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SIMULTANEOUS STABILIZATION OF PERIODIC ORBITS AND FIXED POINTS IN DELAY-COUPLED LORENZ SYSTEMS

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Abstract

We study two delay-coupled Lorenz systems and demonstrate unified chaos control by noninvasive time-delayed coupling. Both an unstable periodic orbit and an unstable fixed point of the system can be stabilized close to a subcritical Hopf bifurcation. Using a multiple scales method, the systems are reduced to Hopf normal forms, and an analytical approach for stabilizing a periodic orbit as well as a fixed point of the system is developed. As a result, the equations for the characteristic exponents are derived in an analytical form, revealing the range of coupling parameters for successful stabilization. Finally, we illustrate the results with numerical simulations, which show good agreement with the theory.

Key words

Delay-coupled networks, Lorenz system, chaos control, unstable periodic orbit, unstable fixed point, subcritical Hopf bifurcation.

1 Introduction

Time-delayed feedback control (DFC), proposed by Pyragas [Pyragas, 1992], is a simple and convenient method to stabilize unstable periodic orbits (UPOs) occurring in a dynamical system. Since the DFC uses only the difference of the current and the delayed state where the time delay is given by the period of the UPO, the control is non-invasive and is applicable to systems whose equations of motion are unknown. Due to this convenience, the algorithm of DFC has been applied to quite diverse experimental systems and theoretical advances have also been made [Schöll and Schuster, 2008; Pyragas, 2006; Schikora *et al.*, 2006; Popovych *et al.*, 2006; Erneux and Kalmár-Nagy, 2007; Sieber and Krauskopf, 2007; Peil *et al.*, 2007; Sieber *et al.*, 2008; Omel'chenko *et al.*, 2008; Dahms *et al.*, 2008; Schöll *et al.*, 2009; Schneider *et al.*, 2009; Brandstetter *et al.*, 2010; Kyrychko *et al.*, 2009; Kehrt *et al.*, 2009; Dahms *et al.*, 2010; Gjurchinovski and Urumov, 2010; Heil *et al.*, 2001; Oliver *et al.*, 2011; Rosin *et al.*, 2011; Otto *et al.*, 2012].

However, it was commonly believed that torsion-free UPOs or, more precisely, UPOs with an odd-number of real Floquet multipliers larger than unity could not be stabilized by DFC [Nakajima, 1997; Just *et al.*, 1997]. To overcome this limitation, modified control schemes such as a half-period delay [Nakajima and Ueda, 1998] or the introduction of an unstable controller [Pyragas, 2001; Pyragas and Pyragas, 2006] were proposed for stabilizing UPOs in the Lorenz system that is a representative for the so-called odd number limitation [Nakajima and Ueda, 1998].

Fiedler *et al.* refuted this alleged odd-number theorem by a counterexample using the normal form of a subcritical Hopf bifurcation [Fiedler *et al.*, 2007], and Postlethwaite and Silber directly applied this refuting mechanism to UPOs created by subcritical Hopf bifurcations in the 3-variable Lorenz system [Postlethwaite and Silber, 2007], and generalized it to n-dimensional dynamical systems [Brown *et al.*, 2011]. When a time delay is introduced, however, reducing the Lorenz system to the standard normal form via the center manifold theory is a nontrivial task because the dynamics takes place in an infinite-dimensional phase space [Pyragas and Pyragas, 2006]. In Ref. [Choe *et al.*, 2010; Choe *et al.*, 2011], we have shown for a network of normal forms describing subcritical Hopf bifurcations that UPOs can be stabilized in-phase synchronously by delayed coupling. This has been extended to networks of Lorenz systems by a reduction using the method of multiple scales [Choe *et al.*, 2012]. Odd-number orbits have also been stabilized near a fold bifurcation [Fiedler *et al.*, 2008], in two coupled Hopf normal form oscillators [Fiedler *et al.*, 2010], and with more general feedback matrices [Flunkert and Schöll, 2011]. Experimental implementations were also given [von Loewenich *et al.*, 2010; Schikora *et al.*, 2011]. Recently, Hooton and Amann analyzed in detail the stabilization mechanism and presented a corrected form of the original odd-number theorem [Hooton and Amann, 2012].

Besides the control of UPOs, the stabilization of unstable steady states (USSs) has become a field of increasing interest. Although the field of controlling chaos deals mainly with the stabilization of UPOs, the problem of stabilizing USSs could be of practical importance in experimental situations where chaotic or periodic oscillations cause degradation in performance. One of the methods to control an USS introduced by Pyragas *et al.* uses the difference between the current state and a low-pass filtered version, in which an unstable degree of freedom was added to the feedback loop of the Lorenz system to overcome the topological limitation, similar to that of a time-delay feedback controller [Pyragas *et al.*, 2002; Pyragas *et al.*, 2004]. A DFC scheme in a diagonal coupling form was analytically investigated by Hövel *et al.* [Hövel and Schöll, 2005] and Yanchuk *et al.* [Yanchuk *et al.*, 2006] for the purpose of stabilizing USSs. For stronger couplings, diffusively coupled limit-cycle oscillators with time delay exhibit a coupling induced stabilization of an USS, by means of amplitude death of the oscillations [Ramana Reddy *et al.*, 1998; Ramana Reddy *et al.*, 1999].

In a previous study [Choe *et al.*, 2007], we proposed a time-delayed coupling method which makes it possible to stabilize not only UPOs but also USSs in two delay-coupled Hopf normal form systems as a result of conversion of stability. In this paper, we extend this idea to coupled Lorenz system as a representative of more complex dynamical systems, from the viewpoint of chaos control. We consider two delay-coupled Lorenz systems and develop a systematic analytical approach for delayed coupling control of dynamical systems close to a subcritical Hopf bifurcation. Using the multiple scales method [Nayfeh and Balachandran, 1995], the system is reduced to the normal form of a Hopf bifurcation, and equations for the characteristic exponents are derived in analytical form, thereby revealing the coupling parameters for successful stabilization. As a result, we can show that both UPO and USS in Lorenz systems can be stabilized by delay-coupling. Finally, we illustrate the results with numerical simulations of the Lorenz system close to a subcritical Hopf bifurcation using the multiple scales method [Nayfeh and Balachandran, 1995; Choe *et al.*, 2012].

The paper is organized as follows. In Sec. 2 we present our model and an outline of the stability diagram. We introduce a reduction of our model to two delay-coupled Hopf normal form systems using the method of multiple scales. In Sec. 3, we derive analytical stability conditions and confirm their validity by direct numerical simulations. Finally, in Sec. 4, we draw conclusions.

2 Two Delay-Coupled Lorenz Systems

We consider the following model of two delay-coupled Lorenz systems:

$$\dot{\mathbf{x}}_1 = \mathbf{F}(\mathbf{x}_1;\rho_1) + kH(\mathbf{x}_2 - \mathbf{x}_2^*), \tag{1a}$$

$$\dot{\mathbf{x}}_2 = \mathbf{F}(\mathbf{x}_2;\rho_2) - kH(\mathbf{x}_1 - \mathbf{x}_{1,\tau}), \tag{1b}$$

where

$$\mathbf{F}(\mathbf{x};\rho) = \begin{pmatrix} \sigma y - \sigma x \\ \rho x - y - xz \\ xy - \mathrm{b}z \end{pmatrix}$$

describes the Lorenz system with state vector $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ and the real parameters σ, ρ and b. We select the standard set of the parameter values, i.e., $\sigma = 10$, $\mathrm{b} = 8/3$, and the bifurcation parameter ρ is assumed to vary [Lorenz, 1963]. k denotes the coupling strength, the 3×3 matrix H is the connectivity matrix that determines which components of the vector $\mathbf{x}_j$ enter the coupling, and τ is the delay time. We abbreviate the delayed variables $\mathbf{x}_1(t-\tau)$ by $\mathbf{x}_{1,\tau}$. $\mathbf{x}_2^*$ denotes a steady state of the $\mathbf{x}_2$-system as discussed below.

It is well known [Lorenz, 1963; Sparrow, 1982] that the original Lorenz equations, i.e., $k = 0$, demonstrate different dynamical regimes on variation of the bifurcation parameter ρ, which are associated with the existence and stability of several equilibrium states. In brief, the system dynamics can be characterized by three regimes. For $0 < \rho < 1$, there exists the only stable fixed point at the origin $\mathbf{x} = (0, 0, 0)$. For $\rho > 1$, the origin becomes a saddle and two additional symmetrical stable fixed points $\mathbf{x} = \mathbf{x}_\pm^*(\rho) \equiv (\pm\sqrt{\mathrm{b}(\rho-1)}, \pm\sqrt{\mathrm{b}(\rho-1)}, \rho - 1)$ appear. For $\rho > \rho_H \approx 24.7368$, the steady states become unstable through the subcritical Hopf bifurcation at $\rho = \rho_H$. Just below this bifurcation point, for $\rho = \rho_H + \epsilon$, $\epsilon < 0$, there are two small unstable limit cycles $\tilde{\mathbf{x}}_\pm(t)$ surrounding the stable steady states $\mathbf{x}_\pm^*$. At the same values of the parameter ρ there exists a strange attractor and thus the system is multistable depending on initial conditions, i.e., the phase trajectory may either be attracted to one of the steady states or exhibit chaotic behavior on the strange attractor. The periodic orbit exists for $13.926 < \rho < \rho_H$; at the lower boundary it collides with a fixed point in a homoclinic bifurcation.

In the following, the value of the bifurcation parameter for the $\mathbf{x}_2$-system, which plays the role of the controller, is chosen in the range $\rho_2 > \rho_H$, and thus its fixed points become USSs. On the other hand, the $\mathbf{x}_1$-system to be controlled might exhibit UPO and USS according to the choice of parameter values of $\rho_1 < \rho_H$ and $\rho_1 > \rho_H$, respectively. Our aim is to stabilize an UPO or USS in system $\mathbf{x}_1$ and a USS in system $\mathbf{x}_2$ by choosing the proper delay-time and coupling strength.

We transform the variables of the coupled system using the eigenvectors of the fixed point at the bifurcation point as a basis for a new coordinate system. Then, applying the method of multiple scales we eliminate the decaying mode and obtain the normal form for oscillating modes with delayed-coupling. The details of this normal form reduction can be found in the Appendix.

We arrive at dynamical equations for the variables $Z_1, Z_2 \in \mathbb{C}$ in the normal form as

$$\begin{aligned}\dot{Z}_1 =&\big(\lambda_1 + i\omega_1 + (b_R + ib_I)\,|Z_1|^2\big)Z_1 \qquad (2a)\\ &+ Ke^{i\beta}Z_2,\\ \dot{Z}_2 =&\big(\lambda_2 + i\omega_2 + (b_R + ib_I)\,|Z_2|^2\big)Z_2 \qquad (2b)\\ &- Ke^{i\beta}\big(Z_1 - Z_{1,\tau}\big),\end{aligned}$$

where the coupling parameters $K, \beta \in \mathbb{R}$ follow from the coupling strength k, and the real parameters λ_1, ω_1, λ_2, ω_2, b_R, and b_I follow from the parameters of the Lorenz system and are derived in the Appendices, Eq. (34). The coupling parameters K and β follow from the coupling strength k and the choice of the matrix H. In this paper, we choose a matrix

$$H = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}. \qquad (3)$$

which is the simplest control scheme, since only one variable, i.e., the z-variable, is coupled. The rescaled coupling strength $K = |c|k$ includes the parameter $|c|$, which can also be found in Eq. (34).

For $b_R > 0$, Eqs. (2) without coupling ($K = 0$) describe the normal forms of a subcritical Hopf bifurcation with transformed bifurcation parameter $\lambda_j = \epsilon_j a_R$, where the scaling coefficient a_R can be found in Eq. (34). UPOs with radius $r_j^* = \sqrt{-\lambda_j/b_R}$ and period $T_j = 2\pi/\Omega_j = 2\pi/(\omega_j - \lambda_j b_I/b_R)$ exist for $\lambda_j < 0$. The Floquet exponents of the UPO are determined by $\Lambda_j^0 = -2\lambda_j$. For $\lambda_j > 0$, clearly, there is no limit cycle, and the origin $Z_j = 0$ is a USS with the characteristic exponents $\lambda_j + i\omega_j$.

3 Linear Stability Analysis

In this section, we analyze Eqs. (2) and demonstrate stabilization of both an UPO and an USS by using the delayed coupling. In the previous study [Choe *et al.*, 2007], we analyzed Eqs. (2) in the case that both the shear b_I and the coupling phase β are zero. In the normal form reduction of the Lorenz system, both b_I and β are non-zero, requiring a different approach, which we present here.

The reduced system Eq. (2) in the Hopf normal form admits an analytical derivation of the equation for the Floquet exponents and the stability conditions. In addition, the numerical analysis of the original system of nonlinear differential-difference Eqs. (1) is performed to confirm the analytical results. The bifurcation parameter of system $\mathbf{x}_2$ is fixed at $\rho_2 > \rho_H$, i.e., $\epsilon_2 > 0$, while the system $\mathbf{x}_1$ takes the parameter value either at $\rho_1 < \rho_H$ or $\rho_1 > \rho_H$ when stabilization of an UPO or an USS is considered, respectively.

3.1 Stabilization of UPO and USS for $\rho_1 < \rho_H$ and $\rho_2 > \rho_H$

First, consider stabilization of a UPO in the *reduced* system Eq. (2) with $\lambda_1 < 0$ and $\lambda_2 > 0$. The time delay τ is chosen to be equal to the period of the UPO, which allows for noninvasive control of the dynamical system.

Calculating the Floquet exponents Λ of the UPO is not straightforward since Eqs. (2a) and (2b) should be linearized around a UPO and a USS, respectively. Introducing the real amplitude r_1 and phase φ_1 as $Z_1(t) = r_1(t)e^{i\varphi_1(t)}$, we obtain the following equations:

$$\dot{r}_1 = (\lambda_1 + b_R r_1^2)r_1 + K\mathrm{Re}\left[e^{i(\beta-\varphi_1)}Z_2\right], \qquad (4a)$$

$$\dot{\varphi}_1 = \omega_1 + b_I r_1^2 + \frac{K}{r_1}\mathrm{Im}\left[e^{i(\beta-\varphi_1)}Z_2\right], \qquad (4b)$$

$$\dot{Z}_2 = (\lambda_2 + i\omega_2)Z_2 - Ke^{i\beta}\left(r_1 e^{i\varphi_1} - r_{1,\tau}e^{i\varphi_{1,\tau}}\right), \qquad (4c)$$

where $\varphi_{1,\tau} = \varphi_1(t-\tau)$ and the cubic term of Z_2 was neglected since we confine ourselves to the behavior close to the USS.

Using the ansatz $r_1(t) = r_1^*\big(1 + \delta r_1(t)\big)$, $\varphi_1(t) = \Omega_1 t + \delta\varphi_1(t)$ and $Z_2 = 0 + r_1^*\delta z_2$, expanding Eqs. (4) to linear order in the small deviations δr_1, $\delta\varphi_1$ and δz_2 around the periodic orbit, we obtain

$$\delta\dot{r}_1 =\Lambda_1^0\delta r_1 + K\mathrm{Re}\big(e^{i(\beta-\Omega_1 t)}\delta z_2\big), \qquad (5a)$$

$$\delta\dot{\varphi}_1 =2b_I\,(r_1^*)^2\,\delta r_1 + K\mathrm{Im}\big(e^{i(\beta-\Omega_1 t)}\delta z_2\big), \qquad (5b)$$

$$\begin{aligned}\delta\dot{z}_2 =&\big(\lambda_2 + i\omega_2\big)\delta z_2 \qquad (5c)\\ &+ Ke^{i(\beta+\Omega_1 t)}\big[\delta r_{1,\tau} - \delta r_1 + i(\delta\varphi_{1,\tau} - \varphi_1)\big],\end{aligned}$$

where $\Lambda_1^0 = \lambda_1 + 3b_R\,(r_1^*)^2 = -2\lambda_1$ is the Floquet exponent for the UPO of the decoupled free system.

As a next step, we use a transformation of $Z_2(t)$ to co-rotating complex coordinates $\zeta(t) = e^{i(\beta-\Omega_1 t)}Z_2$ and as above $\zeta = r_1^*\delta\zeta$, i.e., $\delta\zeta(t) = e^{i(\beta-\Omega_1 t)}\delta z_2$. This yields for Eq. (5c)

$$\begin{aligned}\delta\dot{\zeta} =&(\lambda_2 + i\Delta\bar{\omega})\delta\zeta\\ &+ Ke^{2i\beta}\left[\delta r_{1,\tau} - \delta r_1 + i(\delta\varphi_{1,\tau} - \delta\varphi_1)\right] \qquad (6)\end{aligned}$$

with $\Delta\bar{\omega} = \omega_2 - \Omega_1$. This expression can be further rewritten in terms of real-valued coordinates $\delta\zeta = \delta\zeta_R + i\delta\zeta_I$ as

$$\begin{pmatrix} \delta\dot{\zeta}_R \\ \delta\dot{\zeta}_I \end{pmatrix} = \begin{pmatrix} \lambda_2 & -\Delta\bar{\omega} \\ \Delta\bar{\omega} & \lambda_2 \end{pmatrix} \begin{pmatrix} \delta\zeta_R \\ \delta\zeta_I \end{pmatrix} + K \begin{pmatrix} \cos 2\beta & -\sin 2\beta \\ \sin 2\beta & \cos 2\beta \end{pmatrix} \begin{pmatrix} \delta r_{1,\tau} - \delta r_1 \\ \delta\varphi_{1,\tau} - \delta\varphi_1 \end{pmatrix}. \quad (7)$$

Since the coefficient matrices of Eqs. (5a), (5b) and (7) do not depend on time, the Floquet exponents of the periodic orbit are simply given by the eigenvalues Λ of the characteristic equation

$$\begin{vmatrix} \Lambda_1^0 - \Lambda & 0 & K & 0 \\ 2b_I\,(r_1^*)^2 & -\Lambda & 0 & K \\ K\chi_c(\Lambda) & -K\chi_s(\Lambda) & \lambda_2 - \Lambda & -\Delta\bar{\omega} \\ K\chi_s(\Lambda) & K\chi_c(\Lambda) & \Delta\bar{\omega} & \lambda_2 - \Lambda \end{vmatrix} = 0, \quad (8)$$

where we used the abbreviations $\chi_c(\Lambda) = \left(e^{-\Lambda\tau} - 1\right)\cos(2\beta)$ and $\chi_s(\Lambda) = \left(e^{-\Lambda\tau} - 1\right)\sin(2\beta)$ for notational convenience. Clearly, the characteristic equation admits the solution $\Lambda = 0$, which corresponds to the trivial Floquet mode of the UPO with Floquet multiplier 1. In the absence of the coupling, $K = 0$, Eq. (8) yields the quartic equation

$$\Lambda(\Lambda - \Lambda_1^0)\left[(\Lambda - \lambda_2)^2 + \Delta\bar{\omega}^2\right] = 0.$$

In the approximate case of diagonal coupling ($\beta = 0$) and $\omega_2 = \Omega_1$, i.e., $\Delta\bar{\omega} = 0$, the characteristic equation (8) factorizes:

$$\left[\Lambda(\Lambda - \lambda_2) + K^2(1 - e^{-\Lambda\tau})\right] \times \left[(\Lambda - \Lambda_1^0)(\Lambda - \lambda_2) + K^2(1 - e^{-\Lambda\tau})\right] = 0. \quad (9)$$

In order to appreciate the behavior of the roots at a rough estimate, we reduce Eq. (9) to a polynomial equation using an expansion $e^{-\Lambda\tau} \approx 1 - \Lambda\tau$ for small $|\Lambda|\tau$ as follows as

$$\Lambda(\Lambda + \kappa - \lambda_2)\left[\Lambda^2 + (\kappa - \Lambda_1^0 - \lambda_2)\Lambda + \Lambda_1^0\lambda_2\right] = 0, \quad (10)$$

where $\kappa = K^2\tau$. The roots of the second and third factors give the nontrivial Floquet exponents, of which the former crosses into the left half plane at $\kappa = \kappa_1 \equiv \lambda_2$ and the latter show the behavior as considered in Ref. [Pyragas *et al.*, 2002]: For $\kappa = 0$, the Floquet exponents are Λ_1^0 and λ_2. With the increase of κ, they approach each other on the real axis, then collide at $\kappa = \Lambda_1^0 + \lambda_2 - 2\sqrt{\Lambda_1^0\lambda_2}$ and form a complex conjugate pair in the complex plane. At $\kappa = \kappa_2 \equiv \Lambda_1^0 + \lambda_2$, they cross symmetrically into the left half plane (inverse Hopf bifurcation). Taking into account $\kappa_2 > \kappa_1$ due to $\Lambda_1^0 = -2\lambda_1 > 0$, the dominant Floquet exponents are determined by quadratic equation,

$$\Lambda^2 + (\kappa - \Lambda_1^0 - \lambda_2)\Lambda + \Lambda_1^0\lambda_2 = 0, \quad (11)$$

which provides the mechanism of stabilization of UPO and yields the stability condition $K^2\tau > -2\lambda_1 + \lambda_2$.

We validate these qualitative estimates using the polynomial approximation through the solutions of the full transcendental equation Eq. (8) and, more accurately, through the numerical calculation of the variational equations of the exact systems Eqs. (1).

We determine the *exact* Floquet exponents by linearization of Eqs. (1) around UPO for $\rho_1 < \rho_H$ and $\rho_2 > \rho_H$:

$$\delta\dot{\mathbf{x}}_1 = \tilde{A}_1(t)\delta\mathbf{x}_1 + kH\delta\mathbf{x}_2, \quad (12a)$$

$$\delta\dot{\mathbf{x}}_2 = \bar{A}_2\delta\mathbf{x}_2 - kH\left(\delta\mathbf{x}_1 - \delta\mathbf{x}_{1,\tau}\right), \quad (12b)$$

where $\tilde{A}_1(t) = D_{\mathbf{x}}\mathbf{F}(\tilde{\mathbf{x}}_1(t);\rho_1)$ and $\bar{A}_2 = D_{\mathbf{x}}\mathbf{F}(\mathbf{x}_2^*;\rho_2)$. $D_{\mathbf{x}}\mathbf{F}$ denotes the matrix of first partial derivatives of $\mathbf{F}$ with respect to the vector arguments. Here $\tilde{A}_1(t)$ is the Jacobian taken on the UPO $[\tilde{x}_1(t), \tilde{y}_1(t), \tilde{z}_1(t)] = [\tilde{x}_1(t+\tau), \tilde{y}_1(t+\tau), \tilde{z}_1(t+\tau)]$, thus it is a τ-periodic 3×3 matrix and $\delta\mathbf{x} = (\delta\mathbf{x}_1, \delta\mathbf{x}_2)$ denotes a small deviation from the periodic orbit $\tilde{\mathbf{x}}(t) = (\tilde{\mathbf{x}}_1(t), \mathbf{x}_2^*)$ which is a solution of the decoupled system.

The Floquet exponents of the exact variational Eqs. (12) have been calculated by the algorithm described in Ref. [Pyragas, 2002]. According to the Floquet theory, solutions of Eqs. (12) can be decomposed into eigenfunctions

$$\delta\mathbf{x} = e^{\Lambda t}\mathbf{w}(t), \quad \mathbf{w}(t) = \mathbf{w}(t+\tau),$$

and the delay term can be eliminated, $\delta\mathbf{x}(t-\tau) = e^{-\Lambda\tau}\delta\mathbf{x}(t)$. The characteristic equation for the Floquet exponents reads

$$\det\left\{\Psi(\Lambda,\tau) - e^{\Lambda\tau}I\right\} = 0, \quad (13)$$

where I is the 6×6 identity and $\Psi(\Lambda, t)$ is the fundamental matrix of Eqs. (12) that is defined by the equalities

$$\dot{\Psi}(\Lambda,t) = \left[\tilde{A}(t) + G(\Lambda)\right]\Psi(\Lambda,t), \quad \Psi(\Lambda,0) = I$$

with $\tilde{A}(t) = \begin{pmatrix} \tilde{A}_1(t) & \mathbf{0} \\ \mathbf{0} & \bar{A}_2 \end{pmatrix}$ and $G(\Lambda) = k\begin{pmatrix} 0 & 1 \\ (e^{-\Lambda\tau} - 1) & 0 \end{pmatrix} \otimes H$. Here, $\mathbf{0}$ is the 3×3 null matrix and $\otimes$ is the direct product.

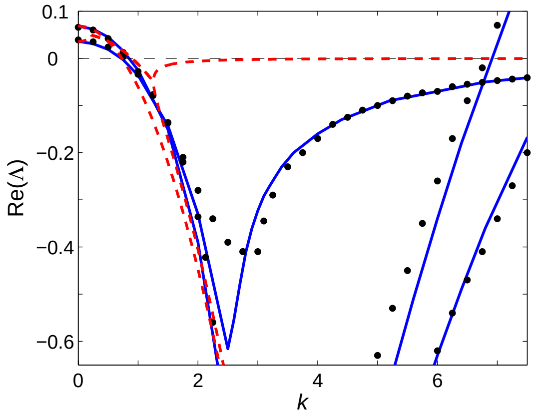

Figure 1. (Color online) Real parts of leading Floquet exponents Λ of UPO in delay-coupled Lorenz systems as a function of coupling strength k. The dashed (red) and solid (blue) lines denote the solutions of the polynomial Eq. (10) and the transcendental Eq. (8), respectively. Dots correspond to $\mathrm{Re}\ \Lambda$ obtained from the exact variational Eqs. (12). Parameters are given by $\rho_1 = 24.144$ (i.e., $\lambda_1 = -0.01792$, $\omega_1 = 9.5164$), $\rho_2 = 27$ (i.e., $\lambda_2 = 0.0684$, $\omega_2 = 10.03466$), $\tau = 0.674$, and Eqs. (3) and (34).

In Fig. 1, we compare the real parts of the Floquet exponents Λ as a function of the coupling gain *k*, which were determined by three different methods, namely, (i) using the solutions of the approximate quadratic equation (10) in dashed (red) line, (ii) by solving the transcendental equation (8) in solid (blue) line, and (iii) by solving Eq. (13) for the exact Floquet exponents of the system (1) in the black dots. Indeed, we see that there exists an interval of coupling gain *k* for which the real parts of Λ are all negative, so that both an UPO of Eq. (2a) and an USS of Eq. (2b) become stable. The parameters are given by $\rho_1 = 24.144$ (i.e., $\lambda_1 = -0.01792$, $\omega_1 = 9.5164$), $\rho_2 = 27$ (i.e., $\lambda_2 = 0.0684$, $\omega_2 = 10.03466$), $\tau = 0.674$, and Eqs. (3) and (34).

To verify the validity of the linear stability analysis, we have numerically investigated the original system Eqs. (1). The results of direct numerical integration of Eqs. (1) with z-coupling given by Eq. (3) are presented in Fig. 2. Without the coupling ($k = 0$), the two Lorenz systems demonstrate chaotic behavior on the strange attractor. When the coupling perturbation of $k = 2.5$ is applied at $t = 60$, the system $\mathbf{x}_1$ approaches an UPO, while the system $\mathbf{x}_2$ converges into an USS (Fig. 2(a) and (b), respectively). After a transient process, the coupling perturbation vanishes (Fig. 2(c)) and thus our delay-coupling method allows for noninvasive control of UPO. We observed in numerical simulations that the basin of attraction for stabilizing the UPO and USS includes the whole phase space in contrast to the situation of Hopf normal form systems [Choe *et al.*, 2007] and Lorenz systems with unstable controller [Pyragas and Pyragas, 2006], which can be explained by the properties of the strange attractor of two chaotic systems.

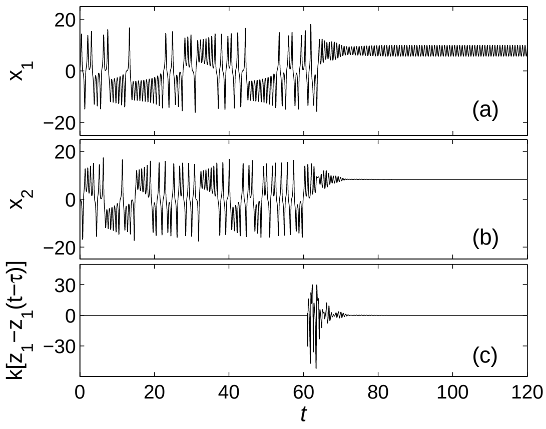

Figure 2. Dynamics of (a) variable x_1, (b) variable x_2, and (c) the delayed coupling perturbation $k[z_1(t) - z_1(t-\tau)]$. The coupling control with $k = 2.5$ is switched on at $t = 60$. The values of the parameters are $\rho_1 = 24.144$, $\rho_2 = 27$ and $\tau = 0.674$, and the connectivity matrix H is given by Eq. (3).

3.2 Stabilization of Two USSs for $\rho_1 > \rho_H$ and $\rho_2 > \rho_H$

We now consider the problem for stabilizing USSs in both systems. First, the linear stability of the USS of the reduced system (2) is analyzed. We linearize Eqs. (2) around $Z_1 = Z_2 = 0$ to obtain the characteristic equation for the eigenvalue Λ

$$\det\{A_0(\Lambda) - \Lambda I\} = 0, \tag{14}$$

where the linearized matrix A_0 is given by $A_0(\Lambda) = \begin{bmatrix} \lambda_1 + i\omega_1 & Ke^{i\beta} \\ -Ke^{i\beta}(1 - e^{-\Lambda\tau}) & \lambda_2 + i\omega_2 \end{bmatrix}$ and I is the 2×2 identity matrix. The bifurcation parameters λ_1 and λ_2 are positive and the delay time is chosen as $\tau = 2\pi/\bar{\omega}$ with $\bar{\omega} = (\omega_1 + \omega_2)/2$.

The matrix $A_0(\Lambda)$ remains invariant under the transformation $\Lambda \mapsto \Lambda + i(\omega_1 + \omega_2)/2$ due to $e^{i\bar{\omega}\tau} = 1$, and Eq. (14) can be rewritten in the form

$$\left(\Lambda - \lambda_1 + i\frac{\Delta\omega}{2}\right)\left(\Lambda - \lambda_2 - i\frac{\Delta\omega}{2}\right) + K^2 e^{2i\beta}\left(1 - e^{-\Lambda\tau}\right) = 0. \tag{15}$$

(Note that the stability is determined only by the real part of Λ.) This is a transcendental equation with an infinite number of complex roots and we are interested how the eigenvalues move from the right half-plane for $K = 0$ into the left half-plane with increasing value of K.

Now, for a rough estimate, we again use an approximation $e^{-\Lambda\tau} \approx 1 - \Lambda\tau$ for $|\Lambda|\tau \ll 1$. Furthermore, if the diagonal coupling, $\beta = 0$, is assumed, then Eq. (15) is reduced, for $\omega_1 = \omega_2$ or $\lambda_1 = \lambda_2$, to a quadratic equation with real coefficients. For $\omega_1 = \omega_2$, for example, we obtain the simple equation characterizing

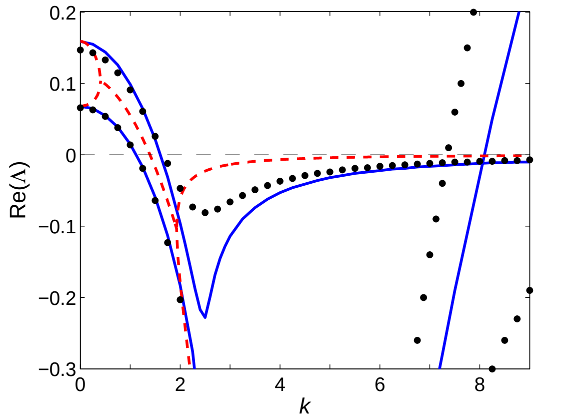

Figure 3. (Color online) Real parts of the eigenvalues Λ of USSs in delay-coupled Lorenz systems as a function of coupling strength k. The dashed (red) and solid (blue) lines denote the solutions of the polynomial Eq. (16) and the transcendental Eq. (14), respectively. Dots correspond to $\mathrm{Re}\ \Lambda$ obtained from the exact characteristic Eqs. (18). Parameters are given by $\rho_1 = 30$ (i.e., $\lambda_1 = 0.159$ and $\omega_1 = 10.579$), $\rho_2 = 27$ (i.e., $\lambda_2 = 0.068$ and $\omega_2 = 10.035$), $\tau = 0.61$, and Eqs. (3) and (34).

stability as

$$\Lambda^2 + (\kappa - \lambda_1 - \lambda_2)\Lambda + \lambda_1\lambda_2 = 0, \qquad (16)$$

where $\kappa = K^2\tau$. Note that Eq. (16) coincides with Eq. (11) and yields the stability condition $K^2\tau > \lambda_1 + \lambda_2$. This means that stabilization of USS could also be explained with the same mechanism as in the case of UPO, and the stability condition of USS reads $K^2\tau > \lambda_1 + \lambda_2$.

Next, we determine the *exact* eigenvalues Λ of the fixed points $(\mathbf{x}_1^*, \mathbf{x}_2^*)$ by linearization of Eqs. (1) for ρ_1,$\rho_2 > \rho_H$:

$$\delta\dot{\mathbf{x}}_1 = \bar{A}_1\delta\mathbf{x}_1 + kH\delta\mathbf{x}_2, \qquad (17a)$$
$$\delta\dot{\mathbf{x}}_2 = \bar{A}_2\delta\mathbf{x}_2 - kH\left(\delta\mathbf{x}_1 - \delta\mathbf{x}_{1,\tau}\right), \qquad (17b)$$

which yields the characteristic equation

$$\det\left\{\bar{A}(\Lambda) - \Lambda I_6\right\} = 0, \qquad (18)$$

where $\bar{A}_1 = D_{\mathbf{x}}\mathbf{F}(\mathbf{x}_1^*;\rho_1)$, $\bar{A}_2 = D_{\mathbf{x}}\mathbf{F}(\mathbf{x}_2^*;\rho_2)$, $\bar{A}(\Lambda) = \begin{bmatrix} \bar{A}_1 & kH \\ (e^{-\Lambda\tau}-1)kH & \bar{A}_2 \end{bmatrix}$, and I_6 is the 6×6 identity matrix.

Figure 3 shows the real parts of the eigenvalues Λ as a function of the coupling gain k, which were determined by different characteristic equations: The dashed line (red), solid line (blue) and dots (black) correspond to the eigenvalues obtained from the approximate quadratic polynomial Eq. (16), the reduced transcendental Eq. (14) and the exact transcendental Eqs. (18), respectively. There exists an interval of coupling gain k for which the largest real parts of Λ are negative, so that the two USSs become stable. The parameters are given by $\rho_1 = 30$ (i.e., $\lambda_1 = 0.159$ and $\omega_1 = 10.579$), $\rho_2 = 27$ (i.e., $\lambda_2 = 0.068$ and $\omega_2 = 10.035$), $\tau = 0.61$, and Eqs. (3) and (34).

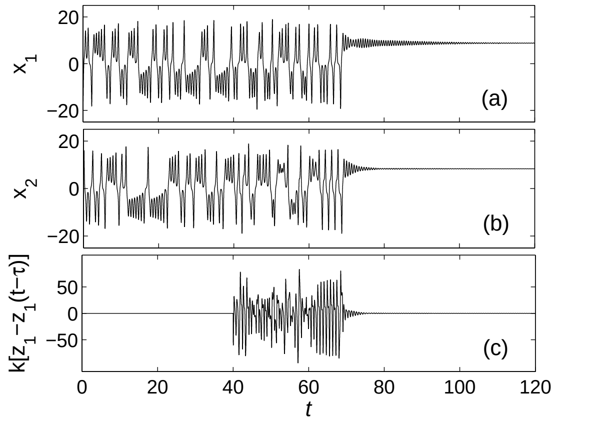

Figure 4. Dynamics of (a) variable x_1, (b) variable x_2, and (c) the delayed coupling perturbation $k[z_1(t) - z_1(t-\tau)]$. The coupling control with $k = 2.5$ is switched on at $t = 40$. The values of the parameters are $\rho_1 = 24.144$, $\rho_2 = 27$ and $\tau = 0.61$, and the connectivity matrix H is given by Eq. (3).

Direct integration of the original system (1) with the above parameter values also confirms the results of linear stability analysis (Fig. 4). Initially, the decoupled system ($k = 0$) is in a chaotic regime, and the two USSs are stabilized (amplitude death) when the coupling of $k = 2.5$ is switched on at $t = 40$ (Figs. 4(a) and (b)). As seen from Fig. 4(c), the perturbation vanishes as the stabilization of the USSs is attained. Therefore, our delay-coupling method allows for noninvasive control.

4 Conclusions

We have demonstrated a unified control method for stabilizing both a periodic orbit and a fixed point of the Lorenz system close to a subcritical Hopf bifurcation by noninvasive delayed coupling of two systems. We have developed systematic analytical approaches for reducing the system into Hopf normal forms and for stabilizing a periodic orbit and a fixed point using the multiple scales method and linear stability analysis. As a result the characteristic equations for Floquet exponents of the UPO and for eigenvalues of the USS have been derived in analytical form, which reveal the coupling parameters for successful stabilization.

To verify the validity of the linear stability analysis, we have performed numerical simulations of the original system, which show good agreement with the theory, i.e., the time-delay coupling method is capable of stabilizing not only UPOs but USSs as well in the Lorenz systems for a wide interval of the coupling strength. In particular, two Lorenz systems with this coupling exhibit a global basin of attraction for stabilizing an UPO and an USS at the same time due to the nature of the strange attractor, which is in striking contrast to the situation of the delay-coupled Hopf normal form systems [Choe *et al.*, 2007] and the Lorenz system with unstable controller [Pyragas and Pyragas, 2006].

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A Transforming the System Variables

In order to reduce the coupled Lorenz systems Eq. (1) to coupled Hopf normal forms, we first shift the origins of both phase space and bifurcation parameter to a fixed point, e.g., $\mathbf{x}_j^*(\rho) = \mathbf{x}_{j+}^*$, and to the Hopf bifurcation point ρ_H, respectively, by using the transformations $\mathbf{x}_j(t) = \mathbf{x}_j^* + \mathbf{u}_j(t)$ and $\rho_j = \rho_H + \epsilon_j$, respectively, with $j = 1, 2$. We rewrite Eq. (1) in the form

$$\dot{\mathbf{u}}_1 = A^{(0)}\mathbf{u}_1 + \epsilon_1 A^{(1)}\mathbf{u}_1 + N(\mathbf{u}_1) + kH\mathbf{u}_2, \quad (19a)$$

$$\dot{\mathbf{u}}_2 = A^{(0)}\mathbf{u}_2 + \epsilon_2 A^{(1)}\mathbf{u}_2 + N(\mathbf{u}_2) - kH(\mathbf{u}_1 - \mathbf{u}_{1,\tau}), \quad (19b)$$

where $A^{(0)}$ is the Jacobian evaluated at the point $\mathbf{x}_j^*(\rho_H)$, $\epsilon_j A^{(1)}$ is a small deviation due to the shift of the parameter ρ_j from the bifurcation point ρ_H, and $N(\mathbf{u}_j)$ defines the nonlinear part. The Lorenz system yields the matrices

$$A^{(0)} = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -p \\ p & p & -\mathrm{b} \end{pmatrix}, \; A^{(1)} = q \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix},$$
$$N(\mathbf{u}_j) = \begin{pmatrix} 0 \\ -\mathrm{u}_{j1}\mathrm{u}_{j3} \\ \mathrm{u}_{j1}\mathrm{u}_{j2} \end{pmatrix}, \quad (20)$$

where $p = \sqrt{\mathrm{b}(\rho_H - 1)}$, $q = \sqrt{\mathrm{b}/(\rho_H - 1)}/2$, and the approximation $\sqrt{\mathrm{b}(\rho - 1)} - \sqrt{\mathrm{b}(\rho_H - 1)} \approx \epsilon\sqrt{\mathrm{b}/(\rho_H - 1)}/2$ was used.

Let Φ be the matrix that transforms the matrix $A^{(0)}$ into Jordan canonical form, i.e., the columns of the matrix Φ are the eigenvectors $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ of the matrix $A^{(0)}$ corresponding to the eigenvalues γ_1, γ_2, γ_3, respectively. Solving the eigenvalue problem for the matrix $A^{(0)}$, we obtain three eigenvalues

$$\gamma_1 = \gamma_2^* \equiv i\omega_0 \approx 9.624i, \; \gamma_3 \approx -13.666, \quad (21)$$

and the matrix $\Phi = [\mathbf{p}_1\, \mathbf{p}_2\, \mathbf{p}_3]$ reads

$$\Phi = \begin{pmatrix} 0.268 + 0.306i & 0.268 - 0.306i & 0.863 \\ -0.027 + 0.564i & -0.027 - 0.564i & -0.316 \\ 0.7187 & 0.7187 & -0.395 \end{pmatrix}. \quad (22)$$

Then, under the transformation $\mathbf{u}_j(t) = \Phi\mathbf{v}_j(t) = \sum_{m=1}^{3} \mathrm{v}_{j,m}(t)\mathbf{p}_m$, Eq. (19) yields the eigenmode equations as follows

$$\dot{\mathbf{v}}_1 = J\mathbf{v}_1 + \epsilon_1 A\mathbf{v}_1 + B\mathbf{v}_1^2 + \epsilon_1 k_1 C\mathbf{v}_2, \quad (23a)$$

$$\dot{\mathbf{v}}_2 = J\mathbf{v}_2 + \epsilon_2 A\mathbf{v}_2 + B\mathbf{v}_2^2 - \epsilon_2 k_2 C(\mathbf{v}_1 - \mathbf{v}_{1,\tau}), \quad (23b)$$

where $J = \Phi^{-1}A^{(0)}\Phi = \mathrm{diag}\,(\gamma_1, \gamma_2, \gamma_3)$, $A = \Phi^{-1}A^{(1)}\Phi$, and $C = \Phi^{-1}H\Phi$ are the 3×3 similarity matrices of $A^{(0)}$, $A^{(1)}$ and H, respectively. The notation $B\mathbf{v}_j^2 \equiv \Phi^{-1}N(\Phi\mathbf{v}_j)$ can be regarded as the product of the 3×6 matrix B and the column vector with $\mathrm{P}_3^2 = 6$ elements, $\mathbf{v}_j^2 \equiv (\mathrm{v}_{j1}^2,\ \mathrm{v}_{j1}\mathrm{v}_{j2},\ \mathrm{v}_{j2}^2,\ \mathrm{v}_{j1}\mathrm{v}_{j3},\ \mathrm{v}_{j2}\mathrm{v}_{j3},\ \mathrm{v}_{j3}^2)^T$. The coupling strength in Eqs. (23) was rescaled as $k = \epsilon_j k_j$ in order that the influences of the coupling terms are of the same order as the bifurcation parameter term $\epsilon_j A\mathbf{v}_j$.

Since a pair of eigenvalues is complex conjugate, i.e., $\gamma_2 = \gamma_1^*$, the corresponding eigenvectors are also complex conjugate: $\mathbf{p}_2 = \mathbf{p}_1^*$. Moreover, the amplitudes of the corresponding eigenmodes have to be complex conjugate, $\mathrm{v}_{j2} = \mathrm{v}_{j1}^*$, in order to provide the *real* valued solution for $\mathbf{u}_j(t)$. Therefore, it is sufficient to observe only one of the two eigenmodes, e.g., v_{j1}.

In the numerical simulations in this paper, we use z-coupling with the connectivity matrix (3). According to the expressions (20) and (3), the matrices A, B and C are given in Fig. 5.

B Application of the Multiple Scales Method

We now approach the task of simplifying Eq. (23), i.e., reducing the dimensionality and eliminating the nonlinearity in the term $B\mathbf{v}_j^2$ as far as possible. For doing that, we apply an approximation, the method of multiple scales [Nayfeh and Balachandran, 1995], seeking an expansion of the form

$$\mathrm{v}_{j1} = \sum_{l=1}^{3} \mu^l \xi_{jl}(T_0, T_1, T_2) + O(\mu^4), \quad (25a)$$

$$\mathrm{v}_{j3} = \sum_{l=1}^{3} \mu^l \eta_{jl}(T_0, T_1, T_2) + O(\mu^4), \quad (25b)$$

where the time scales T_l are defined by $T_l = \mu^l t$ and μ is a small positive dimensionless parameter that is artificially introduced to establish the different orders of magnitude. The results obtained are independent of this parameter, and it is ultimately absorbed back into

$$A = \begin{pmatrix} 0.03+0.18i & 0.049+0.005i & 0.047-0.044i \\ 0.049-0.005i & 0.03-0.18i & 0.047+0.044i \\ 0.042-0.048i & 0.042+0.048i & -0.06 \end{pmatrix}, \tag{24a}$$

$$B = \tag{24b}$$

$$\begin{pmatrix} -0.26+0.25i & 0.25+0.31i & 0.09+0.02i & 0.07+0.6i & -0.07+0.16i & -0.21-0.27i \\ 0.09-0.02i & 0.25-0.31i & -0.26-0.25i & -0.07-0.16i & 0.07-0.6i & -0.21+0.27i \\ 0.15+0.05i & 0.07 & 0.15-0.05i & 0.27-0.18i & 0.27+0.18i & -0.07 \end{pmatrix},$$

$$C = \begin{pmatrix} 0.4348+0.0459i & 0.4348+0.0459i & -0.239-0.0252i \\ 0.4348-0.0459i & 0.4348-0.0459i & -0.239+0.0252i \\ -0.2373 & -0.2373 & 0.1304 \end{pmatrix}. \tag{24c}$$

Figure 5. Matrices for numerical simulations.

the solution, which is equivalent to setting it equal to unity in the final analysis. In terms of the T_l, the time derivative becomes

$$\frac{d}{dt} = D_0 + \mu D_1 + \mu^2 D_2 + \cdots \tag{26}$$

where $D_l = \partial/\partial T_l$. Also, the parameter ϵ is ordered as $\epsilon_j = \mu^2 \epsilon'_j$, so that the influences of the nonlinear terms, coupling terms, and the bifurcation parameter term $\epsilon_j A\mathbf{v}_j$ are of the same order.

Substituting Eqs. (25) and (26) into Eqs. (23), and equating coefficients of powers of μ, we obtain the following hierarchy of equations. For the first order $O(\mu)$ we get:

$$D_0\xi_{j1} - i\omega_0\xi_{j1} = 0, \tag{27a}$$
$$D_0\eta_{j1} - \gamma_3\eta_{j1} = 0. \tag{27b}$$

The nondecaying solution of Eqs. (27) is

$$\xi_{j1} = W_j(T_1, T_2)e^{i\omega_0 T_0}, \tag{28a}$$
$$\eta_{j1} = 0, \tag{28b}$$

where W_j is determined by imposing the solvability conditions at the next levels of the approximation.

For the second order $O(\mu^2)$ we obtain:

$$D_0\xi_{j2} - i\omega_0\xi_{j2} = b_{11}\xi_{j1}^2 + b_{12}\xi_{j1}\xi_{j1}^* + b_{13}\xi_{j1}^{*2} - D_1\xi_{j1}, \tag{29a}$$

$$D_0\eta_{j2} - \gamma_3\eta_{j2} = b_{31}\xi_{j1}^2 + b_{32}\xi_{j1}\xi_{j1}^* + b_{33}\xi_{j1}^{*2}. \tag{29b}$$

Substituting Eqs. (28) into Eqs. (29) and eliminating the source of secular terms, we have $D_1 W_j = 0$ or $W_j = W_j(T_2)$. Then, the solutions of Eqs. (29) are

$$\xi_{j2} = \frac{b_{11}W_j^2 e^{2i\omega_0 T_0}}{i\omega_0} - \frac{b_{12}W_j W_j^*}{i\omega_0} - \frac{b_{13}W_j^{*2}e^{-2i\omega_0 T_0}}{3i\omega_0}, \tag{30a}$$

$$\eta_{j2} = \tag{30b}$$
$$-\left(\frac{b_{31}W_j^2 e^{2i\omega_0 T_0}}{\gamma_3 - 2i\omega_0} + \frac{b_{32}W_j W_j^*}{\gamma_3} + \frac{b_{33}W_j^{*2}e^{-2i\omega_0 T_0}}{\gamma_3 + 2i\omega_0}\right),$$

where the general solutions of the homogeneous equations of (29) were omitted since they have no influence on the source of secular terms in the next level.

Finally, we obtain for the third order $O(\mu^3)$:

$$\begin{aligned} D_0\xi_{13} - i\omega_0\xi_{13} &= \epsilon'_1(a_{11}\xi_{11} + a_{12}\xi_{11}^*) + 2b_{11}\xi_{11}\xi_{12} \\ &+ b_{12}(\xi_{11}\xi_{12}^* + \xi_{11}^*\xi_{12}) + 2b_{13}\xi_{11}^*\xi_{12}^* \\ &+ b_{14}\xi_{11}\eta_{12} + b_{15}\xi_{11}^*\eta_{12} - D_2\xi_{11} \\ &+ \epsilon'_1 k_1(c_{11}\xi_{21} + c_{12}\xi_{21}^*), \end{aligned} \tag{31a}$$

$$\begin{aligned} D_0\xi_{23} - i\omega_0\xi_{23} &= \epsilon'_2(a_{11}\xi_{21} + a_{12}\xi_{21}^*) + 2b_{11}\xi_{21}\xi_{22} \\ &+ b_{12}(\xi_{21}\xi_{22}^* + \xi_{21}^*\xi_{22}) + 2b_{13}\xi_{21}^*\xi_{22}^* \\ &+ b_{14}\xi_{21}\eta_{22} + b_{15}\xi_{21}^*\eta_{22} - D_2\xi_{21} \\ &+ \epsilon'_2 k_2\left[c_{11}(\xi_{11,\tau} - \xi_{11})\right. \\ &\qquad \left. + c_{12}(\xi_{11,\tau}^* - \xi_{11}^*)\right], \end{aligned} \tag{31b}$$

where $\xi_{11,\tau} = \xi_{11}(T_0 - \tau,\ T_1 - \mu\tau,\ T_2 - \mu^2\tau)$.

Substituting Eqs. (30) into Eq. (31) and eliminating the terms that produce secular terms, we obtain equations for the slowly varying amplitude

$$\frac{dW_1}{dT_2} = (\epsilon'_1 a + b\,|W_1|^2)W_1 + \epsilon'_1 k_1 c W_2, \tag{32a}$$

$$\frac{dW_2}{dT_2} = (\epsilon'_2 a + b\,|W_2|^2)W_2 + \epsilon'_2 k_2 c(\tilde{W}_{1,\mu^2\tau} - W_1), \tag{32b}$$

where $\tilde{W}_{1,\mu^2\tau} = e^{-i\omega_0\tau} W_1(T_2 - \mu^2\tau)$ and the complex parameters are given as follows:

$$a = a_{11}, \tag{33a}$$

$$b = \frac{i}{\omega_0}\left(b_{11}b_{12} - b_{12}b_{12}^* - \frac{2}{3}b_{13}b_{13}^*\right) \tag{33b}$$

$$- \frac{b_{14}b_{32}}{\gamma_3} - \frac{b_{15}b_{31}}{\gamma_3 - 2i\omega_0},$$

$$c = c_{11}. \tag{33c}$$

Multiplying Eq. (32) with $e^{i\omega_0 T_0}$ and setting $\mu = 1$, we finally arrive at two delay-coupled Hopf bifurcation systems as Eq. (2), where $Z_j(t) = e^{i\omega_0 t} W_j(t)$, $\lambda_j = \epsilon_j a_R$, $\omega_j = \omega_0 + \epsilon_j a_I$, $a_R = \mathrm{Re}(a)$, $a_I = \mathrm{Im}(a)$, $b_R = \mathrm{Re}(b)$, $b_I = \mathrm{Im}(b)$, $\beta = \arg(c)$ and $K = k|c|$.

The set of parameters a, b, c and ω_0 determines the coupled system Eq. (2) of Hopf normal forms. According to Eqs. (24) and (33), the values of the parameters are given as

$$a_R = 0.03022, \tag{34a}$$

$$a_I = 0.18145, \tag{34b}$$

$$b_R = 0.00256, \tag{34c}$$

$$b_I = -0.02765, \tag{34d}$$

$$|c| = 0.4372, \tag{34e}$$

$$\beta = 0.105244, \tag{34f}$$

$$\omega_0 = 9.624. \tag{34g}$$

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