ESTIMATION PROBLEM FOR LINEAR IMPULSIVE CONTROL SYSTEMS UNDER UNCERTAINTY

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Abstract

The paper deals with the state estimation problem for impulsive control system described by linear differential equations containing impulsive terms (or measures). The problem is studied under uncertainty conditions with set — membership description of uncertain variables, which are taken to be unknown but bounded with given bounds (e.g., the model may contain unpredictable errors without their statistical description). Basing on the techniques of approximation of the generalized trajectory tubes by the solutions of usual differential systems without measure terms and using the techniques of ellipsoidal calculus we present here a new state estimation algorithms for the studied impulsive control problem. The examples of construction of such ellipsoidal external estimates of reachable sets and trajectory tubes of linear impulsive control systems are given.

Key words

Impulsive control, differential inclusions, reachable sets, estimation.

1 Introduction

The topics of this paper come from the theory of dynamical systems with unknown, but bounded uncertainties (the case of the so-called "set-membership" description of uncertainties) [Kurzhanski, 1977; Kurzhanski and Valyi, 1997; Filippova, 2005]. Number of researches is devoted to the different aspects of the theory of optimization of dynamic systems with generalized (impulse) control [Filippova, 2005; Dykhta and Sumsonuk, 2000; Zavalischin and Sesekin, 1991].

In this paper the impulsive control and estimation problem for a dynamic systems with unknown but bounded initial states is studied. Such problems arise from mathematical models of dynamical and physical systems for which we have an incomplete description of time dependence of their generalized coordinates.

We discuss the approaches to solution concepts for such uncertain dynamical systems based on ideas of well known discontinuous time substitution [Rishel, 1965] and the techniques of differential inclusions theory [Filippov, 1985]. The approaches are based on the techniques of approximation of the discontinuous generalized trajectory tubes by the solutions of usual differential systems without measure terms.

Furthermore in this paper we use the well known results of the theory [Kurzhanski and Valyi, 1997; Chernousko, 1988] of ellipsoidal estimating of states of dynamical control systems with classical (measurable) controls and develop these results to find the upper set-valued bounds for reachable sets of linear impulsive control problem.

The ellipsoidal estimates of the reachable set of the linear impulsive control systems were obtained in [Vzdornova and Filippova, 2006; Vzdornova, 2007]. The algorithms of ellipsoidal estimates of such impulsive systems are based on the techniques of external and internal ellipsoidal approximations of a convex hull of the union of a family of ellipsoids.

In this paper we suggest another approach to the construction of ellipsoidal estimates of the reachable set of the linear impulsive control systems. To solve this problem we study the related differential inclusion of a classical type (without measure or impulsive controls) and find ellipsoidal estimates of its trajectory tubes projections of which coincide with the upper ellipsoidal bounds for reachable sets of the linear impulsive control system.

2 Problem Formulation

Consider a dynamical linear control system described by a differential equation with impulsive control $u(\cdot)$

$$\begin{gathered} dx = A(t)xdt + b(t)du, \\ x \in \mathbb{R}^n,\ x(t_0 - 0) = x_0,\ t \in [t_0, T]. \end{gathered} \tag{1}$$

Here we assume that $A(t)$ is a continuous $n\times n$ - matrix function on $[t_0, T]$, $b(t)$ is n-vector continuous function.

The initial value x_0 is unknown but bounded with a given bound.

$$x_0 \in \mathcal{X}_0 = \mathcal{E}(a, R), \tag{2}$$

where

$$\mathcal{E}(a, R) = \{x_0 {\in} \mathbb{R}^n | (x_0 - a)' R^{-1} (x_0 - a) \leq 1\},$$

R is a symmetric positive definite $n\times n$ matrix, $a \in \mathbb{R}^n$ is a center of the ellipsoid $\mathcal{X}_0$.

The impulsive control $u(t)$ ($u(\cdot) : [t_0, T] \to \mathbb{R}$) is continuous from the right, with bounded variation

$$\operatorname*{Var}_{t\in[t_0,T]} u(t) = \sup_{\{t_i\}} \sum_{i=1}^{k} |u(t_i) - u(t_{i-1})| \leq \mu, \tag{3}$$

where $u = (u_1, \ldots, u_n)$; $t_i : t_0 {<} \ldots {<} t_k = T$ and μ is a given positive number. We assume also that $u(t)$ is increasing on $[t_0, T]$.

Denote $\mathcal{U}$ the class of impulsive functions $u(t)$ that satisfied (3).

The solution $x(t) = x(t; t_0, u, x_0)$ of the control system (1) under constraints (2)–(3) has form

$$x(t; t_0, u, x_0) = X(t)x_0 + \int_{t_0}^{t} X(t)X^{-1}(\tau)b(\tau)du(\tau),$$

where $X(t)$ is the Cauchy matrix solution $\dot{X} = A(t)X$ ($X(0) = I$).

Denote

$$\mathcal{X}(t; t_0, \mathcal{X}_0) = \bigcup_{u(\cdot)\in\mathcal{U}} \bigcup_{x_0\in\mathcal{X}_0} x(t; t_0, u, x_0).$$

The set $\mathcal{X}(t) = \mathcal{X}(t; t_0, \mathcal{X}_0)$ is actually the reachable set of the impulsive differential system (1) from the initial set $\mathcal{X}_0$ at the instant t under restriction (3) for all possible admissible controls $u(\cdot)$.

The main problem of the paper is to find the ellipsoidal estimate $\mathcal{E}(a^+(T), Q^+(T))$ for the reachable set $\mathcal{X}(T)$ basing on the special structure of the data $\mathcal{X}_0$ and restriction (3) on the impulsive control.

3 Main Results

Basing on results of ellipsoidal calculus [Chernousko, 1988; Kurzhanski and Valyi, 1997] developed for linear uncertain systems and discrete-time versions of the funnel equations [Panasyuk, 1990] we present the modified state estimation approaches that allow to solve the problem.

3.1 The Approach

Let us introduce a new time variable and a new state coordinate [Rishel, 1965; Vinter and Pereira, 1988]:

$$\eta(t) = t + \int_{t_0}^{t} du(t), \qquad \tau(\eta) = \inf\{t \mid \eta(t) \geq \eta\}.$$

Consider the auxiliary differential inclusion [Filippova, 2005]

$$\frac{d}{d\eta}\begin{pmatrix} z \\ \tau \end{pmatrix} \in G(\tau, z), \tag{4}$$

$$z(t_0) = x_0 \in \mathcal{X}_0, \;\; \tau(t_0) = t_0, \;\; t_0 \leq \eta \leq T + \mu.$$

Here

$$G(\tau, z) {=} \bigcup_{0\leq\nu\leq1} \left\{ (1{-}\nu) \begin{pmatrix} A(\tau)z \\ 1 \end{pmatrix} {+} \nu \begin{pmatrix} b(\tau) \\ 0 \end{pmatrix} \right\}. \tag{5}$$

Denote $w = \{z, \tau\}$ the extended state vector of the system (4) and denote $w(\eta, t_0, w_0)$ the solution of the differential inclusion (4). Consider the trajectory tube of this differential inclusion:

$$\mathcal{W}(\eta) = \bigcup_{w_0\in\mathcal{X}_0\times\{t_0\}} w(\eta, t_0, w_0), \;\; t_0 \leq \eta \leq T + \mu. \tag{6}$$

From the properties of trajectory tubes of ordinary differential inclusion and the properties of the system (4) we conclude that the following theorem is valid.

Theorem 1 [Filippova, 2005]. *The reachable set $\mathcal{X}(T)$ is the projection of $\mathcal{W}(T + \mu)$ at the subspace of variables z:* $\quad \mathcal{X}(T) = \pi_z \mathcal{W}(T + \mu)$.

Applying Theorem 1 to construct the ellipsoidal estimates of the reachable set $\mathcal{X}(T)$ we need to construct first the ellipsoidal estimates of $\mathcal{W}(T + \mu)$.

It should be noted that the technique of ellipsoidal calculus can not be applied directly because the set-valued function $G(\tau, z)$ is nonlinear on state variables. We propose here an algorithm for estimating reachable sets $\mathcal{W}(\eta)$ based on the theory of integral funnel equations [Panasyuk, 1990]. The presented algorithm is similar to Euler's numerical scheme for finding set-valued states of differential systems.

3.1.1 Ellipsoidal Estimation of $\mathcal{W}(T + \mu)$

Let us consider the particular case of the funnel equation related to (4)–(5) [Filippova, 2005; Panasyuk, 1990]:

$$\lim_{\sigma\to+0}\sigma^{-1}h\Big(\mathcal{W}(\eta+\sigma),\bigcup(w+\sigma G(w_{n+1},w_1,\ldots,w_n)|\; w=(w_1,\ldots,w_{n+1})\in\mathcal{W}(\eta)\Big)=0,\Big)$$

$$\mathcal{W}(t_0)=\mathcal{W}_0,\ \eta\in[t_0,T+\mu], \tag{7}$$

here $h(A,B)$ is the *Hausdorff distance* between compact sets A, $B\subseteq\mathbb{R}^n$.

Under above mentioned assumptions the following theorem is true (details may be found in [Filippova, 2005; Panasyuk, 1990]).

Theorem 2. [Filippova, 2005; Panasyuk, 1990] *The nonempty compact-valued function $\mathcal{W}(\eta)$ is the unique solution to the evolution equation* (7).

From the properties of the solutions of the evolution equation (7) we conclude that the following theorem is valid.

Theorem 3. *For all $\sigma>0$ the following inclusion holds*

$$\mathcal{W}(t_0+\sigma)\subseteq\bigcup_{0\le\nu\le1}\begin{pmatrix}\mathcal{E}(a^+(t_0,\sigma,\nu),Q^+(t_0,\sigma,\nu))\\ t_0+\sigma(1-\nu)\end{pmatrix}+ +o(\sigma)B_*(0,1),\ \ \lim_{\sigma\to+0}\sigma^{-1}o(\sigma)=0, \tag{8}$$

where

$$B_*(0,1)=\{x\in\mathbb{R}^{n+1}|\ \|x\|\le1\},$$

$$a^+(t_0,\sigma,\nu)=(I+\sigma(1-\nu)A(t_0))a+\sigma\nu b(t_0),$$

$$Q^+(t_0,\sigma,\nu)=(I+\sigma(1-\nu)A(t_0))R(I+\sigma(1-\nu)A(t_0))'. \tag{9}$$

Proof. The proof of this theorem is carried out under the scheme of proof of Theorem 3 [Filippova and Berezina, 2008].

Remark. In the paper [Vzdornova and Filippova, 2006] we constructed the upper estimates of the union of ellipsoids under condition that ellipsoids are nondegenerate. Here the set

$$\mathcal{W}(t_0,\sigma)=\bigcup_{0\le\ \nu\le1}\mathcal{W}(t_0,\sigma,\nu) \tag{10}$$

with

$$\mathcal{W}(t_0,\sigma,\nu)=\begin{pmatrix}\mathcal{E}(a^+(t_0,\sigma,\nu),Q^+(t_0,\sigma,\nu))\\ t_0+\sigma(1-\nu)\end{pmatrix} \tag{11}$$

in (8) is the union of degenerate ellipsoids in the extended space $\mathbb{R}^{n+1}$ for each parameter ν.

So we fix an arbitrary $\epsilon>0$ and put the degenerated ellipsoid $\mathcal{W}(t_0,\sigma,\nu)$ into nondegenerated ellipsoid $\mathcal{E}_\epsilon(w(t_0,\sigma,\nu),O_\epsilon(t_0,\sigma,\nu))$:

$$\mathcal{W}(t_0,\sigma,\nu)\subseteq\mathcal{E}_\epsilon(w(t_0,\sigma,\nu),O_\epsilon(t_0,\sigma,\nu)), \tag{12}$$

$$w(t_0,\sigma,\nu)=\begin{pmatrix}a^+(t_0,\sigma,\nu)\\ t_0+\sigma(1-\nu)\end{pmatrix},$$

$$O_\epsilon(t_0,\sigma,\nu)=\begin{pmatrix}Q^+(t_0,\sigma,\nu) & 0\\ 0 & \epsilon^2\end{pmatrix}.$$

Therefore for any $\epsilon>0$ the following inclusion is true

$$\mathcal{W}(t_0,\sigma)\subset\mathcal{W}_\epsilon(t_0,\sigma)=\bigcup_{0\le\nu\le1}\mathcal{E}_\epsilon(w(t_0,\sigma,\nu),O_\epsilon(t_0,\sigma,\nu)). \tag{13}$$

3.1.2 Auxiliary Results. In order to construct the external estimate of $\mathcal{W}_\epsilon(t_0,\sigma)$, we consider two auxiliary problems.

Auxiliary Problem AP1. Find the ellipsoid $E^+_\epsilon(w^+(t_0,\sigma),O^+(t_0,\sigma))$ such that

$$\mathcal{W}_\epsilon(t_0,\sigma)\subset E^+_\epsilon(w^+(t_0,\sigma),O^+(t_0,\sigma)). \tag{14}$$

Therefore from (13) we will have also

$$\mathcal{W}(t_0,\sigma)\subset E^+_\epsilon(w^+(t_0,\sigma),O^+(t_0,\sigma)). \tag{15}$$

Remark. Because the set $\mathcal{W}(t_0,\sigma)$ is compact and functions $a^+(t_0,\sigma,\nu)$ and $Q^+(t_0,\sigma,\nu)$ are continuous, the equality

$$\lim_{\epsilon\to+0}h(\mathcal{W}(t_0,\sigma),\mathcal{W}_\epsilon(t_0,\sigma))=0$$

is true.

Auxiliary Problem AP2. Given two ellipsoids $\mathcal{E}(a_1,Q_1)$ and $\mathcal{E}(a_2,Q_2)$, ($a_i\neq0$, $Q_i=Q_i'>0$, $i=1,2$), find an external ellipsoid $\mathcal{E}(a^+,Q^+)$ such that

$$\mathcal{E}(a_1,Q_1)\cup\mathcal{E}(a_2,Q_2)\subseteq\mathcal{E}(a^+,Q^+).$$

The following lemmas are true.

Lemma 1. *For given a_1, a_2, Q_1, Q_2 ($a_i\neq0$, $Q_i=Q_i'>0$, $i=1,2$) the following inclusions hold*

$$\mathcal{E}(a_1,Q_1)\subset\mathcal{E}(a_1,Q_1)\cup\mathcal{E}(2a_2-a_1,Q_1)\subset$$
$$\subset\mathcal{E}(a_1,Q_1)+\mathcal{E}((a_2-a_1),(a_2-a_1)(a_2-a_1)')\subset$$
$$\subset\mathcal{E}(a_2,(1+p)Q_1+(1+p^{-1})(a_2-a_1)(a_2-a_1)')=$$
$$=\mathcal{E}(a_2,\tilde{Q}_1),\ \ p\in(0,((a_2-a_1)'Q_1^{-1}(a_2-a_1))^{\frac{1}{2}}].$$

Proof. For any matrix $Q = Q' > 0$ and any vector $a \in R^n$, $a \neq 0$ the following formula is valid [Vazhentsev, 2004]

$$\operatorname{co}(\mathcal{E}(0,Q) \cup \mathcal{E}(2a,Q)) = \mathcal{E}(0,Q) + \mathcal{E}(a,aa'). \quad (16)$$

The matrix of the ellipsoid $\mathcal{E} = \mathcal{E}(a, Q^+)$ which contains the sum $\mathcal{E}(0,Q) + \mathcal{E}(a,aa')$ may be found as in [Chernousko, 1988; Kurzhanski and Valyi, 1997]:

$$Q^+ = (p^{-1}+1)Q + (p+1)aa', \quad (17)$$

where parameter $p \in (0, (a'Q^{-1}a)^{\frac{1}{2}}]$. The proof of this lemma following from formulaes (16)–(17).

Lemma 2. [Vzdornova and Filippova, 2006] *The inclusion is true*

$$\mathcal{E}(a_2, \tilde{Q}_1) \cup \mathcal{E}(a_2, Q_2) \subset E(a_2, Q^+) = E(a^+, Q^+), \quad (18)$$

$$Q^+ = \tilde{Q}_1^{\frac{1}{2}} M' \tilde{O}^+ M \tilde{Q}_1^{\frac{1}{2}}, \quad \tilde{O}^+ = \operatorname{diag}\{\mu_1, \ldots, \mu_n\},$$

where $\mu_i = \max\{1, \lambda_i\}$, λ_i *are the eigenvalues of matrix* $\tilde{Q}_1^{-1} Q_2$, M *is the orthogonal matrix such that*

$$M' \tilde{Q}_1^{-\frac{1}{2}} Q_2 \tilde{Q}_1^{-\frac{1}{2}} M = \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}.$$

Proof. Details of the proof may be found in [Vzdornova and Filippova, 2006; Vzdornova, 2007].

Lemma 3. *The ellipsoid* $E(a^+, Q^+)$ *defined in* (18) *is the solution of the Auxiliary Problem AP2*:

$$\mathcal{E}(a_1, Q_1) \cup \mathcal{E}(a_2, Q_2) \subset E(a^+, Q^+).$$

Proof. The proof of this lemma follows directly from the construction of the ellipsoid $E(a^+, Q^+)$.

The following example illustrates the construction of the external ellipsoidal estimate of the union of two ellipsoids with different centers and matrices (the solution of Auxiliary Problem AP2).

Example 1. Consider two ellipsoids $\mathcal{E}(a_1, Q_1)$ and $\mathcal{E}(a_2, Q_2)$. Here $a_1 = (0, 0.1)$, $a_2 = (0.1, 0)$,

$$Q_1 = \begin{pmatrix} 1.09 & 0.9 \\ 0.9 & 9 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}.$$

Figure 1 shows the ellipsoid $\mathcal{E}(a_1, Q_1)$ (it is marked by number **1**), ellipsoid $\mathcal{E}(a_2, Q_2)$ (it is marked by number **2**) and the external ellipsoid $E(a^+, Q^+) \supset \mathcal{E}(a_1, Q_1) \cup \mathcal{E}(a_2, Q_2)$ (it is marked by number **3**).

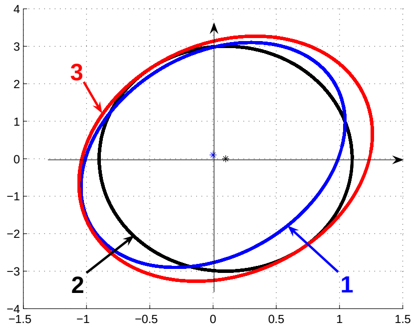

Figure 1. Ellipsoidal estimates of two ellipsoids with different centers

Algorithm of ellipsoidal estimation of $\mathcal{W}_\epsilon(t_0, \sigma)$ is given below.

Algorithm 1.
We fix arbitrary $\epsilon > 0$ and $\sigma > 0$. Subdivide the segment $[0,1]$ into subsegments $[\nu_j, \nu_{j+1}]$ where $\nu_j = jh$ $(j = 0, \ldots, m)$, $h = 1/m$, $\nu_0 = 0$, $\nu_m = 1$.

For given $\mathcal{X}_0 = \mathcal{E}(a, R)$ we find parameters $a^+(t_0, \sigma, \nu_j)$, $Q^+(t_0, \sigma, \nu_i)$ of ellipsoids $\mathcal{E}(a^+(t_0, \sigma, \nu_j), Q^+(t_0, \sigma, \nu_j))$ defined in (9) (Theorem 3) for $j = 0, \ldots, m$.

Given

$$\{\mathcal{E}(a^+(t_0, \sigma, \nu_j), Q^+(t_0, \sigma, \nu_j)) |\, j = 0, \ldots, m\}$$

we find $m+1$ ellipsoids

$$\mathcal{E}_\epsilon(w(t_0, \sigma, \nu_j), O_\epsilon(t_0, \sigma, \nu_j)), \quad j = 0, \ldots, m \quad (19)$$

using (12) in the extended space $\mathbb{R}^{n+1}$.
To solve the Auxiliary Problem AP1 (14) we need to find the ellipsoid $E_\epsilon^+(w^+(t_0, \sigma), O^+(t_0, \sigma))$ so that

$$\bigcup_j \mathcal{E}_\epsilon(w(t_0, \sigma, \nu_j), O_\epsilon(t_0, \sigma, \nu_j)) \subset$$
$$\subset E_\epsilon^+(w^+(t_0, \sigma), O^+(t_0, \sigma)).$$

Step 1. Consider the ellipsoids defined in (19) for $j = 0$ and $j = 1$, namely

$$\mathcal{E}_\epsilon(w(t_0, \sigma, \nu_0), O_\epsilon(t_0, \sigma, \nu_0)),$$
$$\mathcal{E}_\epsilon(w(t_0, \sigma, \nu_1), O_\epsilon(t_0, \sigma, \nu_1)).$$

Basing on the solution of the Auxiliary Problem AP2 we find the ellipsoid $E_\epsilon^{+1}(w^+(t_0, \sigma), O^+(t_0, \sigma))$ such that

$$\begin{gathered}\mathcal{E}_\epsilon(w(t_0,\sigma,\nu_0),O_\epsilon(t_0,\sigma,\nu_0)) \cup \\ \cup\, \mathcal{E}_\epsilon(w(t_0,\sigma,\nu_1),O_\epsilon(t_0,\sigma,\nu_1)) \subset \\ \subset E^{+1}_\epsilon(w^+(t_0,\sigma),O^+(t_0,\sigma)).\end{gathered}$$

Step 2. We take two ellipsoids:

$$\begin{gathered}E^{+1}_\epsilon(w^+(t_0,\sigma),O^+(t_0,\sigma)), \\ \mathcal{E}_\epsilon(w(t_0,\sigma,\nu_2),O_\epsilon(t_0,\sigma,\nu_2)),\ (j=2).\end{gathered}$$

As at the Step 1 we find the ellipsoid $E^{+2}_\epsilon(w^+(t_0,\sigma),O^+(t_0,\sigma))$ such that

$$\begin{gathered}E^{+1}_\epsilon(w^+(t_0,\sigma),O^+(t_0,\sigma)) \cup \\ \cup\, \mathcal{E}_\epsilon(w(t_0,\sigma,\nu_2),O_\epsilon(t_0,\sigma,\nu_2)) \subset \\ \subset E^{+2}_\epsilon(w^+(t_0,\sigma),O^+(t_0,\sigma)).\end{gathered}$$

Step 3. Next steps continue iterations 1–2.
At the end of the process we will get the external estimate $E^{+}_\epsilon(w^+(t_0,\sigma),O^+(t_0,\sigma))$

$$\begin{gathered}\bigcup_j \mathcal{E}_\epsilon(w(t_0,\sigma,\nu_j),O_\epsilon(t_0,\sigma,\nu_j)) \subset \\ \subset E^{+}_\epsilon(w^+(t_0,\sigma),O^+(t_0,\sigma)).\end{gathered}$$

Therefore we will have the estimate of the reachable set $\mathcal{W}_\epsilon(t_0,\sigma)$

$$\mathcal{W}_\epsilon(t_0,\sigma) \subset E^{+}_\epsilon(w^+(t_0,\sigma),O^+(t_0,\sigma))$$

which provides the solution of Auxiliary Problem AP1 (14).

3.2 Algorithm of Ellipsoidal Estimation of reachable set $\mathcal{X}(T)$.

Basing on the previous results we may formulate the following scheme that gives the external ellipsoidal estimate of $\mathcal{X}(T)$ of the system (1).

Algorithm 2. We fix arbitrary $\epsilon > 0$ and subdivide the time segment $[t_0, T+\mu]$ into subsegments $[t_i, t_{i+1}]$ where $t_i = t_0 + i\sigma$ ($i = 1,\dots,k$), $\sigma = (T+\mu-t_0)/k$, $t_k = T+\mu$.

Step 1. Consider the time segment $[t_0, t_1]$. We take the initial set $\mathcal{X}_0 = \mathcal{E}(a,R)$ in the Algorithm 1 and find the ellipsoid $E^{+1}_\epsilon(w^+(\sigma),O^+(\sigma))$ such that

$$\mathcal{W}(t_0,\sigma) \subseteq E^{+1}_\epsilon(w^+(\sigma),O^+(\sigma)),$$

where sets $\mathcal{W}(t_0,\sigma)$ are defined in (10).

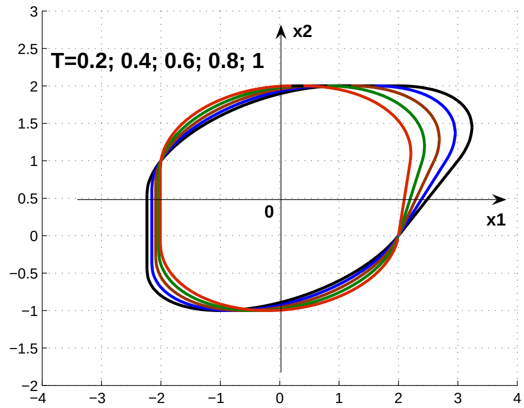

Figure 2. The dynamics of reachable sets $\mathcal{X}(T)$

Step 2. Consider the next time interval $[t_1, t_2]$. The ellipsoid $E^{+1}_\epsilon(w^+(\sigma),O^+(\sigma))$ is considered as the start ellipsoid at the moment t_1 for the Algorithm 1. Apply Algorithm 1 again. The resulted set $E^{+2}_\epsilon(w^+(\sigma),O^+(\sigma))$ will be the start ellipsoid for the next moment t_2 in the Algorithm 1.

Step 3. Repeat Step 2 for each moment $t_i : t_i = t_0 + i\sigma$ ($i = 2,\dots,k$). At the end of the process the ellipsoid $E^{+}_\epsilon(w^+,O^+)$ will be obtained so that $\mathcal{W}(T+\mu) \subseteq E^{+}_\epsilon(w^+,O^+)$.

Step 4. By Theorem 1 find the projection of the ellipsoid $E_\epsilon(w^+,O^+)$ at the subspace of variables $\{z_1,\dots,z_n\}$

$$\mathcal{E}(a^+(T),Q^+(T)) = \pi_z E_\epsilon(w^+,O^+).$$

Therefore we will have the external estimate $\mathcal{E}(a^+(T),Q^+(T))$ of the reachable set $\mathcal{X}(T)$ of system (1) from initial set $\mathcal{X}_0$ under restriction (3).

Example 2. Consider the following impulsive control system:

$$\begin{cases} dx_1(t) = x_2(t)dt, \\ dx_2(t) = du_2(t), \end{cases} \quad 0 = t_0 \le t \le T. \tag{20}$$

The impulsive control $u(t)$ is continuous from the right, with variation $\mathrm{Var}_{t\in[0,T]}\, u(t) \le 1$. We assume also that $u(t)$ is increasing on $[0,T]$.
The initial states x_0 are unknown but belong to the following ellipsoid $X_0 = \mathcal{E}(0,R)$,

$$x_0 \in X_0 = \mathcal{E}(0,R), \qquad R = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}.$$

The exact reachable set are presented at Figure 2 for some values of T. The tube of trajectories of the system (20) is indicated at Figure 3.

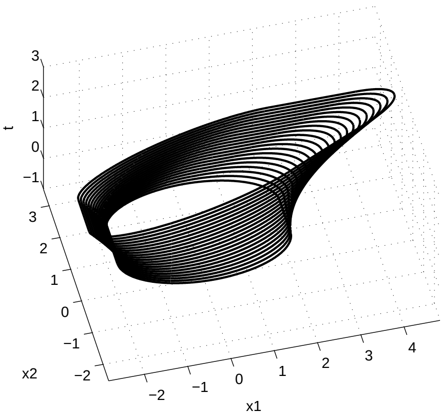

Figure 3. Trajectory tube $\mathcal{X}(t)$ for $t \in [0, 1.7]$

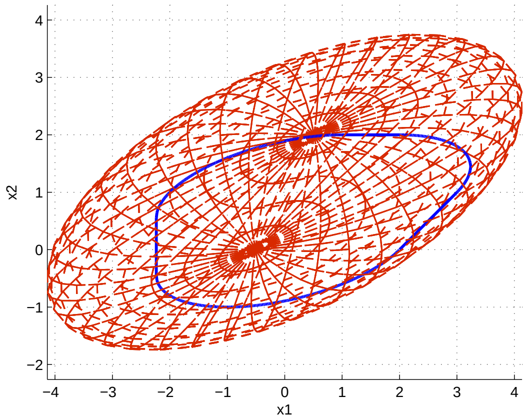

Figure 4. The estimate of the reachable set $\mathcal{X}(T)$ for $T = 1$

Figure 4 illustrates the external estimation algorithm 2. The external ellipsoidal estimates and exact reachable set are presented at Figure 4.

4 Conclusion

We consider the problems of state estimation for dynamical control systems with impulsive control and with unknown but bounded initial state.

Basing on results of ellipsoidal calculus developed for linear uncertain systems and discrete-time versions of the funnel equations we present the modified state estimation approaches that allow to solve the problem.

Suggested approach opens the way to solve the problem of estimating of the uncertain states of impulsive control systems under state constraints.

Examples and numerical results related to procedures of set-valued approximations of trajectory tubes and reachable sets are also presented.

Acknowledgements

The research was supported by the Russian Foundation for Basic Researches (RFBR) under the Project No. 09-01-00223, Collaboration Program in Fundamental Research of UB RAS and SB RAS ”Qualitative and Numerical Analysis of Evolution Equations and Control Systems” and Research Grant of UB RAS.

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