Abstract
A state estimation problem is an important part of the more general control problem under incomplete information. In many cases control strategies are built on the base of various algorithms of state estimation. In this work, estimation problems for linear systems are considered under mixed disturbances. The determined disturbances are constrained by convex and compact sets, and the random ones are standard Wiener processes. Random information sets (multiestimates) are defined. In the absence of random components the multiestimates coincide with information sets from the theory of guaranteed estimation. The structure of multiestimates is considered: they are the sum of a random vector and a determinate set, which depend on parameters. In turn, the given set of parameters unequivocally defines the conditional and unconditional probability of inclusion of the multiestimate in a covering set. Special cases are considered, and the question on the form of the covering set is discussed. A modified problem is considered under communication constraints, in which the limited capacity of the digital data link and random errors in a communication channel are taken into consideration.

Key words
Multiestimates, communication channel, estimation, control, filtering.

1 Introduction
There are many control problems under incomplete information in which state estimation algorithms are used. A motion correction problem is one of them [Kurzhanski, 1977]. To solve this problem it is necessary at some instant to find a set of state vectors compatible with measurements. After that on the rest of time interval, a control is chosen to minimize a terminal cost uniformly for all the initial conditions from the found set. Hence, algorithms for set estimates are necessary and useful for control. For linear multistage systems with the mixed uncertainty including both determined and random disturbances, in [Katz and Kurzhanski, 1975] estimates in the form of sets were offered. However these estimates were not reduced to information sets from [Kurzhanski, 1977; Kurzhanski and Vályi, 1996] in the absence of random disturbances. In this connection, in [Anan’ev, 2000] the concept of random information set for systems with discrete time and with the mixed uncertainty is entered. The entered sets are already reduced to earlier known in the absence of random disturbances, but generally demand the further processing, as they depend on random and not observable parameters. In work [Anan’ev, 2007], a generalization of the random information sets named for brevity multiestimates is offered for multistage stochastic inclusions. An inclusion of the multiestimates in a covering set taking into account the communication constraints is considered in [Anan’ev, 2008]. The received results in many respects lean on [Savkin and Petersen, 2003; Huber, 1981; Matheron, 1975]. Various aspects of estimation theory with uncertain disturbances are examined in [Schweppe, 1973; Chernousko, 1994]. In this work, linear continuous time systems are considered mostly in a case when the determined disturbances are subordinated to geometrical restrictions and the random disturbances are Wiener processes. At the same time, it makes sense to give in the beginning the general statement of a problem and to plan the scheme of its decision. A realization of this scheme for rather simple case is offered below.

Let \((\Omega, \mathcal{F})\) be a standard measurable space, and \(P\) be a probability on \(\mathcal{F}\). Consider two functions \(x(d, \omega)\) and \(y(d, \omega)\) mapping \(D \times \Omega\) in \(U\) and \(S\) respectively, where \(D, U, S\) are separable metrizable spaces. The function \(x(d, \omega)\) not observable in experiment is supposed continuous in \(d\) and measurable in \(\omega\). The observable function \(y(d, \omega)\) has analogous properties. Let’s enter the random closed set of compatible parameters:
for all $d \in D_s$, $y(d) = s$, and $p(d, s, A) = E[I_A | y(d, .) = s]$ is a regular conditional probability [Shiryaev, 1991].

In a case when functions $x, y$ do not depend on parameter $d$, the Problem is reduced to construction of a confidence set with conditional probability of inclusion not smaller than $1 - \varepsilon$. If functions $x, y$ depend only on parameter $d$, the Problem coincides with one of construction of the information set in the theory of guaranteed estimation [Kurzhanski, 1977; Kurzhanski and Vályi, 1996]. In Problem, not all aspects and the difficulties meeting in specific cases are reflected far. In particular, for evolutionary systems it is rather important to find the recurrent procedure of construction of covering sets. Besides, the Problem can be complicated due to the presence of communication constraints. In this case, the information to the Center of Processing Information (CPI) may arrive during the discrete instants by the words consisting of integers and having the limited length. The communication channel for simplicity is often supposed noiseless and not having delay. The coding device in a communication channel is exploited for an information transfer about parameters of the multiestimate and of the signal. In CPI the information is decoded and utilized for the approximated restoration of a covering set. Other schemes are used for channels with noise. One of them we suggest at the end of the paper.

2 Linear-Gaussian Continuous Systems

Consider a linear system in Itô’s form:

$$\begin{align*}
\dot{x}_t &= (A(t)x_t + v(t))dt + dξ(t), \quad t \in [0, T], \\
\dot{y}_t &= (C(t)x_t + w(t))dt + dη(t), \quad y_0 = 0,
\end{align*}$$

where $x(t) \in R^n$ is the unmeasured vector, $y(t) \in R^n$ is the measured one. The initial state $x_0$ has Gaussian distribution $x_0 \sim N(\bar{x}_0, \gamma_0)$ with mean value $\bar{x}_0$ and the covariance $\gamma_0$. Uncertain parameters in (1) are restricted by the convex and compact constraints $\bar{x}_0 \in \bar{X}_0$, $v(t) \in V, w(t) \in W$. The random processes $ξ(t), η(t)$ in (1) are supposed to be Wiener ones with zero means and $\text{cov}(dξ, dξ) = Q(t)dt$, $\text{cov}(dη, dη) = R(t)dt$. The processes $ξ(t), η(t)$ are independent and do not depend on the initial vector $x_0$.

Recall that for any matrix $C$ the pseudoinverse matrix $C^−$ is defined by conditions $CC^−C = C$, $C^−C = MC^−$, where $L, M$ are matrices of the suitable size, the symbol ‘$\parallel$’ means the transposition, [Liptser and Shiryaev, 2000]. Let $\text{ker}C = \{ x : Cx = 0 \}$, $\text{im}C = \{ y : y = Cx, \exists x \}$, then the following equality $\{ x : Cx \in W \} = C^−(W \cap \text{im}C) \parallel \text{ker}C$ takes place.

We introduce purely random processes defined by

$$\begin{align*}
\dot{x}^θ(t) &= A(t)x^θ(t)dt + dξ(t), \quad x^θ_0 \sim N(0, γ_0), \\
\dot{y}^θ(t) &= C(t)x^θ(t)dt + dη(t), \quad y^θ_0 = 0, \quad t \in [0, T],
\end{align*}$$

determine functions for which

$$\begin{align*}
\dot{x}^θ(t) &= A(t)x^θ(t)dt + v(t), \quad x^θ_0 = 0, \\
\dot{y}^θ(t) &= C(t)x^θ(t)dt + w(t), \quad y^θ_0 = 0, \quad t \in [0, T].
\end{align*}$$

It is clear that $x(t) = x^θ(t) + x^v(t), y(t) = y^θ(t) + y^w(t)$. Using the function $y^θ(t) = \{ y^θ(s) : s \in [0, T] \}$ we construct the determinate information set, which is denoted by $X^θ(t, y)$ [Kurzhanski, 1977]. To this end for given $Δ > 0$ and for any $t \in [0, T]$, consider the finite sequence

$$y^θ_{1−}(t) = \{ y^θ(Δ), \ldots, y^θ(j^θΔ), y^θ(t) \}$$

consisting of $j^θ + 1$ elements, where $j^θ = \lfloor t/Δ \rfloor$.

Here $\lfloor \cdot \rfloor$ denotes the integer part of a real number. Let $X(t, s)$ be the fundamental matrix of the first equation in (1).

Let’s put $(j^θ + 1)Δ = t$ by definition and introduce the designations:

$$\begin{align*}
C^θ_1 &= \{ \int_{(i−1)Δ}^{iΔ} C(s,X(s,(i−1)Δ))ds, \quad X^θ_1 = X(iΔ, (i−1)Δ), \\
W^θ_1 &= \{ \int_{(i−1)Δ}^{iΔ} w(s)ds : w(s) \in W \} \subseteq R^n, \\
V^θ_1 &= \{ \int_{(i−1)Δ}^{iΔ} \dot{X}(iΔ, s)v(s)ds : v(s) \in V \} \subseteq R^{n+m}, \\
C^θ &\equiv C^θ_1, \quad X^θ_1 = \{ X^θ_1, D \}, \quad D = \{ [I_n, O_{n×m}], I_n \} = \text{a unity matrix,} \\
O_{m×n} &\text{is a zero matrix. Using this notation we can write the multistage system}
\end{align*}$$

$$\begin{align*}
x^θ_1 &= X^θ_1x^θ_{i−1} + Dv, \quad v \in V^θ, \quad y^θ(iΔ) = y^θ((i−1)Δ) + C^θ_1x^θ_{i−1} + Fv + w, \quad w \in W^θ_i, \\
x^θ_{i+1} &= X^θ_1x^θ_i, \quad y^θ(0) = 0, \quad i = 1, \ldots, j^θ + 1.
\end{align*}$$

From now on, we use the recurrent sets defined by the equations:

$$\begin{align*}
\dot{X}^θ_i &= \bigg( \dot{X}^θ_{i−1} \times V^θ \bigg) \cap \bigg( C^θ_1 \big( (y^θ_{i−1} − W^θ_i) \big) \bigg) & i = 1, \ldots, j^θ + 1.
\end{align*}$$

\textbf{Theorem 1.} The following convergence takes place in Hausdorff metric uniformly with respect to $t \in [0, T]$: $\lim_{Δ \rightarrow 0} X^θ_{j^θ + 1} = X^θ(t, y)$.

\textbf{Theorem 1} follows from the compactness of the initial set and the weak compactness of disturbances.

\textbf{Definition.} The sets

$$X(t, y) = X^θ(t, y) + x^θ(t), \quad t \in [0, T],$$

(6) are called the random information sets or the multiestimates.

The sets $X^θ(t, y)$ and (6) coincide in the absence of random components. Owing to the law of averages and Tchebychev’s inequality [Shiryaev, 1991] we have $\sum_{k=1}^N y(t)(k)/N \rightarrow E y(t) = y(t)$ if $N \rightarrow \infty$, where $y(t)(k)$ are realizations of measured vectors. Unfortunately, repetition of experiences and ways of specification of multiestimates on the basis of given relation, as a rule, are impossible. Therefore in the present work other methods connected with construction of covering set $Z(t)$, depending on the measured signal $y^t(·)$ and providing inclusion $X(t, y) \subseteq Z(t)$ with
given conditional probability irrespective of realization of uncertain parameters \( \{x_0, v(\cdot), w(\cdot)\} \), are offered.

### 3 Constructing of covering set

In system (1), the sum \( y(t) = y^0(t) + y^1(t) \) is measured, and it is obviously impossible to find out the summand \( y^1(t) \), on which the determinant set in multiparameter (6) is under construction. Let’s act as follows. Let the symbol \( \gamma^i \) mean the set of all signals in system (3) on the segment \([0, t]\). It is convex compact set in the space of all continuous vector functions. Fixing \( y^1(t) \in \gamma^i \) and marking out the random element \( y^0 = y(\cdot) \) - \( y^1(t \) can be found, the conditional distribution \( \text{Law}(x^0(t) \mid y^0, y^1(t)) = N(m^0(t), \gamma(t)) \) for the vector \( x^0(t) \) from (2). According to Kalman’s theory [Liptser and Shiryaev, 2000; Shiryaev, 1991] the parameters \( m^0(t), \gamma(t) \) satisfy the equations

\[
\begin{align*}
\dot{m}^0 &= A(t)m^0dt + K(t)(dy^0 - C(t)m^0dt), \\
\dot{\gamma} &= \gamma(t)C(t)R^{-1}(t), \\
\gamma(0) &= \gamma_0.
\end{align*}
\]

(7)

Let \( k(t) \) be a vector satisfying the first equation in (7), when the signal \( y^0 \) is replaced by \( y^1 \). We have the obvious equality \( m^0(t) = k(t) + \alpha(t) \) that follows from the relation \( y^0(t) = y(t) - y^1(t) \). We form the set \( \mathcal{Z}(t) = k(t) + \bigcup_{x_0}^y \mathcal{Y}(a(t) + \lambda^1(t), y(t)) \) and use it as a covering one for the multiparameter \( \mathcal{X}(t, y) \) from (6). Now our goal is to count up the conditional probability \( P(\mathcal{X}(t, y) \subset \mathcal{Z}(t) \mid y^0(t), y^1(t)) \). In force (6), (8) we have the equality of events

\[
\{\mathcal{X}(t, y) \subset \mathcal{Z}(t)\} = \{x^0(t) - m^0(t) \in \mathcal{D}(t)\},
\]

(9)

where

\[
\mathcal{D}(t) = \left\{ x : x + \alpha(t) + \lambda^1(t), y(t) \right\} \subset \bigcup_{x_0}^y \mathcal{Y}(a(t) + \lambda^1(t), y(t)).
\]

(10)

The set \( \mathcal{D}(t) \) depends on the concrete realization \( y^1(t) \) and contains zero at any such realization. Thus, the required conditional probability subject to (9), (10) is equal to

\[
P(\mathcal{X}(t, y) \subset \mathcal{Z}(t) \mid y^0(t), y^1(t)) = \int_{\mathcal{D}(t)} f(x, \gamma(t))dx,
\]

(11)

where \( f(x, \gamma) \) is the density of a gaussian distribution with zero mean and the covariance \( \gamma \). As the distribution can be singular, we give an expression of the type (11), which is true and for the singular case. Consider the matrix representation \( \gamma = T\Lambda^{1/2}/T^t \), where \( T \) is the orthogonal matrix, \( \Lambda^{1/2} = [\text{diag}\{\lambda_1^{1/2}, \ldots, \lambda_k^{1/2}\};O_{(n-k)\times k}] \). Here \( k = \text{rank} \gamma, \lambda \) are eigenvalues that distinct from zero of the matrix \( \gamma \). Then equality (11) is transformed to the following:

\[
P(\mathcal{X}(t, y) \subset \mathcal{Z}(t) \mid y^0(t), y^1(t)) = \int_{\mathcal{A}(t)} f(x, I_k)dx,
\]

(11)'

where \( I_k \) is the unity \( k \times k \) matrix.

The value \( y^1 \) is actually realized together with \( y^0 \) and is unknown. Therefore, it is possible to guarantee the inclusion \( \mathcal{X}(t, y) \subset \mathcal{Z}(t) \) with conditional probability not smaller than the minimum of value (11) ((11)') over all \( y^1(t) \in \gamma^i \). In order to provide the high probability of covering of the multiestimates, consider the concentration ellipsoid \( \mathcal{E}(t) = \{ x \in \text{im}(\gamma(t) : x^t\gamma(t)x \leq l^2 \} \) such that \( P(x^0(t) - m^0(t) \in \mathcal{E}(t)) = \int_{\{x \in \text{im}(\gamma(t)) \}} f(x, \mathcal{E}(t))dx \geq 1 - \varepsilon \), \( k = \text{rank} \gamma(t) \). Then we have the equality of the events

\[
\{\mathcal{X}(t, y) \subset \mathcal{Z}(t) + \mathcal{E}(t)\} = \{x^0(t) - m^0(t) \in \mathcal{D}(t)\},
\]

(12)

where

\[
\mathcal{D}(t) = \left\{ x : x + \alpha(t) + \lambda^1(t), y(t) \right\} \subset \bigcup_{x_0}^y \alpha(t) + \lambda^1(t), y(t)) + \mathcal{E}(t)\}.
\]

(13)

Whereas \( \mathcal{E}(t) \subset \mathcal{D}(t) \), the inequality

\[
P(\mathcal{X}(t, y) \subset \mathcal{Z}(t) + \mathcal{E}(t) \mid y^0(t), y^1(t)) \geq 1 - \varepsilon
\]

(14)

is valid for any signal \( y^1(t) \).

### 4 Evolutionary equations for covering set with the account of confidence sets

In this section, we try to reduce the exhaustive search of uncertain signals \( y^i(t) \) under construction of covering set for a multiestimat and simultaneously to construct the evolutionary covering set. Note that \( y^0(t) \sim N(0, J(t)) \), where matrices \( J(t) \) are defined by the evolutionary relations

\[
\begin{align*}
J &= C(t)L + L^tC(t) + R(t), \\
L &= A(t)L + PC(t), \\
\hat{L} &= A(t)L + PC(t), \\
\hat{P} &= A(t)P + PA^t(t) + Q(t), \\
P(t) &= \gamma(t).
\end{align*}
\]

For any \( t \in [0, T] \) let us choose the concentration ellipsoid \( \mathcal{Y}(t) = \{ x \in \text{im}(J(t) : x^tJ(t)x \leq l^2 \} \) such that \( P(y^0(t) \in \mathcal{Y}(t)) = \int_{\{x \in \text{im}(\gamma(t)) \}} f(x, \mathcal{Y}(t))dx > 0.99, k = \text{rank} \gamma(t) \). Then it is possible to assume practically that \( y^1(t) \in \mathcal{Y}(t) \). Therefore, it is possible to expand inclusion (15):

\[
y^1(t) \subset \mathcal{Y}(t) \subset y^1(t) \subset y^0(t) - \mathcal{Y}(t), \quad s \in [0, t].
\]

(15)'

With the account of restrictions (15) consider an evolutionary construction of such covering set, which in the absence of random parameters, coincides with \( \mathcal{X}(t, y) \). Let us pass to the approximating discrete scheme (4), (5). Purely random system (2) in discrete scheme can be written as

\[
\begin{align*}
x_i &= X_i^\Delta x_{i-1} + \zeta_i^\Delta, \quad \text{cov}(\zeta_i^\Delta, \zeta_j^\Delta) = Q_i^\Delta, \\
y_i &= C_i^\Delta x_i + \eta_i^\Delta, \quad \text{cov}(\eta_i^\Delta, \eta_j^\Delta) = R_i^\Delta,
\end{align*}
\]

where \( \{\zeta_i^\Delta\} \) and \( \{\eta_i^\Delta\} \) are random vectors, \( x_i \) and \( y_i \) are random signals, \( C_i^\Delta \) is a given matrix, \( Q_i^\Delta \) and \( R_i^\Delta \) are covariance matrices. Let us replace the conditional distribution of random vectors \( \{\zeta_i^\Delta\} \) and \( \{\eta_i^\Delta\} \) by corresponding empirical distributions. Then we receive the deterministic covering set: the value \( y^1(t) \) is realized together with \( y^0(t) \) and \( \zeta_i^\Delta, \eta_i^\Delta \) are given.
\[ S^\Delta = \int_{[t^\Delta]} X(i\Delta, s) Q(s) \left( \int_{J^\Delta} C(l) \right) X(t, s) dt \]  

Note that \( y_i^0 \sim N(0, J_i) \), where matrices \( J_i \) defined by the recurrent relations  
\[
J_i = C_i^\Delta P_{i-1} C_i^\Delta + R_i^\Delta, \quad P_i = \gamma_i, \quad i \geq 1.
\]
Choosing the concentration ellipsoid \( Y_i^\Delta(l) = \{ x \in \mathbb{R}^m : \mu^1, i^2 \leq 1 \} \) such that \( \| f(x, \xi_i) \| \leq R \). 
Introduce the sets  
\[
\begin{align*}
\hat{X}_0^\Delta &= \hat{X}_0^\Delta(\hat{X}_0, y_0), \quad \hat{X}_0^\Delta = \hat{X}_0, \\
\hat{Y}_0^\Delta(l, y) &= \hat{Y}_0^\Delta(\{ y \mid \| y \| \leq \mu \}), \quad \hat{Y}_0^\Delta(l, y) = \{ y : \| y \| \leq \mu \}.
\end{align*}
\]
Introduce the sets  
\[
\begin{align*}
\hat{X}_0^\Delta &= \hat{X}_0^\Delta(\hat{X}_0, y_0), \quad \hat{X}_0^\Delta = \hat{X}_0, \\
\hat{Y}_0^\Delta(l, y) &= \hat{Y}_0^\Delta(\{ y \mid \| y \| \leq \mu \}), \quad \hat{Y}_0^\Delta(l, y) = \{ y : \| y \| \leq \mu \}.
\end{align*}
\]
This theorem is established by induction. The same as in (12) \- (14), it is possible to add the ellipsoid \( E_i(l) \) to set \( \tilde{Z}_i \) in order to obtain the guaranteed result of type (14).

Note that for any given signal \( y^{11} \) in (3) the \( k_{i} \Delta \rightarrow 1 \) converges almost surely to the \( k(l) \) when \( \Delta \to 0 \). Thus, taking into consideration Theorem 1 we come to

**Theorem 3.** Suppose \( d(\cdot, \cdot) \) is the Hausdorff metric for compact sets. Then \( d(\tilde{Z}_{i+1}, \tilde{Z}_i) \) converges almost surely to zero when \( \Delta \to 0 \). Here \( \tilde{Z}_i(l) \) is defined by (8), where the set \( Y^{11} \) is replaced by (15). The same convergence takes place for the multiset \( X^{11}_{\Delta+1}(y) \) and \( X(l, y) \).

### 5 Example

Consider the system of Ito’s equations:
\[
\begin{align*}
&dx = d\xi, \quad l \in [0, T], \quad \text{Law}(x_0) = N(x_0, \gamma_0), \\
&dy = dy + (x + w)dt, \quad y_0 = 0, \quad \| w(t) \| \leq \nu, \\
&x, y \in \mathbb{R}^2, \quad \| x \| \leq \mu.
\end{align*}
\]

Here \( w(\cdot) \) is an uncertain function; \( \xi(t), \eta(t) \) are independent Wiener processes with \( \text{cov}(\xi(t), \xi(t)) = Q dt, \quad \text{cov}(\eta(t), \eta(t)) = R dt \). Let’s divide the segment \([0, T] \) into \( n \) equal parts and set \( \Delta = T/n, t_i = \Delta, \quad i = 1, \ldots, n \). Integrating the system by steps with the help of Cauchy’s formula we get
\[
\begin{align*}
x_i &= x_{i-1} + \xi, \quad \xi \sim \xi(t_i) - \xi(t_{i-1}), \\
y_i &= \Delta x_i + \eta_i + w_i, \quad \| w_i \| \leq \nu\Delta, \quad \text{cov}(\eta_i, \eta_i) = R\Delta + Q\Delta^2/3, \\
&\text{cov}(\xi(t), \eta(t)) = Q\Delta/2, \quad \text{cov}(\xi(t), \eta(t)) = Q\Delta/2, \quad i = 1, \ldots, n.
\end{align*}
\]

Here \( C = \Delta I_2, \quad V = \{ 0 \}, \quad \Delta \xi = \{ w : \| w \| \leq \nu\Delta \}, \quad \widetilde{X}_0 = \{ x : \| x \| \leq \mu \} \). The determinant component is constant: \( x_i^1 = \widetilde{X}_0 \), and the determinant sets are of the form \( X^\Delta = \widetilde{X}_0 \cap \gamma_{i-1}^1(y_i - W)/\Delta \). Let us write system (18):
\[
\begin{align*}
m_i^0 &= m_i^0 - K_{i-1}(y_i - D m_i^0), \quad m_i^0 = 0; \\
K_{i-1} &= (Q\Delta^2/2 + \gamma_{i-1}\Delta)(R\Delta + Q\Delta^2/3 + \Delta^2 \gamma_{i-1})^{-1}, \quad \gamma_i = \gamma_{i-1} + Q\Delta K_{i-1}(Q\Delta^2/2 + \gamma_{i-1}/\Delta), \quad i \geq 1.
\end{align*}
\]

Let, for example, \( Q = q^2 I_2, \quad R = r^2 I_2, \quad \gamma_0 = \gamma I_2 \). Then \( \gamma_i \leq \gamma_0 \), where \( \gamma = q \sqrt{r^2 + q^2\Delta^2/T} \), \( K = 2\sqrt{3}/(\sqrt{3} + 12r^2q^2 + \Delta^2)I_2 \). The vector \( m_0 \) submits to the stable system \( m_i^0 = m_i^0 + K(y_i - D m_i^0) \). The influence of initial conditions in this system under large \( i \) is negligible. The system of equations (16) gives \( J_i = (\Delta^2 \gamma + r^2 \Delta + q^2\Delta^2(3 - 2/3))I_2 \). For any \( i \) we choose \( i \) in the ellipsoid \( Y_i(l) = \{ x : \mu x \leq x_i \} \) so that \( P(y_i^0 \in Y_i(l)) = 1 - \exp(-\mu^2/2) \geq 0.99 \). Therefore, \( l > 2\sqrt{\ln \mu} \). Let us set numerical parameters: \( \mu = 1, \quad \nu = 0.5, \quad q = 2.06, \quad r = 0.01, \quad T = 10, \quad \Delta = 0.1, \quad n_1 = 100 \). Then \( \gamma = 0.0246, \quad K = 2.1774 I_2, \quad m_i^0 = 0.7823 m_i^0 + 2.1774 q^2 \). We write the system:
\[
\begin{align*}
\hat{X}_0^\Delta &= \hat{X}_0^\Delta \cap 10(Y_i^\Delta(l, y) - W^\Delta), \\
Y_i^\Delta(l, y) &= \{ \| x \| \leq 0.15 \} \cap (y_i - Y_i(l)).
\end{align*}
\]

It follows from this that the covering set can be constructed by formulas \( \tilde{Z}_i = \hat{X}_0^\Delta + A_i + \hat{X}_0^\Delta \).
It is necessary to add the concentration ellipsoid $E_t(l) = \{x : \|x\| \leq \sqrt{\eta t}\}$ to $\bar{Z}_t$ in order to obtain the guaranteed result of type (14), where $l$ is defined by $1 - \exp(-l^2/2) = 1 - \varepsilon$, $l = \sqrt{-2\ln \varepsilon}$. The typical picture of covering is shown on Fig.1. Here the green spot is a multieestimate $X^{\Delta}(y)$, the blue spot is a covering set $Z_t$, the red cross is a true state, and the red star is true Kalman’s estimate.

Fig. 1

6 Accounting of communication constraints

We have chosen above the concentration ellipsoid so that $P(y_0 | y_0) \in Y^{\Delta}(l) > 0.99$. Therefore, the inclusion $y_i = y_i^0 + y_i^{\Delta} \in Y^{\Delta}(l) + Y^{\Delta}$ is practically valid. Equations (1) and characteristics of noises are known in CPI. Hence, for the construction of covering set for a multieestimate, it is necessary to transfer a signal observed on object in CPI as precisely as possible. We lay aside the important question on size $\Delta$ and consider it already chosen.

6.1 Noisless constraints

In this subsection we generalize the approach of [Savkin and Petersen, 2003] for our statistically uncertain situation. Note that in [Savkin and Petersen, 2003] only determined linear systems with quadratic limitations were considered.

For vector $y \in R^m$ consider the sup-norm $\|y\|_{\infty} = \max |y_j|$, where $y_j$ are coordinates of $y$. The obvious inequality $\|y\|_{\infty} \leq \|y\|$ is valid, where $\| \cdot \|$ is Euclidean norm. Corresponding norms for matrices are designated similarly. Let $B_a = \{y : \|y\|_{\infty} \leq a\}$ be a sphere of radius $a$. Given natural number $q$, we divide the sphere $B_a$ into $q^m$ subspheres of type $I_{j_1} \times \cdots \times I_{j_m}$, where indexes $j_i$ independently vary in the set $\{1, \ldots, q\}$ and $I_{j_i} = \{y_i : -a + (j-1)a/q \leq y_i < -a + 2j-1a/q\}$, $i = 1, \ldots, m$, $j = 1, \ldots, q - 1$; $I_{j_i} = \{y_i : -a - 2a/q \leq y_i \leq -a\}$. The vector $y \in B_a$ is coded by the sequence $y(y) = (j_1, \ldots, j_m)$, if $y \in I_{j_1} \times \cdots \times I_{j_m}$. On the contrary, each set $(j_1, \ldots, j_m)$ of natural numbers independently varying in the set $\{1, \ldots, q\}$ is assigned the vector $\gamma(j_1, \ldots, j_m)$ with coordinates

$$
\gamma_i = -a + (2j_i - 1)a/q, \quad i = 1, \ldots, m.
$$

This vector is the geometric center of the set $I_{j_1} \times \cdots \times I_{j_m}$.

Let $a = \max \|y\|_{\infty} : y \in Y^{\Delta}(l) + Y^{\Delta}$, $i = 1, \ldots, n_1$. In the instant $i$, the coding-decoding is given by the formulas $(j_1, \ldots, j_m) = \eta(y_i)$, if $y_i \in B_a; \bar{y}_i = \gamma(j_1, \ldots, j_m)$. By construction we receive

$$
\|y_i - \bar{y}_i\|_{\infty} \leq a/q, \quad y_i \in \bar{y}_i + B_a/q.
$$

Therefore,

$$
Y^{\Delta}(l, y) \subset (\bar{y}_i + B_a/q - Y^{\Delta}(l)) \cap Y^{\Delta} = \bar{Y}(l, y).
$$

For constructing of covering sets in CPI, the sets from (24) are used in relations (19) – (22) instead of ones from (17). Thus, Theorem 2 remains valid.

Let us choose the value $q$ so that values of the vector $k_i$ and the vector $\bar{k}_i$ defined from the equation

$$
\bar{k}_i = X^{\Delta} \bar{k}_{i-1} + K_{i-1}(\bar{y}_i - C^{\Delta}\bar{k}_{i-1}), \quad \bar{k}_0 = 0,
$$

differed slightly. Suppose also that matrices $A, C, Q, R$ in (1) are constant. Introduce the matrices $\bar{A} = X^{\Delta} - S^{\Delta}R^{-\Delta}C^{\Delta}, \quad \bar{B} = (Q^{\Delta} - S^{\Delta}R^{-\Delta} - S^{\Delta})/2$.

Let the following conditions be fulfilled: (a) the pair $(C^{\Delta}, \bar{A})$ is observable; (b) the pair $(A^{\Delta}, \bar{B})$ is controllable; (c) $R^{\Delta} > 0$. Then it is known [Liptser and Shiryaev, 2000] that $\lim_{t \to \infty} \gamma(t) = 0$. In addition, the matrix $K = (S^{\Delta} + X^{\Delta}C^{\Delta})/(R^{\Delta} + C^{\Delta}C^{\Delta}) - 1$ is stable, i.e. $\|K\|_1 < 1$. Suppose that the number $a$ defined above is bounded for all $n_1$, and let $\pi_{n_1}$ be a number such that $\|K_{i}\|_1 \leq \beta < 1$ for all $i > \pi_{n_1}$.

**Theorem 4.** Let the conditions (a) – (c) be fulfilled. For any $\varepsilon > 0$ and $i > \pi_{n_1}$ we choose the number $q$ so that

$$
\delta \cdot \eta_i(1 - \beta)/\varepsilon < \varepsilon,
$$

where $\|K_{i}\|_1 \leq \delta, \forall i, \quad \delta_1 = \max_{1 \leq i \leq \pi_{n_1}} (\|A_{i}\|_1 + \delta\|C_{i}\|_1)^{\varepsilon}$.

Then for all $i > \pi_{n_1}$ we have $\|\bar{k}_i - k_i\| < \varepsilon$.

The theorem is established by comparison of equations (25) and (18) with the account of inequality (23).

If in relation (19) we replace the vector $k_i$ on $\bar{k}_i$, the received set will differ from (19) in Hausdorff’s metric also on value $\varepsilon$.

Note that inequality (26) establishes a connection between the accuracy of approximation of covering sets (it is defined by parameter $\varepsilon$) and the constraint on the capacity of the data transfer channel (is defined by parameter $q$).

6.2 Constraints with Gaussian noise

One-dimensional Gaussian channels were considered in [Liptser and Shiryaev, 2000]. Here we generalize the results for multidimensional case and adapt these. This is not quite trivial. Consider the system in $R^{2a}$ exited by ‘white noise’:

$$
z_i = A_i z_{i-1} + \chi_i + K_{i-1} y_i, \quad z_0 = [0; x_{0a}],
$$

where $A_i = [X^{\Delta} - \bar{K}_{i-1}C^{\Delta}; \bar{K}_{i-1}C^{\Delta}0_a; X^{\Delta}]$, $z_i = [k_i; x_i]$, $C_{i} = [K_{i}; 0_{m_1, m_2}]$, $x_{i} = [K_{i}; 0_{m_1, m_2}]$. We will use **Encoder Class** of the form

$$
\theta_i = A_i^{\theta}(\theta) + B_i(C_{i-1} z_{i-1} + \zeta_i),
$$

where $C_i =$
$(\lambda_1(\theta)\hat{\beta}^-_k)^{1/2}[I_n,0_n]; A^0$ and $\lambda$ are nonanticipating values, $\text{det}B_i \neq 0; \zeta_i$ is standard Gaussian vector independent of $\chi_i$. The matrix $\beta_i$ is the left upper cell of matrix $B_i$ that satisfies the Riccati equation

$$\beta_i = A_i(\beta_{i-1} - \beta_{i-1}C_{i-1}'(I_n + C_{i-1}\beta_{i-1}C_{i-1}')^{-1}C_{i-1}\beta_{i-1}A_i + Q_i).$$

(28)

The parameters of encoders are subject to the energy constraints

$$E[\|B_{i+1}^{-1}A_i^0(\theta) + (\lambda_1(\theta)\hat{\beta}^-_k)^{1/2}k_i\|^2] \leq \mathbb{P}, \ i \geq 1. \ (29)$$

Generalizing the result from [Liptser and Shiryaev, 2000] we obtain

Theorem 5. If the determine vector $y_i^k$ is known, the mean-square optimal encoder is received when $\lambda_i = \mathbb{P}/\text{rank} \hat{\beta}_i$ and $B_{i+1}^{-1}A_i^0(\theta) = -\hat{C}_i\hat{z}_i$, where $\hat{z}_i = E(\hat{z}_i \mid \theta^n, \theta^{11})$. The value $\hat{z}_i$ satisfies the corresponding Kalman filter, and its first component $k_i$ is optimal decoder.

As the value $y_i^k \in \mathcal{Y}_m$ is unknown, we use the minimax scheme as follows. First, we define the value

$$\delta_i = \mathbb{P}/(\max_{\theta^n, \theta^{11}} \max_{y_i^k} p_i^0(\hat{\beta}^-_i|p_i + \text{rank} \hat{\beta}_i), \ y_i^k \in \mathcal{Y}_m, \ y_i^k \in \mathcal{Y}_m \times \cdots \times \mathcal{Y}_m, \ i \geq 1. \ (30)$$

Here $\hat{\beta}_i$ is as above the left upper cell of matrix $\beta_i$ (28), where now $\lambda_i = \delta_i$ and $p_i$ is the first component of the vector $\mathcal{P}_i$ satisfying the equation

$$\mathcal{P}_i = K_{i-1}(y_i^k - y_i^k) + A_i(\mathcal{P}_{i-1} - \beta_{i-1}C_{i-1}'(I_n + C_{i-1}\beta_{i-1}C_{i-1}')^{-1}C_{i-1}\mathcal{P}_{i-1}), \ \mathcal{P}_0 = 0. \ (31)$$

If we consider the recursively defined vector $y_i^k$ as

$$y_i^k = \text{argmin}_{y_i^k} p_i^0(\hat{\beta}^-_i|p_i), \ (32)$$

then we get

Theorem 6. Let the values $\beta_i, \delta_i$, and $y_i^k$ be defined by (28), (30), (32). Then the encoder of the form

$$\hat{z}_i = K_{i-1}y_i^k + A_i(\hat{z}_{i-1} + \beta_{i-1}C_{i-1}'(I_n + C_{i-1}\beta_{i-1}C_{i-1}')^{-1}C_{i-1}\hat{z}_{i-1}), \ (33)$$

ensures energy constraints (29) for any determine vector $y_i^k \in \mathcal{Y}_m$. The corresponding decoder $k_i$ is the first component of the vector $\hat{z}_i$, for which

$$\hat{z}_i = K_{i-1}(\hat{z}_{i-1} + \beta_{i-1}C_{i-1}'(I_n + C_{i-1}C_{i-1}')^{-1}B_i)^{-1}B_i^\top \theta_i + C_{i-1}\mathcal{P}_{i-1}^{-1}k_i. \ (34)$$

Here $\hat{z}_i + \mathcal{P}_i = \hat{z}_i$, and the error of restoring of $k_i$ is equal to $E[|k_i - k_i|^2 = tr \hat{\beta}_i$.

For equations (27), (33), and (34) with unknown vector $y_i^k$ we can apply the same method as above for covering of the multistimate in CPI. Note that we have to use the inclusion $y_i^k \in \mathcal{Y}_m$ as the vector $y_i$ used in the object is unknown in CPI.

7 Conclusion

In this paper, a construction of estimators in the form of sets has been considered for phase states of linear control systems. The given estimators can be used for the solution of control problems with incomplete information. The system was supposed to be linear and subject to influence of disturbances of the mixed nature. The deterministic components of disturbances laid in convex and compact sets, and random ones were Wiener processes. According to measurement random information sets (multistimates) were under construction, for which were defined covering them sets. Last sets provide the inclusion with high conditional probability irrespective of multistimation parameters. Results has been illustrated on an example. A modification of the problem has been considered at presence of communication constraints. It was studied two kinds of communication channels: silent, supposing a discrete information transfer by words of the limited length, and the channel with Gaussian disturbances. The relations characterizing the accuracy of the parameter transfer for the construction of sets covering the multistimates has been received.

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References


