UNI-DIRECTIONAL SYNCHRONIZATION AND FREQUENCY DIFFERENCE LOCKING INDUCED BY A HETEROCLINIC RATCHET

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Abstract

A system of four coupled oscillators that exhibits unusual synchronization phenomena has been analyzed. Existence of a one-way heteroclinic network, called heteroclinic ratchet, gives rise to uni-directional (de)synchronization between certain groups of cells. Moreover, we show that locking in frequency differences occur when a small white noise is added to the dynamics of oscillators.

Key words

Heteroclinic ratchets, synchronization, coupled oscillators, uni-directional desynchronization.

1 Introduction

Phase oscillators are used as approximations for the phase dynamics of coupled limit cycle oscillators in the case of weak coupling [Kuramoto, 1984; Pikovsky et al., 2001; Hoppensteadt and Izhikevich, 1997]. They exhibit synchronization and clustering phenomena [Kuramoto, 1984; Sakaguchi and Kuramoto, 1986], even if coupling function consists of the first harmonic only. If the second harmonic of the coupling function is considered, it is possible to observe switchings between different clusterings as a result of an asymptotically stable robust heteroclinic cycle [Hansel et al., 1993]. It is known that heteroclinic cycles are not structurally stable but they may exist robustly for coupled systems. This is due to the existence of robust invariant subspaces for certain coupling structures that may support robust heteroclinic connections that are saddle-to-sink on the invariant subspaces and form a heteroclinic cycles [Krupa, 1997; Ashwin and Field, 1999; Aguiar et al., 2011]. Existence of robust heteroclinic cycles or more generally heteroclinic networks in a system of three and four globally coupled phase oscillators have been analyzed in [Ashwin et al.,

2008] and in [Ashwin et al., 2006], where an extreme sensitivity phenomenon to detuning of natural frequencies has been observed. Namely, oscillators loose synchrony even for very small detuning of natural frequencies. [Karabacak and Ashwin, 2010] have considered the third harmonic of the coupling function and observed one-way heteroclinic networks, which are called heteroclinic ratchets. A heteroclinic ratchet is a heteroclinic network that, for some axis, contains trajectories winding in one direction only. Heteroclinic ratchets give rise to extreme sensitivity to detuning of certain sign. Namely, synchronization of a pair of oscillators is possible only when the natural frequency of a certain oscillator is larger than the other. We call this phenomenon uni-directional (de)synchronization.

In the sequel, we identify a heteroclinic ratchet for a system of four coupled oscillators in Section 2. Although the system is less complicated than the original ratcheting system considered in [Karabacak and Ashwin, 2010], it exhibits more complicated dynamics: uni-directional synchronization between groups of oscillators, explained in Sections 3 and 4. We believe that this is due to the asymmetric structure in the coupling (see Figure 1).

2 A Model of Four Coupled Oscillators That Supports Heteroclinic Networks

Consider the following well-known model of coupled phase oscillators:

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^N c_{ij} g(\theta_i - \theta_j).$$
(1)

Here, $\theta_i \in \mathbb{T} = [0, 2\pi)$ denotes the phase of oscillator i and ω_i is its natural frequency. The connection matrix $\{c_{ij}\}$ represents the coupling. $c_{ij} = 1$ if oscillator i receives an input from oscillator j and $c_{ij} = 0$ otherwise.

Since $g(\cdot)$ is a 2π -periodic function, it can be written as a sum of Fourier harmonics: $g(x) = \sum_{k=1}^{\infty} r_k \sin(kx + \alpha_k)$. Without loss of generality, we may set K = N and $r_1 = -1$ by a scaling of time. Let us choose the following coupling function with two harmonics only:

$$g(x) = -\sin(x + \alpha_1) + r_2\sin(2x)$$
 (2)

The model (1) is used as the approximate phase dynamics of weakly coupled limit cycle oscillators, and the weak coupling gives rise to a \mathbb{T}^1 phase-shift symmetry in the phase model (1). Hence, the dynamics of (1) is invariant under the phase shift

$$(\theta_1, \theta_2, \dots, \theta_N) \mapsto (\theta_1 + \epsilon, \theta_2 + \epsilon, \dots, \theta_N + \epsilon)$$

for any $\epsilon \in \mathbb{T}$. Below, this symmetry is used to reduce the dynamics to an (N-1)-dimensional phase difference system.

Let us consider the coupled phase oscillator system (1) with the coupling structure given in Figure 1. This gives rise to the following system:

$$\dot{\theta}_{1} = \omega_{1} + g(\theta_{1} - \theta_{2}) + g(\theta_{1} - \theta_{3})
 \dot{\theta}_{2} = \omega_{2} + g(\theta_{2} - \theta_{1}) + g(\theta_{2} - \theta_{4})
 \dot{\theta}_{3} = \omega_{3} + g(\theta_{3} - \theta_{2}) + g(0)
 \dot{\theta}_{4} = \omega_{4} + g(\theta_{4} - \theta_{1}) + g(\theta_{4} - \theta_{2}).$$
(3)

Defining phase difference variables as $\phi_1 := \theta_1 - \theta_2$, $\phi_2 := \theta_2 - \theta_4$ and $\phi_3 = \theta_3 - \theta_4$, we obtain the following dynamical system for phase differences:

$$\dot{\phi_1} = \omega_{13} + g(\phi_1 + \phi_3 - \phi_2) + g(\phi_1) - g(\phi_3 - \phi_2) - g(0) \dot{\phi_2} = \omega_{24} + g(-\phi_1 - \phi_3 + \phi_2) + g(\phi_2) - g(-\phi_3 - \phi_1) - g(-\phi_2) \dot{\phi_3} = \omega_{34} + g(\phi_3 - \phi_2) + g(0) - g(-\phi_3 - \phi_1) - g(-\phi_2).$$
(4)

 ω_{ij} denotes the detuning between oscillator i and oscillator j, namely $\omega_{ij} = \omega_i - \omega_j$.

We first assume identical oscillators, that is

 $\omega_1 = \dots = \omega_4 = \omega \Longrightarrow \omega_{ij} = 0 \quad \forall i, j.$ (5)

Oscillators with different natural frequencies will be considered in Section 3.



Figure 1. An asymmetric coupled cell system: this gives the coupled system of form (3).



Figure 2. Two balanced colorings of the coupled cell system in Figure 1. These give rise to 2-dimensional invariant subspaces X_1 and X_2 of the system (4).

2.1 Invariant Subspaces

The assumption that the oscillators are identical makes it possible to use the balanced coloring method [Stewart et al., 2003] to obtain invariant subspaces of the system (3). A coloring of cells in a coupled cell system is called balanced if cells with identical color receives the same number of inputs from cells of any given color. A balanced coloring gives rise to an invariant subspace obtained by assuming that the cells of same color have identical states. The converse of this statement is also true. Namely, for a given coupling structure, a coloring is balanced if the corresponding subspace is invariant under any system having that coupling structure. Therefore, the invariant subspaces obtained by the balanced coloring method are robust under the perturbations that preserve the coupling structure. For an introduction to this theory, see [Golubitsky and Stewart, 2006].

Using the balanced coloring method the invariant subspaces of the coupled cell system given in Figure 1 can be found as in Table 1. Using the above-mentioned phase difference reduction, the corresponding invariant subspace in \mathbb{T}^3 for the system (4) are also listed in Table 1. Note that for the system (4), there are only two 2-dimensional invariant subspaces, namely X_1 and X_2 . Balanced colorings for these invariant subspaces are given in Figure 2. The invariant subspaces X_1 , X_2 and their intersection X_3 can support a robust heteroclinic cycle (see [Ashwin et al., 2011] for robustness criteria of heteroclinic cycles).

Balanced	Invariant subspaces of the system (3)		
colorings	on \mathbb{T}^4 and system (4) and on \mathbb{T}^3		
$\{1 2 3 4\}$	$X_0 = \mathbb{T}^4$		
	$ar{X}_0 = \mathbb{T}^3$ (whole space)		
$\{1 2 34\}$	$X_1 = \{\theta \in \mathbb{T}^4 \mid \theta_2 = \theta_4\}$		
	$\bar{X}_1 = \{\phi \in \mathbb{T}^3 \mid \phi_2 = 0\} \ (\phi_1 - \phi_3 \text{ plane})$		
$\{12 3 4\}$	$X_2 = \{\theta \in \mathbb{T}^4 \mid \theta_1 = \theta_3\}$		
	$\bar{X}_2 = \{\phi \in \mathbb{T}^3 \phi_1 = 0\} \ (\phi_2 - \phi_3 \text{ plane})$		
$\{12 34\}$	$X_3 = \{ \theta \in \mathbb{T}^4 \mid \theta_1 = \theta_3, \theta_2 = \theta_4 \}$		
	$\bar{X}_3 = \{\phi \in \mathbb{T}^3 \phi_1 = \phi_2 = 0\} \ (\phi_3 \text{ axis})$		
$\{134 2\}$	$X_4 = \{\theta \in \mathbb{T}^4 \mid \theta_1 = \theta_3 = \theta_4\}$		
	$\bar{X}_4 = \{\phi \in \mathbb{T}^3 \phi_1 = \phi_3 = 0\} \ (\phi_2 \text{ -axis})$		
{1234}	$X_5 = \{\theta \in \mathbb{T}^4 \mid \theta_1 = \theta_2 = \theta_3 = \theta_4\}$		
	$ar{X}_5=\{(0,0,0)\}$ (origin)		

Table 1. Balanced colorings of the coupled system given in Figure 1. The corresponding invariant subspaces for the system (3) and the corresponding reduced invariant subspaces for the system (4) are given.

2.2 Existence and Stability of a Heteroclinic Ratchet

A heteroclinic ratchet (first defined in [Karabacak and Ashwin, 2010]) is a heteroclinic network that contains a heteroclinic cycle winding in some direction and does not contain another heteroclinic cycle winding in the opposite direction. To be precise, a heteroclinic cycle $C \subset \mathbb{T}^N$ parametrized by x(s) $(x: [0,1) \to \mathbb{T}^N)$ is winding in some direction if there is a projection map $P: \mathbb{R}^N \to \mathbb{R}$ such that the parametrization $\bar{x}(s)$ $(\bar{x}: [0,1) \to \mathbb{R}^N)$ of the lifted heteroclinic cycle $\bar{C} \subset \mathbb{R}^N$ satisfies $\lim_{s\to 1} P(\bar{x}(s)) - P(\bar{x}(0)) = 2k\pi$ for some positive integer k. A heteroclinic cycle winding in the opposite direction would satisfy the same condition for a negative integer k (see [Ashwin and Karabacak, 2011] for general properties of heteroclinic ratchets).

As discussed above, the system (4) may have a robust heteroclinic network on the invariant subspaces X_1 and X_2 . Such a heteroclinic network should be connecting saddles on $X_3 = X_1 \cap X_2$. Reducing the equations in (4) to X_3 and considering identical natural frequencies, we get

$$\dot{\phi}_3 = g(\phi_3) - g(-\phi_3).$$
 (6)

This system is \mathbb{Z}_2 -equivariant, and therefore it can admit a codimension-1 pitchfork bifurcation of the zero solution under some nondegeneracy conditions. The saddles emanating from this bifurcation are on X_3 and they are of the form $p = (0, 0, p_3)$ and $q = (0, 0 - p_3)$. The value p_3 can be obtained by solving



Figure 3. Phase portraits of the system (4) on invariant subspaces \bar{X}_1 (a) and \bar{X}_2 (b) are illustrated for parameters given in (8). Red lines indicate robust heteroclinic trajectories. Thick red lines are the winding heteroclinic trajectories. The robust heteroclinic ratchet formed by these winding and non-winding heteroclinic trajectories and the saddles p and q is shown in (c).

 $g(p_3) - g(-p_3) = 0$ as

1

$$p_3 = \cos^{-1}\left(\frac{\cos\alpha_1}{2r_2\cos\alpha_2}\right).$$
 (7)

In order to show that there exist heteroclinic connections between p and q we use the simulation program XPPAUT [Ermentrout, 2002]. We identify a heteroclinic ratchet for the parameter values

$$\alpha_1 = -5, \quad r_2 = 0.15 \tag{8}$$

(see Figure 3). On X_1 , the heteroclinic ratchet contains a non-winding trajectory and a trajectory winding along $+\phi_1$ and $-\phi_3$ directions (see Figure 3a). On X_2 , it contains a non-winding trajectory and a trajectory winding along $+\phi_2$ direction (see Figure 3b). These four connections and the saddles p and q form a heteroclinic ratchet (see Figure 3c). For the parameters given in (8), p can be found as (0, 0, 0.3315). Considering the Jakobian of (4) at p, we can find the eigenvalues of the saddle p as $\lambda_1^{(p)} = -0.3112, \, \lambda_2^{(p)} = 0.2967$ and $\lambda_3^{(p)} = -0.0636$. Similarly, the eigenvalues of q =-p = (0, 0, 0.3315) can be found as $\lambda_1^{(q)} = 0.3130$ and $\lambda_2^{(q)} = -0.3276$ and $\lambda_3^{(q)} = -0.0636$. These eigenvalues of p and q correspond to the eigenvectors $v_1 = (1,0,0) \in X_1, v_2 = (0,1,0) \in X_2$ and $v_3 =$ $(0,0,1) \in X_3$. A heteroclinic cycle is attracting if the saddle quantity, defined as the absolute value of the ratio between the product of the eigenvalues corresponding to the expanding connections and the product of the



Figure 4. A solution of the system (4) converging to the heteroclinic ratchet. Initial states are chosen as (2, 1, 0.5). The solution shows the peculiar property for heteroclinic cycles that the residence time near equilibria increases as $t \to \infty$, before ϕ_1 and ϕ_2 get locked at zero due to the precision errors.

eigenvalues corresponding to the contracting connections is smaller than 1 [Melbourne, 1989]. Hence, we can conclude that the heteroclinic ratchet for the system (4) with parameters given in (8) is asymptotically stable since the saddle quantity $\left|\frac{\lambda_2^{(p)}\lambda_1^{(q)}}{\lambda_1^{(q)}\lambda_2^{(p)}}\right| = 0.9532$ is less than one.

A solution of (4) converging to the heteroclinic ratchet can be seen in Figure 4. The increase in the residence time near equilibria is typical for a solution converging to a heteroclinic network. Winding of ϕ_1 and ϕ_3 occur at the same time, respectively in positive and negative directions, due to the winding heteroclinic trajectory on X_1 (see Figure 3a). Winding of ϕ_2 occur in the positive direction due to the winding heteroclinic trajectory on X_2 (see Figure 3b). Since at each turn the solution gets closer to the equilibria p and q, after some time, ϕ_1 and ϕ_2 get locked at zero due to the precision errors. Note that, the invariant subspaces X_1 and X_2 serve as barriers, and therefore no solution can pass through them. For this reason the solution winds in ϕ_1 and ϕ_3 directions only one time. Since winding in $-\phi_3$ direction occurs together with the winding in $+\phi_1$ direction, this also happens only one time. However, these barriers can be broken by noise and/or detuning of natural frequencies leading to the uni-directional synchronization phenomenon.

3 Uni-directional Synchronization in the Model

We say that oscillators i and j are (frequency) synchronized if the observed frequency differences

$$\Omega_{ij} = \lim_{t \to \infty} \frac{|\theta_i^L - \theta_j^L|}{t},\tag{9}$$

is equal to zero. Here $\theta_i^L \in \mathbb{R}$ is the lifted phase variable for $\theta_i \in \mathbb{T}$. It is know that coupled oscillators can get frequency synchronized when the distance between



Figure 5. Uni-directional synchronization with respect to ω_{13} . The main graph shows the frequency differences Ω_{13} , Ω_{24} and Ω_{34} for (3) with parameters given in (8) as a function of detuning ω_{13} when $\omega_{24} = \omega_{34} = 0$. Oscillators are frequency synchronized when $\omega_{13} \leq 0$ and the synchronization fails for oscillator pairs (13) and (24) whenever $\omega_{13} > 0$. The insets show time evolution of the phase differences ϕ_i for a positive value of ω_{13} .



Figure 6. Uni-directional synchronization with respect to ω_{24} . The frequency differences Ω_{13} , Ω_{24} and Ω_{34} for (3) with parameters given in (8) are shown as a function of detuning ω_{24} when $\omega_{13} = \omega_{34} = 0$. Oscillators are frequency synchronized when $\omega_{24} \leq 0$ and the synchronization fails for the oscillator pair (24) whenever $\omega_{24} > 0$.

their natural frequencies, namely $|\omega_{ij}| := |\omega_i - \omega_j|$ is small enough. If frequency synchronization of oscillators *i* and *j* occurs only when a specific one of the oscillators has greater natural frequency, namely for a certain sign of ω_{ij} , we say that synchronization is uni-directional. Uni-directional synchronization phenomenon has been shown to occur for oscillator pairs when an asymptotically stable heteroclinic ratchet exists in the phase space [Karabacak and Ashwin, 2010; Ashwin and Karabacak, 2011].

For the system (4), we investigate the effect of detunings ω_{13} , ω_{24} and ω_{34} on the synchronization of oscillators, respectively in Figure 5, 6 and 7. Due to the winding connections in the heteroclinic ratchet, unidirectional synchronization occurs for detunings ω_{13} and ω_{24} . However, because of the connection winding both in $+\phi_1$ and $-\phi_3$ directions, a positive detuning



Figure 7. Bi-directional synchronization (the usual case) with respect to ω_{34} . The frequency differences Ω_{13} , Ω_{24} and Ω_{34} for (3) with parameters given in (8) are shown as a function of detuning ω_{34} when $\omega_{13} = \omega_{24} = 0$. Oscillators are frequency synchronized when $|\omega_{34}|$ is small enough and the synchronization fails for the oscillator pair (34) for large values of $|\omega_{34}|$.

Detuning Direction	Natural Frequencies	Winding Direction	Observed Frequencies
$+\omega_{13}$	$\omega^+, \omega, \omega, \omega$	$+\phi_{1}, -\phi_{3}$	$\Omega^+, \Omega^+, \Omega, \Omega^+$
$+\omega_{24}$	$\omega, \omega^+, \omega, \omega$	$+\phi_2$	$\Omega, \Omega^+, \Omega, \Omega$
$+\omega_{34}$	$\omega^+, \omega, \omega^+, \omega$	0	$\Omega, \Omega, \Omega, \Omega$

Table 2. The effect of detunings on the synchronization of oscillators for the system (4). These can be obtained from Figures 5, 6 and 7. Negative detunings have not been considered as they have no effect. ω^+ (Ω^+) represents a number slightly larger than ω (Ω).

 ω_{13} leads to synchronization of oscillators 1, 2 and 4. This is because $\theta_1 - \theta_3 \cong -(\theta_3 - \theta_4) \Longrightarrow \theta_1 \cong \theta_4$. The synchronized groups of oscillators for each detuning case are given in Table 2. It is interesting that the oscillators 1, 2 and 4 get synchronized for a positive detuning of ω_{13} , although the space $\{\theta \in \mathbb{T}^4 \mid \theta_1 = \theta_2 = \theta_4\}$ is not one of the synchronization spaces obtained by the balanced coloring method in Section 2.1. Hence, it is not an invariant subspace.

4 Locking in Frequency Differences

Noise induced uni-directional desynchronization of oscillators has been observed in [Karabacak and Ashwin, 2010] for a coupled system admitting a heteroclinic ratchet. Here, we show that existence of a heteroclinic ratchet for the system (4) leads to a locking in frequency differences when a small noise is applied. Figure 8 shows a solution of the system (4) under white noise with amplitude 10^{-12} . The noisy solution exhibits approximately equal number of windings in ϕ_1 and ϕ_2 directions. This is because the noise is homogeneous and the invariant subspace X_1 (resp. X_2) divides any ϵ -ball around the equilibrium q (resp. p) into two regions of attractions of equal volume for the winding and non-winding trajectories. On the other hand, the



Figure 8. A solution of the system (4) for parameters given in (8) with no detuning and with additive white noise (amplitude= 10^{-12}).

number of windings in ϕ_1 and $-\phi_3$ directions are exactly the same because of the structure of the heteroclinic ratchet in Figure 3.

As a result, the solution gives rise to the following frequency locking between frequency differences:

$$\Omega_{13} = \Omega_{24} = -\Omega_{34}.$$
 (10)

This is in agreement with the simulation results given in Figure 8. Therefore, the observed oscillator frequencies $\Omega_i := \lim_t |\theta_i^L(t)|/t$ are in the following form:

$$\begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ \Omega_4 \end{pmatrix} = \begin{pmatrix} \Omega + \delta \\ \Omega \\ \Omega + \delta \\ \Omega + 2\delta \end{pmatrix}, \quad (11)$$

where δ is a positive number. This type of a result cannot be seen directly from the connection structure of the coupled system, and is a consequence of the particular heteroclinic ratchet that the system admits. Although the noise induces synchronization of oscillators 1 and 3, the balanced coloring method explained in Section 2.1 does not give $\{\theta \in \mathbb{T}^4 \mid \theta_1 = \theta_3\}$ as an invariant subspace.

5 Conclusion

We have analyzed a system of four coupled phase oscillators. The existence of an asymptotically stable heteroclinic ratchet gives rise to uni-directional synchronization of certain groups of oscillators and induce a particular locking in the frequency differences of oscillators when small amplitude white noise is introduced to the system. These phenomena also lead to frequency synchronization of some oscillators, that can not be found by using the balanced coloring method, therefore does not correspond to any synchrony subspace.

For future works, the relation between the connection structure and possible synchronization groups can be studied. Although the synchronization groups can not be inferred from the coupling structure directly, the coupling structure serves to create invariant subspaces on which heteroclinic ratchets can be supported. For this reason, the coupling structure plays an indirect role on the existence of possible synchronization groups. Another direction could be to study bifurcations of heteroclinic ratchets that result in winding periodic orbits on torus. This can explain the effect of small detunings of natural frequencies on the observed frequencies in a complete way.

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