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NOISE-INDUCED TRANSITIONS FOR COEXISTING PERIODIC ATTRACTORS

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Abstract

We consider a dynamical system in the parameter zone admitting two coexisting limit cycles under the transition to chaos via period-doubling bifurcations. Under the random disturbances, noise-induced transitions between two coexisting separate attractors are studied. We suggest a stochastic sensitivity function technique for the analysis of this type transitions. This approach allows to construct the dispersion ellipses of random trajectories for any Poincare sections. Possibilities of our descriptive-geometric method for a detailed analysis of noise-induced transitions between two periodic attractors of Lorenz model are demonstrated.

Key words

Limit cycles; Noise-induced transitions; Lorenz model; Stochastic sensitivity function

1 Introduction

Stochastic fluctuations of nonlinear oscillations play an important role for understanding the corresponding dynamical phenomena in electronic generators, lasers, mechanical, chemical and biological systems ([Stratonovich, 1963]). The various noise-induced transitions through periodic to more complicated regimes are a central problem in a nonlinear stochastic dynamics ([Horsthemke and Lefever, 1984]). The sensitivity analysis of random forced oscillations is a key for investigation of these transitions. Multistable systems exhibit complex dynamics ([Feudel and Grebogi, 2003]) with noise-induced hopping between coexisting attractors and their basins of attraction ([Kraut and Feudel, 2002; Soiza et al, 2007]).

Noise-induced transitions between coexisting stable equilibria of 1D and 2D-systems are well studied. Now a subject of intensive investigation is an analysis of noise-induced transitions between limit cycles of 3D-systems.

Since its invention ([Lorenz, 1963]) the Lorenz system has been a basic model for investigations in many directions. This 3D-model shows a great variety of qualitatively different regimes of behavior ([Sparrow, 1982]). Lorenz model is a classic example of 3D-system with coexisting periodic attractors and transition to chaos via period-doubling bifurcations. It allows to use Lorenz model as a basic tool for the testing of new methods of nonlinear systems analysis.

In this paper, we consider the stochastically forced Lorenz system with coexisting limit cycles for period-doubling bifurcation zone near chaos. The aim of our work is to analyze noise-induced transitions between periodic attractors. Our approach is based on the stochastic sensitivity function technique ([Bashkirtseva and Ryashko, 2000], [Bashkirtseva and Ryashko, 2002], [Bashkirtseva and Ryashko, 2004]).

2 Deterministic and stochastic attractors of the Lorenz model

Consider the stochastic Lorenz model

$$\begin{aligned} \dot{x} &= \sigma(-x+y) + \varepsilon\dot{w}_1 \qquad \sigma = 10,\ b = \tfrac{8}{3} \\ \dot{y} &= rx - y - xz + \varepsilon\dot{w}_2 \\ \dot{z} &= -bz + xy + \varepsilon\dot{w}_3. \end{aligned} \tag{1}$$

Here $w_i(t)$ $(i = 1,2,3)$ are independent standard Wiener processes with Gaussian increments, $E(w_i(t) - w_i(s)) = 0$, $E(w_i(t) - w_i(s))^2 = |t-s|$. Parameter ε is a value of the noise intensity. For $\varepsilon = 0$, an interval $200 < r < 350$ is well-known ([Sparrow, 1982]) as a period doubling bifurcations zone with infinite chain of limit cycles and transition to chaos (see Fig.1).

As the parameter r decreases the deterministic Lorenz system passes the symmetric saddle-node bifurcation and the symmetric stable cycle splits and two stable non-symmetric cycles appear (see Fig. 1 b).

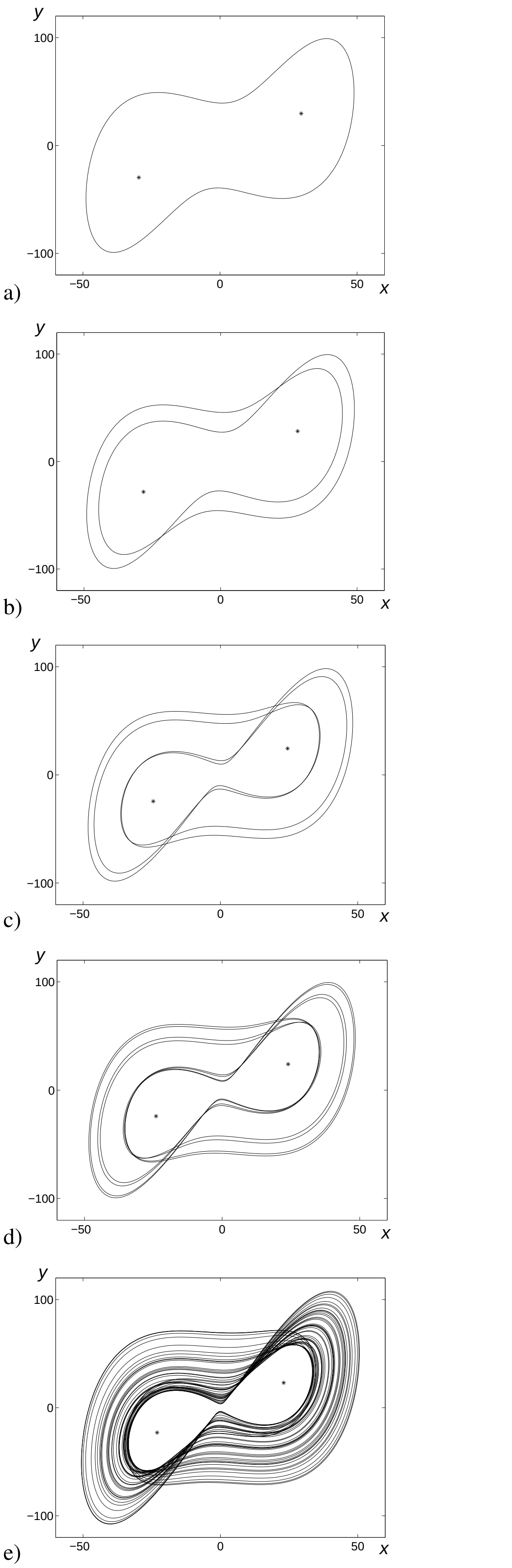

Figure 1. Attractors of deterministic Lorenz model a) $r = 330$; b) $r = 300$; c) $r = 225$; d) $r = 217$; e) $r = 200$.

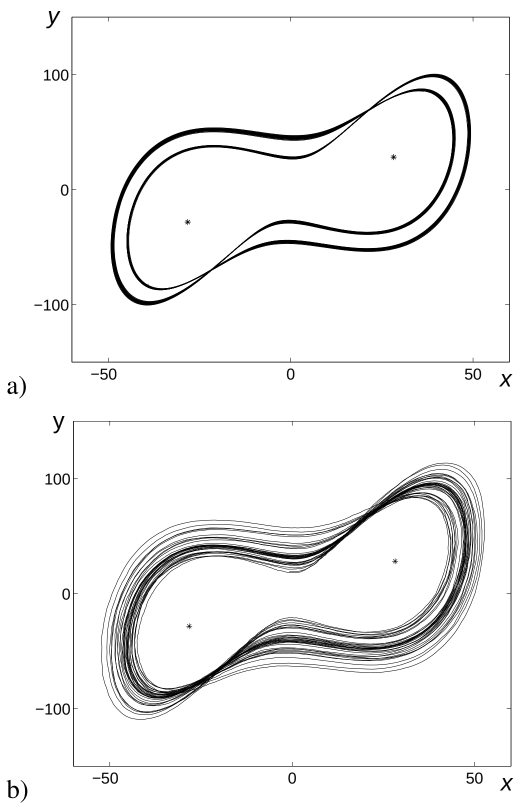

Figure 2. Stochastic attractors of the forced Lorenz model for $r = 300$ a) $\varepsilon = 0.3$; b) $\varepsilon = 2$

At the further decrease of r either of the two cycles demonstrates the standard cascade of period doubling with transition to chaos.

The noise disturbances result in a stochastic deformation of the deterministic unforced attractors. Under the random disturbances the trajectories of a stochastically forced system leave the deterministic attractor and form some bundle around it with a corresponding probabilistic distribution. A dispersion of random states of the stochastically forced system near a deterministic attractor depends on the noise intensity and stability properties of the attractor local parts. In Fig. 2, stochastic attractors of the system (1) for $r = 300$ and $\varepsilon = 0.3,\ 2$ are plotted.

Consider noise-induced transitions between two separate co-existing deterministic cycles. For small noises, random states are concentrated in the small neighborhoods of corresponding deterministic curves (see Fig. 2 a). As a noise intensity grows, the dispersion increases, two separate bundles of random trajectories approach and get mixed up (see Fig. 2 b). Probabilistic characteristics of these noise-induced transitions between basins of attraction for co-existing unforced cycles depends on both the stochastic sensitivity and a spatial arrangement of these deterministic attractors.

The results presented here give us a qualitative description of possible noise-induced transitions. More detailed quantitative analysis of noise-induced transitions between two coexisting cycles will be presented below using the stochastic sensitivity function (SSF) technique.

3 SSF analysis of the stochastic attractors and noise-induced transitions

Consider Ito's stochastic system

$$\dot{x} = f(x) + \varepsilon\sigma(x)\dot{w}. \qquad (2)$$

Here $w(t)$ is a $n-$dimensional Wiener process, $\sigma(x)$ is a matrix function of disturbances with intensity ε. Suppose the system (2) for $\varepsilon = 0$ has a T-periodic solution $x = \xi(t)$ with an exponentially stable phase curve γ.

The random trajectories of the forced system (2) leave the closed curve of deterministic cycle γ and due to cycle stability form some bundle around it.

The probabilistic distribution for the bundle of random trajectories localized near cycle has Gaussian approximation ([Bashkirtseva and Ryashko, 2000], [Bashkirtseva and Ryashko, 2004])

$$\rho \approx Ke^{-\frac{v(x)}{\varepsilon^2}} \approx K \exp\left(-\frac{(\Delta(x), \Phi^+(\gamma(x))\Delta(x))}{2\varepsilon^2}\right)$$

with covariance matrix $\varepsilon^2\Phi(\gamma)$. The covariance matrix characterizes the dispersion of the points of intersection of random trajectories with hyperplane orthogonal to cycle at the point γ. The function $\Phi(\gamma)$ is a stochastic sensitivity function (SSF) of the limit cycle.

We represent a function $\Phi(\gamma)$ in a parametric form. The solution $\xi(t)$ connecting the points of cycle γ with points of an interval $[0, T)$ gives the natural parametrization $\Phi(\xi(t)) = W(t)$. A matrix function $W(t)$ is a solution of the boundary value problem for Lyapunov equation

$$\dot{W} = F(t)W + WF^\top(t) + P(t)S(t)P(t), \qquad (3)$$

with conditions

$$W(t)r(t) \equiv 0, \qquad W(0) = W(T). \qquad (4)$$

Here

$$F(t) = \frac{\partial f}{\partial x}(\xi(t)), \;\; S(t) = \sigma(\xi(t))\sigma^\top(\xi(t)),$$

$$r(t) = f(\xi(t)), \;\; P(t) = P_{r(t)}, \;\; P_r = I - rr^\top/r^\top r,$$

where P_r is a projection matrix onto the subspace orthogonal to the vector $r \neq 0$. Details and mathematical background can be found in ([Bashkirtseva and Ryashko, 2004]).

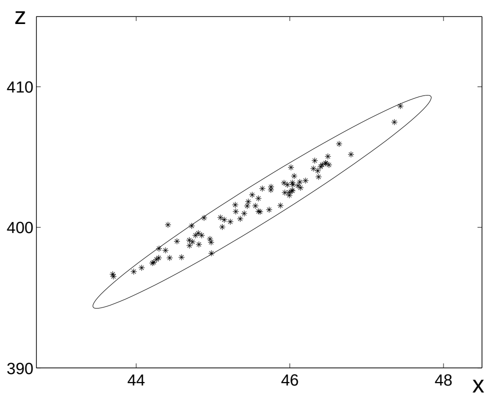

Figure 3. Poincare section of the stochastic cycle and confidence ellipse for $r = 330, \; \varepsilon = 1, \; p = 0.85$

For 3D-cycles, the matrix $W(t)$ has the following decomposition

$$W(t) = \lambda_1(t)v_1(t)v_1^\top(t) + \lambda_2(t)v_2(t)v_2^\top(t). \qquad (5)$$

Here $\lambda_1(t) \geq \lambda_2(t) \geq \lambda_3(t) \equiv 0$ are eigenvalues, and $v_1(t), v_2(t), v_3(t)$ are eigenvectors of the matrix $W(t)$.

The constructive method for computation of this decomposition is presented in ([Bashkirtseva and Ryashko, 2004]).

In a case of non-degenerate noises the functions $\lambda_1(t), \lambda_2(t)$ are strictly positive and determine for any t a dispersion of random trajectories around cycle along vectors $v_1(t), v_2(t)$.

Values $\lambda_1(t)$, $\lambda_2(t)$ determine the size and $v_1(t), v_2(t)$ determine the directions of a dispersion ellipse axes. The equation of this ellipse in a plane orthogonal to the cycle γ at the point $\xi(t)$ looks like

$$(x - \xi(t))^\top W^+(t)(x - \xi(t)) = 2k^2\varepsilon^2,$$

where the parameter k determines a fiducial probability $P = 1 - e^{-k}$.

We apply SSF technique to the analysis of noise-induced transitions between stochastic cycles of the forced Lorenz model (1).

Functions $\lambda_1(t), \; \lambda_2(t), \; v_1(t), \; v_2(t)$ allow to construct an ellipse of the dispersion of random trajectories intersection points with any Poincare section plane. As one can see in Fig. 3 the constructed ellipse precisely reflects a spatial dispersion of the intersection points. For the ellipse plotted in Fig. 3 the value of fiducial probability p equals 0.85.

So, the SSF technique allows to construct the dispersion ellipses for any Poincare sections. These ellipses give us a descriptive-geometric method for the analysis of noise-induced transitions between coexisting limit cycles.

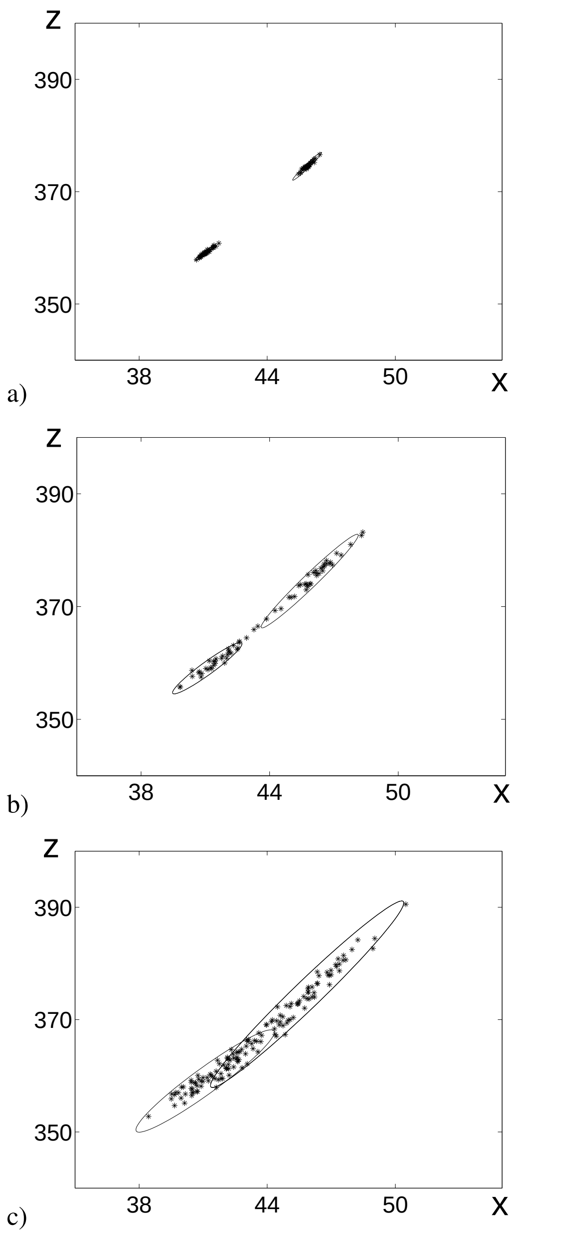

Figure 4. Noise-induced transitions $r = 300$ a) $\varepsilon = 0.3$; b) $\varepsilon = 1$; c) $\varepsilon = 2$

Compare results of SSF technique with direct numerical simulation. Let us track sequential phases of noise-induced transitions between two separate co-existing 3D-cycles with the help of Poincare sections. In Fig. 4, corresponding dispersion ellipses (solid line) found by SSF technique and intersection points (asterisks) of the random trajectories with a half-plane $y = 0,\ x > 0$ for fixed $r = 300$ and various values of noise intensity $\varepsilon = 0.3,\ 1,\ 2$ are plotted.

For $\varepsilon = 0.3$, random trajectories are localized near deterministic orbits (see Fig. 4 a). For $\varepsilon = 1$, intersection points approach (see Fig. 4 b) and for $\varepsilon = 2$ become essentially intermixed (see Fig. 4 c).

These ellipses clearly reflect essential peculiarities of random states distribution near the deterministic cycle. For $\varepsilon = 0.3$, dispersion ellipses are localized near deterministic orbits (see Fig.4a) and lie far from one another. As parameter ε grows, these ellipses approach and begin to intersect (see Fig. 4 b, c). This intersection gives a signal of noise-induced transition beginning. Our ellipses technique based on SSF method connects a level of noise-induced transition with a noise intensity parameter. In fact, a size and a mutual arrangement of dispersion ellipses allows to describe and predict effectively the main features of noise-induced transitions.

Conclusion

We study a basin-hopping phenomenon for systems with multistable states under the random disturbances. This paper has concentrated on the noise-induced transitions in 3D systems with limit cycles on the period-doubling route to chaos. The main probabilistic phenomenon and method of the analysis are presented for the well-known stochastically forced Lorenz model. We study noise-induced transitions between two coexisting separate cycles.

In this paper, we propose a universal theoretical approach to the quantitative and geometrical analysis of the probabilistic mechanism of noise-induced transitions between coexisting 3D-limit cycles. This approach is based on Poincare sections method and stochastic sensitivity function technique. This function provides a constructive approximation of the probabilistic distribution for stochastic 3D-cycles. In the presented paper it was shown that SSF technique is an effective method for noise-induced transitions analysis. This technique allows to find a spatial configuration and sizes of the dispersion ellipses of random trajectories for any Poincare sections. Dispersion ellipses are a plain and useful tool for the study of the probabilistic mechanism of noise-induced phenomena. For small noise, the dispersion ellipses are localized near deterministic cycles and definitely separated. As a noise intensity grows, these ellipses approach one to another and begin to intersect. This intersection marks a noise-induced transition beginning. In fact, a size and a spatial arrangement of dispersion ellipses allow to describe and predict effectively the main features of the noise-induced transitions without huge costs for direct numerical simulations of random trajectories.

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