Sample-Based Minimax Approach and Its Application to Linear Quadratic Optimization under Uncertainty

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During the recent period of time, several techniques have been elaborated to reduce the sensitivity of the standard procedures of estimation, optimization and control to the initial uncertainty in the models under consideration, i.e., to make them robust with respect to the shortage of initial information about the model parameters [1]–[13]. Most of the results are based on the minimax approach, which means that the optimal in a minimax sense strategy minimizes the upper bound of the risk functional that is calculated over some known set of the model parameters containing their true values with acceptable reliability.

In this paper, we study the following optimization problem:

$$x^o \in \arg\min_{x \in X} \mathsf{D}\{\langle \xi, x \rangle\}, \tag{1}$$

where $\mathsf{D}\{\eta\}$ denotes the variance of the random value η;

$$X = \{x \in \mathbb{R}^p \colon Ax = a,\ Bx \leqslant b\}, \tag{2}$$

where $x \in X$ is a vector of strategies subject to optimization; $\xi \in \mathbb{R}^p$ is a random vector of the model perturbations with the expectation $\mathsf{E}\{\xi\} = \mu$ and the covariance matrix $\mathsf{cov}\{\xi, \xi\} = V$. As it follows from (2), the set X of admissible strategies is defined by the systems of linear equations and inequalities with known parameters $A \in \mathbb{R}^{l \times p}$, $a \in \mathbb{R}^l$, $B \in \mathbb{R}^{k \times p}$, and $b \in \mathbb{R}^k$.

From (1) it follows that the risk functional and the corresponding optimization problem take the form

$$x^o \in \arg\min_{x \in X} J(x, V), \tag{3}$$

where

$$J(x, V) = \langle Vx, x \rangle \tag{4}$$

is a quadratic risk functional.

It is well known that the quadratic optimization problem (2)–(4) arises in many important practical situations such as mean-square optimal linear parameter identification [1]–[3], [5], [7], [13]. discrete-time linear filtering and control [1], [3], [5], [9], [14], optimal portfolio selection and investment planning [8], [10]–[12].

In the case of known parameters of the optimization model the desired strategy $\hat{x}$ can be easily calculated using modern numerical algorithms such as Second Order Cone Programming (SOCP) [6], [15] which makes possible to obtain the result in a very effective way. Unfortunately, the model parameters in practice are subject to serious perturbations, and the solutions of the optimization problem are often very sensitive to the above-mentioned perturbations. As an example, the covariance matrix V introduced above is not known as usual neither in estimation framework nor in portfolio and investment planning [2], [7], [10], [11]. In this case it makes sense to transform the original optimization problem into a minimax one:

$$\hat{x} \in \arg\min_{x \in X} \sup_{V \in \mathcal{V}} J(x, V), \tag{5}$$

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where $\mathcal{V}$ is a set of possible values of the covariance matrix V. Since $\mathcal{V}$ defines the restrictions on the uncertain covariance, from now on $\mathcal{V}$ is called the uncertainty set.

Assuming the uncertainty set to be fixed one can further use the approaches based on the duality theory of the minimax programs [1], [8], [13], or the SOCP theory developed in [6], [9]–[11] to solve the minimax problem (5). But, in fact, the described solution is yet incomplete since we still have no effective approach for the uncertainty set $\mathcal{V}$ determination. The very natural way to introduce $\mathcal{V}$ is to construct it as a confidence region for V using the statistical data (measurements) about the random vector ξ behavior. Combining minimax approach with statistical estimation procedures has been pioneered by Goldfarb and Iyengar [10] within the SOCP computational framework. In this paper we suggest to combine the statistical approach together with the minimax optimization techniques developed in [8], [13]. We show that the above-mentioned approach makes possible to elaborate a rather simple numerical optimization algorithm, which is based on the standard quadratic programming method in the general case, and to obtain even the analytical solutions in some special but rather valuable cases. We shall call the approach described the sample-based minimax one (as well as the corresponding strategy $\hat{x}$) since we use sampled data to determine the appropriate uncertainty set $\mathcal{V}$ for the unknown covariance matrix V. The main purposes of the paper are the following ones:

- to derive and examine the sample-based minimax strategy calculation algorithm;
- to study the special cases when the desired strategy can be evaluated analytically;
- to examine the statistical properties of the strategy obtained and to demonstrate its robustness;
- to illustrate the algorithm's performance on the Markowitz optimization problem with uncertain parameters.

Note that the sample-based minimax approach changes radically the nature of the strategies obtained, since they become random, as we use statistical data to formulate the optimization problem, which was originally stated as a deterministic one.

Let us consider the solution of the minimax problem (2), (5) taking into account the following *a priori* and statistical information:

- the random vector of perturbations $\xi \in \mathbb{R}^p$ is normally distributed with the uncertain mean μ and nonsingular covariance V;
- the available multivariate observations $\xi^{(1)}, \dots, \xi^{(n)}$ are mutually independent and follow the Gaussian law $\mathcal{N}(m, V)$.

Let us choose some appropriate confidence threshold β and construct the corresponding confidence region for the uncertain covariance.

Assume that the sample size is large enough for the asymptotic results that follow from the central limit theorem. Consider the sample covariance

$$\widetilde{V}_n = \frac{1}{n-1} \sum_{i=1}^{n} (\xi^{(i)} - \overline{\xi}_n)(\xi^{(i)} - \overline{\xi}_n)^*, \tag{6}$$

where the sample mean is denoted by $\overline{\xi}_n$.

From [16] we can show that $\|I - \widetilde{V}_n^{-1/2} V \widetilde{V}_n^{-1/2}\|_2^2 \cdot (n-1)/2$ converges in distribution to random variable following chi-square law $\chi^2(p(p+1)/2)$ as $n \to \infty$, where $\| \cdot \|_2$ is the Frobenius norm.

Thus the confidence set has the form $\widetilde{\mathcal{V}}^{(n)} = \{V \in \mathbb{R}_+^{p \times p}\colon \|I - \widetilde{V}_n^{-1/2} V \widetilde{V}_n^{-1/2}\|_2 \leqslant \delta^{(n)}\}$, where $\delta^{(n)} = \sqrt{2 \cdot \chi_\beta^2(p(p+1)/2)/(n-1)}$, and $\chi_\beta^2(r)$ denotes the β-quantile of chi-square distribution with r degrees of freedom.

Finally, the confidence set presented above satisfies $\lim\limits_{n \to \infty} \mathsf{P}\{V \in \widetilde{\mathcal{V}}^{(n)}\} = \beta$.

The main result of this paper is the following.

Theorem 1: Given the uncertainty set $\widetilde{\mathcal{V}}^{(n)}$ *the sample-based minimax strategy*

$$\hat{x}_n \in \arg\min_{x\in X} \max_{V\in\widetilde{\mathcal{V}}^{(n)}} J(x,V) \tag{7}$$

coincides with the sample-based optimal solution

$$\widetilde{x}_n \in \arg\min_{x\in X} J(x,\widetilde{V}_n), \tag{8}$$

where $\widetilde{V}_n$ *is the sample covariance.*

Moreover, $\hat{x}_n$ *is* ε_n*-optimal with probability* β, *i.e.*,

$$\lim_{n\to\infty} \mathsf{P}\{J(\hat{x}_n,V) \leqslant J(x^o,V)+\varepsilon_n\} = \beta, \tag{9}$$

where x^o *is the optimal solution corresponding the true covariance matrix* V,

$$\varepsilon_n = 2\,\delta^{(n)} J(\widetilde{x}_n,\widetilde{V}_n). \tag{10}$$

At last, *the minimax risk is equal to*

$$\hat{J}_n = \min_{x\in X} \max_{V\in\widetilde{\mathcal{V}}^{(n)}} J(x,V) = (1+\delta^{(n)}) J(\widetilde{x}_n,\widetilde{V}_n). \tag{11}$$

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