}^0),\ m_0^0 = 0;$
$K_{i-1} = (Q\Delta^2/2 + \gamma_{i-1}\Delta)(R\Delta + Q\Delta^3/3$
$+\Delta^2\gamma_{i-1})^-, \quad \gamma_i = \gamma_{i-1} + Q\Delta - K_{i-1}(Q\Delta^2/2$
$+\gamma_{i-1}\Delta), \quad i \geq 1.$

Let, for example, $Q = q^2 I_2$, $R = r^2 I_2$, $\gamma_0 = \overline{\gamma} I_2$. Then $\gamma_i \equiv \gamma_0$, where $\overline{\gamma} = q\sqrt{r^2 + q^2\Delta^2/12}$, $K = 2\sqrt{3}/(\sqrt{3}\Delta + (12r^2q^{-2} + \Delta^2)^{1/2}) I_2$. The vector m_i^0 submits to the stable system $m_i^0 = m_{i-1}^0 + K(y_i^0 - \Delta m_{i-1}^0)$. The influence of initial conditions in this system under large i is negligible. The system of equations (16) gives $J_i = (\Delta^2\overline{\gamma} + r^2\Delta + q^2\Delta^3(i - 2/3))I_2$. For any i we choose we choose l in the ellipsoid $\mathbf{Y}_i(l) = \{x : x' J_i^{-1} x \leq l^2\}$ so that $P(y_i^0 \in \mathbf{Y}_i(l)) = 1 - \exp(-l^2/2) > 0.99$. Therefore, $l > 2\sqrt{\ln 10}$. Let us set numerical parameters: $\mu = 1$, $\nu = 0.5$, $q^2 = 0.06$, $r^2 = 0.01$, $T = 10$, $\Delta = 0.1$, $n_1 = 100$. Then $\overline{\gamma} = 0.0246$, $K = 2.1774 I_2$, $m_i^0 = 0.7823 m_{i-1}^0 + 2.1774 y_i^0$. We write the system:

$\hat{\mathcal{X}}_i^\Delta = \hat{\mathcal{X}}_{i-1}^\Delta \cap 10(\mathbf{Y}_i^\Delta(l, y) - \mathcal{W}^\Delta), \quad \mathbf{Y}_i^\Delta(l, y)$
$= \{\|x\| \leq 0.15\} \cap (y_i - \mathbf{Y}_i(l)); \mathbf{Y}_i(l) = \big\{x$
$: \|x\| \leq 2\sqrt{\ln 10(\Delta^2\overline{\gamma} + r^2\Delta + q^2\Delta^3(i - 2/3))}\big\}.$

It follows from this that the covering set can be constructed by formulas $\hat{\mathcal{Z}}_i = k_i + \mathcal{A}_i + \hat{\mathcal{X}}_i^\Delta$.

It is necessary to add the concentration ellipsoid $\mathcal{E}_i(l) = \{x : \|x\| \leq \sqrt{\overline{\gamma}}l\}$ to $\hat{\mathcal{Z}}_i$ in order to obtain the guaranteed result of type (14), where l is defined by $1 - \exp(-l^2/2) = 1 - \varepsilon$, $l = \sqrt{-2\ln\varepsilon}$. The typical picture of covering is shown on Fig.1. Here the green spot is a multiestimate $\mathcal{X}_i^{\Delta}(y)$, the blue spot is a covering set $\hat{\mathcal{Z}}_i$, the red cross is a true state, and the red star is true Kalman's estimate.

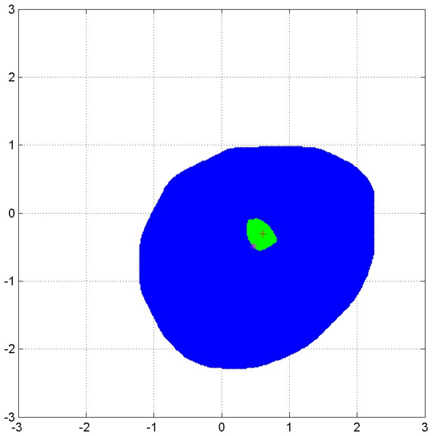

Fig. 1

6 Accounting of communication constraints

We have chosen above the concentration ellipsoid so that $P(y_i^0 \in \mathbf{Y}_i^{\Delta}(l)) > 0.99$. Therefore, the inclusion $y_i = y_i^0 + y_i^{1\Delta} \in \mathbf{Y}_i^{\Delta}(l) + \mathcal{Y}_i^{\Delta}$ is practically valid. Equations (1) and characteristics of noises are known in CPI. Hence, for the construction of covering set for a multiestimate, it is necessary to transfer a signal observed on object in CPI as precisely as possible. We lay aside the important question on size Δ and consider it already chosen.

6.1 Noiseless constraints

In this subsection we generalize the approach of [Savkin and Petersen, 2003] for our statistically uncertain situation. Note that in [Savkin and Petersen, 2003] only determined linear systems with quadratic limitations were considered.

For vector $y \in R^m$ consider the sup-norm $\|y\|_\infty = \max\limits_i |y_i|$, where y_i are coordinates of y. The obvious inequality $\|y\|_\infty \leq \|y\|$ is valid, where $\|\cdot\|$ is Euclidean norm. Corresponding norms for matrices are designated similarly. Let $B_a = \{y : \|y\|_\infty \leq a\}$ be a sphere of radius a. Given natural number q, we divide the sphere B_a into q^m subspheres of type $I_{j_1}^1 \times \cdots \times I_{j_m}^m$, where indexes j_i independently vary in the set $\{1, \dots, q\}$ and $I_j^i = \{y_i : -a + 2(j-1)a/q \leq y_i < -a + 2ja/q\}$, $i = 1, \dots, m$, $j = 1, \dots, q-1$; $I_q^i = \{y_i : a - 2a/q \leq y_i \leq a\}$. The vector $y \in B_a$ is coded by the sequence $\eta(y) = (j_1, \dots, j_m)$, if $y \in I_{j_1}^1 \times \cdots \times I_{j_m}^m$. On the contrary, each set $(j_1, \dots, j_m)$ of natural numbers independently varying in the set $\{1, \dots, q\}$ is assigned the vector $\gamma(j_1, \dots, j_m)$ with coordinates $\gamma_i = -a + (2j_i - 1)a/q$, $i = 1, \dots, m$. This vector is the geometric center of the set $I_{j_1}^1 \times \cdots \times I_{j_m}^m$.

Let $a = \max\{\|y\|_\infty : y \in \mathbf{Y}_i^{\Delta}(l) + \mathcal{Y}_i^{\Delta}, i = 1, \dots, n_1\}$. In the instant i, the coding-decoding is given by the formulas $(j_1, \dots, j_m) = \eta(y_i)$, if $y_i \in B_a$; $\overline{y}_i = \gamma(j_1, \dots, j_m)$. By construction we receive

$$\|y_i - \overline{y}_i\|_\infty \leq a/q, \quad y_i \in \overline{y}_i + B_{a/q}. \qquad (23)$$

Therefore,

$$\mathbf{Y}_i^{\Delta}(l, y) \subset (\overline{y}_i + B_{a/q} - \mathbf{Y}_i^{\Delta}(l)) \cap \mathcal{Y}_i^{\Delta} = \overline{\mathbf{Y}}_i(l, y). \qquad (24)$$

For constructing of covering sets in CPI, the sets from (24) are used in relations (19) – (22) instead of ones from (17). Thus, Theorem 2 remains valid.

Let us choose the value q so that values of the vector k_i and the vector $\overline{k}_i$ defined from the equation

$$\overline{k}_i = X_i^{\Delta}\overline{k}_{i-1} + K_{i-1}(\overline{y}_i - C_i^{\Delta}\overline{k}_{i-1}), \ \overline{k}_0 = 0, \ (25)$$

differed slightly. Suppose also that matrices A, C, Q, R in (1) are constant. Introduce the matrices $\tilde{A} = X^{\Delta} - S^{\Delta}R^{\Delta -}C^{\Delta}$, $B = (Q^{\Delta} - S^{\Delta}R^{\Delta -}S^{\Delta\prime})^{1/2}$. Let the following conditions be fulfilled: (a) the pair $(C^{\Delta}, \tilde{A})$ is observable; (b) the pair (A^{Δ}, B) is controllable; (c) $R^{\Delta} > 0$. Then it is known [Liptser and Shiryayev, 2000] that $\lim\limits_{i\to\infty} \gamma_i = \gamma^0$. In addition, the matrix $K = (S^{\Delta} + X^{\Delta}\gamma^0 C^{\Delta\prime})(R^{\Delta} + C^{\Delta}\gamma^0 C^{\Delta\prime})^{-1}$ is stable, i.e. $\|K\| < 1$. Suppose that the number a defined above is bounded for all n_1, and let $\overline{n}_1$ be a number such that $\|K_i\| \leq \beta < 1$ for all $i > \overline{n}_1$.

Theorem 4. *Let the conditions (a) – (c) be fulfilled. For any $\varepsilon > 0$ and $i > \overline{n}_1$ we choose the number q so that*

$$\delta\delta_1 a/(1-\beta)/q < \varepsilon, \qquad (26)$$

where $\|K_i\| \leq \delta$, $\forall i$, $\delta_1 = \max\limits_{1\leq i\leq \overline{n}_1} (\|A\| + \delta\|C\|)^i$. Then for all $i > \overline{n}_1$ we have $\|\overline{k}_i - k_i\| < \varepsilon$.

The theorem is established by comparison of equations (25) and (18) with the account of inequality (23). If in relation (19) we replace the vector k_t on $\overline{k}_t$, the received set will differ from (19) in Hausdorf's metric also on value ε.

Note that inequality (26) establishes a connection between the accuracy of approximation of covering sets (it is defined by parameter ε) and the constraint on the capacity of the data transfer channel (is defined by parameter q).

6.2 Constraints with Gaussian noise

One-dimensional Gaussian channels were considered in [Liptser and Shiryayev, 2000]. Here we generalize the results for multidimensional case and adapt these. This is not quite trivial. Consider the system in R^{2n} exited by 'white noise':

$$z_i = \mathbf{A}_i z_{i-1} + \chi_i + \mathbf{K}_{i-1}y_i^1, \quad z_0 = [0; x_0^0], \qquad (27)$$

where $\mathbf{A}_i = [X_i^{\Delta} - K_{i-1}C_i^{\Delta}, K_{i-1}C_i^{\Delta}; 0_n, X_i^{\Delta}]$, $z_i = [k_i; x_i^0]$, $\mathrm{cov}(\chi_i, \chi_i) = \mathcal{Q}_i = [K_{i-1}R_i^{\Delta}K_{i-1}', K_{i-1} \times S_i^{\Delta\prime}; S_i^{\Delta}K_{i-1}', Q_i^{\Delta}]$, $\mathbf{K}_i = [K_i; 0_{n\times m}]$, $\chi_i = [K_{i-1}\eta_i^{\Delta}, \xi_i^{\Delta}]$. We will use **Encoder Class** of the form $\theta_i = A_{i-1}^0(\theta) + B_i(\mathbf{C}_{i-1}z_{i-1} + \zeta_i)$, where $\mathbf{C}_i =$

$(\lambda_i(\theta)\tilde{\beta}_i^-)^{1/2}[I_n, 0_n]$; A^0 and λ are nonanticipating values, det$B_i \neq 0$; ζ_i is standard Gaussian vector independent of χ_i. The matrix $\tilde{\beta}_i$ is the left upper cell of matrix β_i that satisfies the Riccati equation

$$\left.\begin{array}{l}\beta_i = \mathbf{A}_i(\beta_{i-1} - \beta_{i-1}\mathbf{C}'_{i-1}(I_n \\ +\mathbf{C}_{i-1}\beta_{i-1}\mathbf{C}'_{i-1})^{-1}\mathbf{C}_{i-1}\beta_{i-1})\mathbf{A}'_i + \mathcal{Q}_i.\end{array}\right\} \quad (28)$$

The parameters of encoders are subject to the energy constraints

$$E\|B_{i+1}^{-1}A_i^0(\theta) + (\lambda_i(\theta)\tilde{\beta}_i^-)^{1/2}k_i\|^2 \leq \mathbb{P},\ i \geq 1. \quad (29)$$

Generalizing the result from [Liptser and Shiryayev, 2000] we obtain

Theorem 5. *If the determinate vector y_i^1 is known, the mean-square optimal* **encoder** *is received when $\lambda_i = \mathbb{P}/rank\tilde{\beta}_i$ and $B_{i+1}^{-1}A_i^0(\theta) = -\mathbf{C}_i\hat{z}_i$, where $\hat{z}_i = E(z_i \mid \theta^{0i}, \theta^{1i})$. The value $\hat{z}_i$ satisfies the corresponding Kalman filter, and its first component $\hat{k}_i$ is optimal* **decoder**.

As the value $y_i^1 \in \mathcal{Y}_i^\Delta$ is unknown, we use the minimax scheme as follows. First, we define recurrently the value

$$\left.\begin{array}{l}\delta_i = \mathbb{P}/\big(\min_{y_i^{1*}}\max_{y^{1i}} p'_i\tilde{\beta}_i^- p_i + \mathrm{rank}\tilde{\beta}_i\big), \\ y_i^{1*} \in \mathcal{Y}_i^\Delta,\ y^{1i} \in \mathcal{Y}_1^\Delta \times \cdots \times \mathcal{Y}_i^\Delta, \quad i \geq 1.\end{array}\right\} \quad (30)$$

Here $\tilde{\beta}_i$ is as above the left upper cell of matrix β_i (28), where now $\lambda_i = \delta_i$, and p_i is the first component of the vector $\overline{p}_i$ satisfying the equation

$$\left.\begin{array}{l}\overline{p}_i = \mathbf{K}_{i-1}(y_i^{1*} - y_i^1) + \mathbf{A}_i\big(\overline{p}_{i-1} - \beta_{i-1}\mathbf{C}'_{i-1}(I_n \\ +\mathbf{C}_{i-1}\beta_{i-1}\mathbf{C}'_{i-1})^{-1}\mathbf{C}_{i-1}\overline{p}_{i-1})\big), \quad \overline{p}_0 = 0.\end{array}\right\} \quad (31)$$

If we consider the recurrently defined vector y_i^{1*} as

$$y_i^{1*} = \mathrm{argmin}\max_{y^{1i}} p'_i\tilde{\beta}_i^- p_i, \quad (32)$$

then we get

Theorem 6. *Let the values β_i, δ_i, and y_i^{1*} be defined by* (28), (30) – (32). *Then the* **encoder** *of the form*

$$\left.\begin{array}{l}\theta_i = B_i\big(\mathbf{C}_{i-1}(z_{i-1} - \tilde{z}_{i-1}) + \zeta_i\big), \\ \tilde{z}_i = \mathbf{K}_{i-1}y_i^{1*} + \mathbf{A}_i\big(\tilde{z}_{i-1} + \beta_{i-1}\mathbf{C}'_{i-1}(I_n + \\ \mathbf{C}_{i-1}\beta_{i-1}\mathbf{C}'_{i-1})^{-1}B_i^{-1}\theta_i\big),\end{array}\right\} \quad (33)$$

ensures energy constraints (29) *for any determinate vector $y_i^1 \in \mathcal{Y}_i^\Delta$. The corresponding* **decoder** *$\hat{k}_i$ is the first component of the vector $\hat{z}_i$, for which*

$$\left.\begin{array}{l}\hat{z}_i = \mathbf{A}_i\big(\hat{z}_{i-1} + \beta_{i-1}\mathbf{C}'_{i-1}(I_n + \\ \mathbf{C}_{i-1}\beta_{i-1}\mathbf{C}'_{i-1})^{-1}(B_i^{-1}\theta_i + \\ \mathbf{C}_{i-1}\overline{p}_{i-1})\big) + \mathbf{K}_{i-1}y_i^1.\end{array}\right\} \quad (34)$$

Here $\hat{z}_i + \overline{p}_i = \tilde{z}_i$, and the error of restoring of k_i is equal to $E\|\hat{k}_i - k_i\|^2 = tr\,\tilde{\beta}_i$.

For equations (27), (33), and (34) with unknown vector y_i^1 we can apply the same method as above for covering of the multiestimate in CPI. Note that we have to use the inclusion $y_i^1 \in \mathcal{Y}_i^\Delta$ as the vector y_i used in the object is unknown in CPI.

7 Conclusion

In this paper, a construction of estimators in the form of sets has been considered for phase states of linear control systems. The given estimators can be used for the solution of control problems with incomplete information. The system was supposed to be linear and subject to influence of disturbances of the mixed nature. The determinate components of disturbances laid in convex and compact sets, and random ones were Wiener processes. According to measurement random information sets (multiestimates) were under construction, for which were defined covering them sets. Last sets provide the inclusion with high conditional probability irrespective of multiestimation parameters. Results has been illustrated on an example. A modification of the problem has been considered at presence of communication constraints. It was studied two kinds of communication channels: silent, supposing a discrete information transfer by words of the limited length, and the channel with Gaussian disturbances. The relations characterizing the accuracy of the parameter transfer for the construction of sets covering the multiestimates has been received.

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