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METHOD OF DETECTING UNSTABLE PERIODIC SPATIO-TEMPORAL STATES OF SPATIAL EXTENDED CHAOTIC SYSTEMS

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Abstract

The method of detection of the unstable periodic spatio-temporal states of spatial extended chaotic systems has been proposed. The application of this method is illustrated by the consideration of two different systems: (i) the fluid model of Pierce diode being one of the fundamental system of the physics of plasmas and (ii) the complex one-dimensional Ginzburg-Landau equation demonstrating different regimes of spatio-temporal chaos.

Key words

spatial extended chaotic systems, unstable orbits

1 Introduction

It is well known that the unstable periodic orbits (UPOs) embedded into chaotic attractors play an important role in the dynamics of the systems with a small number of the degree of freedom (Cvitanović, 1988). The chaotic regime of the system may be characterized by means of the set of UPOs (Carroll, 1999). A universal and powerful tool for exploration of chaotic dynamics (Cvitanović, 1991), UPOs proved to be especially efficient in context of chaotic synchronization (Pikovsky *et al.*, 1997*a*; Pikovsky *et al.*, 1997*b*; Hramov *et al.*, 2005) and the problem of the chaos controlling (Ott *et al.*, 1990). In the last case UPOs may be stabilized by means of the week influence on the system dynamics by the small variation of the control parameter (Ott *et al.*, 1990) or with the delay feedback (Pyragas, 1992).

In the spatial extended systems the unstable periodic spatio-temporal states (UPSTSs) exist (Franceschini *et al.*, 1999) which are similar to the UPOs in the chaotic systems with a small number of the degree of freedom. In particular, the chaotic dynamics of spatial extended systems may be controlled by stabilizing such unstable periodic spatio-temporal states (Boccaletti *et al.*, 1999). Therefore, one of the important problem connected with the study of the spatial extended chaotic system is finding these UPSTSs. It is appropriate to suggest that the methods aimed at the search of UPOs of the dynamical systems with small dimension of phase space may be adapted to the spatial extended systems. The method proposed by D.P. Lathrop and E.J. Kostelich (Lathrop and Kostelich, 1989), as an example, had been used to pick out UPSTSs for the fluid model of Pierce diode (Rempen and Hramov, 2004). This method is based on the obtaining the histograms describing the frequency of system returning to the vicinity of UPOs (in the low-dimensional systems) or UPSTSs (in the spatial extended systems), respectively. Nevertheless, this method applied to spatial extended systems is rather imprecise and time-consuming.

In this report we describe the modification of the method of P. Schmelcher and F. Diakonos (SD–method) (Schmelcher and Diakonos, 1997; Pingel *et al.*, 2001) allowing precise detection of UPSTSs in the spatial extended chaotic systems (Hramov and Koronovskii, 2007). As the sample analyzed spatially extended chaotic systems we consider here the one-dimensional complex Ginzburg-Landau equation (CGLE) and the fluid model of Pierce diode.

2 Detection of UPSTS of chaotic dynamics in the Pierce diode

As the primary system under study we have used the fluid model of Pierce diode (Godfrey, 1987; Hramov and Rempen, 2004) being one of the simplest beam-plasma systems demonstrating chaotic dynamics. It consists of two plane infinite grids pierced by the electron beam. The grids are grounded and the distance between them is L. The entrance space charge density ρ_0 and velocity v_0 are maintained constant. The space between the grids is evenly filled by the neutralizing ions with density $|\rho_i/\rho_0| = 1$. The dynamics of this system is defined by the only parameter, the so-called Pierce parameter $\alpha = \omega_p L/v_0$, where ω_p is the plasma frequency. With $\alpha > \pi$ Pierce instability develops, which leads to the appearance of the virtual cathode. At the same time, with $\alpha \sim 3\pi$, the instability is limited by non-linearity and the regime of complete passing of the electron beam through the diode space can be observed. In this case the system can be described by the partial differential equations:

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=\frac{\partial \varphi}{\partial x},\ \frac{\partial \rho}{\partial t}=-\frac{\partial (v\rho)}{\partial x},\ \frac{\partial^2 \varphi}{\partial x^2}=\alpha^2(\rho-1), \tag{1}$$

with the boundary conditions:

$$v(0,t)=1, \quad \rho(0,t)=1, \quad \varphi(0,t)=\varphi(1,t)=0. \tag{2}$$

In equations (1)–(2) the non-dimensional variables (space charge potential φ, density ρ, velocity v, space coordinate x and time t) are used (see, for example, (Hramov and Rempen, 2004; Filatov *et al.*, 2006)).

One of the core problems related to the spatial extended system consideration is the infinite dimension of the "phase space" W^∞. As a consequence, the state $\mathbf{U}(x,t)$ of the system of study should be considered instead of vector $\mathbf{x}(t)$ in $\mathbb{R}^n$ as in the case of the flow systems. For the system (1) this state $\mathbf{U}(x,t) = (v(x,t), \rho(x,t), \varphi(x,t))^T$ is the vector of the functions characterizing the system dynamics. After the transient finished the set of the states $\mathbf{U}(x,t)$ may be considered as attracting subspace W^s of the infinite–dimensional "phase space" W^∞ of the spatial extended system under study. If the dimension of this subspace is finite, the finite-dimensional space $\mathbb{R}^m$ of variables may be used to describe the dynamics of the spatial extended system.

In is well-known that SD-method was developed to the UPOs detection in the systems with discrete time, although it may be also applied to the flow systems (Pingel *et al.*, 2001) by means of reducing them to maps with the help of Poincaré secant. In order to apply the SD method to an extended system, we assume that its infinite-dimensional phase space possesses the low-dimensional attracting invariant subspace W^s, and the desired solution lies in this subspace. Further, we construct the auxiliary system $\mathbf{y}(t)$ in which the vector field $\mathbf{y}$ is in one-to-one correspondence with W^s.

The stationary states $\mathbf{U}^0(x,t) = \mathbf{U}^0(x)$ of the spatial extended system correspond to the fixed points in the phase space of the auxiliary system, while the periodic spatio-temporal states of (1) are in one-to-one correspondence with the periodic orbits of the finite-dimensional system $\mathbf{y}(t)$. Therefore, UPSTSs of spatial extended system may be found by means of the detection of UPOs of the auxiliary finite-dimensional system.

There are many well-known methods for applying low-dimensional variable space to describe the behavior of the spatial extended system, among which a typical one is the mode expansion method. Therein we propose the use of the variables taken from several points x_i of the extended system space to construct the finite dimensional system

$$\mathbf{y}(t) = (\rho(x_1,t),\ldots,\rho(x_m,t))^T, \tag{3}$$

where m is the dimension of the auxiliary system, $x_i = iL/(m+1)$, $i = \overline{1,m}$. In comparison with the other known methods (for example, Galerkin method), such approach allows us to undergo easily from the spatial extended system state $\mathbf{U}(x,t)$ to the low–dimensional vector $\mathbf{y}(t)$ without any additional calculations.

For the system under study (1) we have estimated the dimension of the auxiliary vector $\mathbf{y}(t)$ as $m = 3$. This assumption is based on the results of the consideration of the finite-dimensional model of the Pierce diode dynamics obtained with the help of Galerkin method (Hramov and Rempen, 2004).

To confirm meeting of the requirements of the one-to-one correspondence between state $\mathbf{U}(x,t)$ of the spatial extended system and vector $\mathbf{y}(t)$ of the constructed auxiliary system with the small number of degree of freedom we have used the neighbour method (Pecora *et al.*, 1995). We have examined that the distance $d(\mathbf{y}_1,\mathbf{y}_2) = ||\mathbf{y}_1 - \mathbf{y}_2||$ between two vectors $\mathbf{y}_1 = \mathbf{y}(t_1)$ and $\mathbf{y}_2 = \mathbf{y}(t_2)$ taken in the arbitrary moments of time t_1 and t_2 is close to zero if and only if the distance $S(\mathbf{U}_1,\mathbf{U}_2)$ between two different states $\mathbf{U}(x,t_1)$ and $\mathbf{U}(x,t_2)$ of the spatial extended system taken in the same moments of time t_1 and t_2 is also small. The distance $S(\mathbf{U}_1,\mathbf{U}_2)$ has been defined as

$$S(\mathbf{U}_1,\mathbf{U}_2) = \left(\int_0^1 \|\mathbf{U_1}(x,t) - \mathbf{U_2}(x,t)\|^2 \, dx\right)^{1/2}, \tag{4}$$

where $||\cdot||$ is Euclidian norm. According to the neighbour method it means that there is the one-to-one correspondence between $\mathbf{U}(x,t)$ and $\mathbf{y}(t)$, therefore we can use the constructed auxiliary low dimensional system $\mathbf{y}(t)$ to find UPTSTs by means of SD–method.

Having constructed the auxiliary flow system (3) we can use SD-method to detect UPOs in it and UPSTSs in the initial spatial extended chaotic system (1), respectively. In $\mathbb{R}^3$ space a plane $\rho(x = 0.25, t) = 1.0$ has been selected as Poincaré secant. Let us denote the vectors $\mathbf{y}(t_n) = (1, \rho(0.5,t_n), \rho(0.75,t_n))^T$ corresponding to the n-th crossing the selected secant surface by the trajectory $\mathbf{y}(t)$ as $\mathbf{y}_n$. Then the description of the system dynamics can be made with the help of the discrete map

$$\mathbf{y}_{n+1} = \mathbf{G}(\mathbf{y}_n), \tag{5}$$

where $\mathbf{G}(\cdot)$ is the evolution operator. Obviously, it is impossible to find the analytical form for the operator $\mathbf{G}$, but numerical integration of the initial system of

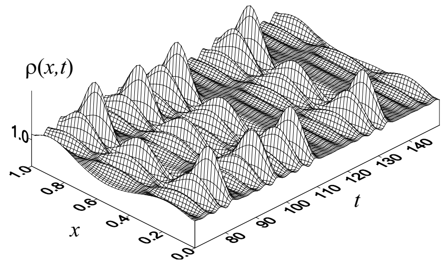

Figure 1. The spatio-temporal dynamics of the charge density $\rho(x, t)$ of the electron beam of Pierce diode. The oscillations for the selected control parameter value $\alpha = 2.858\pi$ are chaotic both in space and time

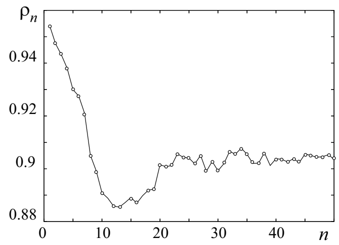

Figure 2. The dependence of $\rho_n(x = 0.75)$ upon the number of iteration of SD–method for the UPSTS of the length 1

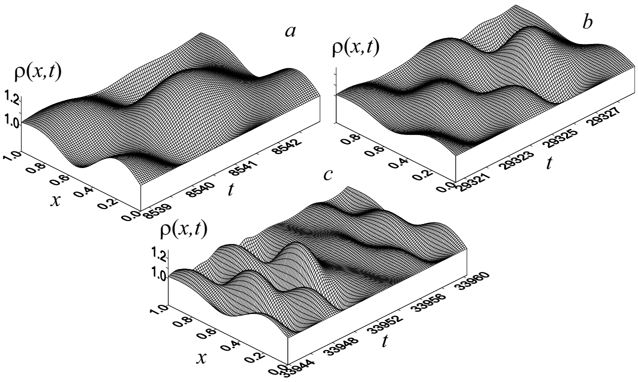

Figure 3. The distribution of space charge density $\rho(x, t)$ corresponding to the unstable spatio-temporal states with the following lengthes p periods T: (*a*) $p = 1, T = 4.2$; (*b*) $p = 2$, $T = 8.3$; (*c*) $p = 4, T = 18.9$

partial differential equations (1) can give us a sequence of values $\{\mathbf{y}\}_n$, generated by the map (5).

SD–method for picking out UPOs in the map (5) supposes consideration of the following map (Pingel *et al.*, 2001):

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \lambda \mathbf{C} \left[\mathbf{G}(\mathbf{y}_n) - \mathbf{y}_n \right], \tag{6}$$

where $\lambda = 0.1$ is the method constant and $\mathbf{C}$ is a certain matrix of the set $\mathbf{C}_k$. Each of matrices C_k should have only one non-vanishing entry $+1$ or -1 in row and column, i.e., they are orthogonal.

In works (Schmelcher and Diakonos, 1997; Pingel *et al.*, 2001) it was shown that map (6) under the appropriate choice of the matrix $\mathbf{C}$ allows to stabilize effectively the unstable saddle periodical orbits of systems (5) and (3). A trajectory of transformed system (6) starting in the domain of attraction of a stabilized fixed point converges to it. Therefore, the UPOs of a chaotic dynamical system (5) can be obtained by iterating the transformed systems (6) using a robust set of initial conditions. In our calculation the matrix

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{7}$$

is suitable to find UPOs in (5).

The transformed system (6) allows to find only the unstable periodic orbits of length 1. To consider UPOs of length p the map

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \lambda \mathbf{C} \left[\mathbf{G}^p(\mathbf{y}_n) - \mathbf{y}_n \right], \tag{8}$$

should be considered instead of (6) where $\mathbf{G}^{(p)}(\cdot)$ is p-times iterated map (5). As far as the spatial extended system and the auxiliary flow system are considered, only the p-th crossing of the Poincaré secant by the trajectory $\mathbf{y}(t)$ should be taken into account.

So, by numerical iteration of the map (8) with different values of p one can find the set of the unstable periodic spatio-temporal states of the extended system (1). However, there is a problem concerning with searching the state $\mathbf{U}(x, t_{n+1})$ at the moment t_{n+1} based on the known vector $\mathbf{y}_{n+1}$. Indeed, we know only the coordinates of the state $\mathbf{y}(t_{n+1})$ in the Poincaré secant but we don't know the corresponding distribution of $\rho(x, t_{n+1})$, $v(x, t_{n+1})$ and $\varphi(x, t_{n+1})$, and, correspondingly, we do not know the state $\mathbf{U}(x, t_{n+1})$ of the extended system (1). However, as we have determined above with the help of the nearest neighbours method the state $\mathbf{y}(t_{n+1})$ in the Poincaré secant uniquely defines the corresponding state $\mathbf{U}(x, t_{n+1})$ belonging to the attracting finite-dimensional subspace W^s of the infite-dimensional phase space W^∞. To obtain this spatial state $\mathbf{U}(x, t_{n+1})$ mentioned above we have used the following procedure. The system of partial differential equations (1) is integrated (and vector $\mathbf{y}(t)$ is calculated) untill some vector $\mathbf{y}(t_s)$ is close to the required one $\mathbf{y}_{n+1}$ with some demanded precision: $||\mathbf{y}_{n+1} - \mathbf{y}(s)|| < 10^{-3}$. When this condition is satisfied, the state $\mathbf{U}(x, t_s)$ corresponding to the found vector $\mathbf{y}(s)$ are considered as the required one $\mathbf{U}(x, t_{n+1})$ and then the next iteration according to (8) should be done.

The spatio-temporal chaotic dynamics of the charge density $\rho(x, t)$ in the Pierce diode is shown in Fig. 1 for $\alpha = 2.858\pi$. Applying the modified SD-method allows to find the demanded periodical time-space states. The convergence of the iteration procedure (8) is illustrated by Fig. 2, which shows the dependence of the space charge density $\rho_n(x = 0.75)$ in the moments of time when the trajectory $\mathbf{y}(t)$ in $\mathbb{R}^3$ space crosses the

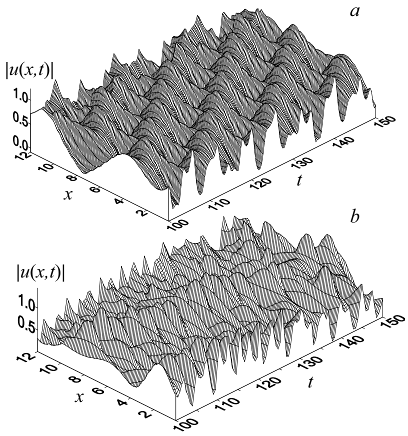

Figure 4. The spatio-temporal dynamics of the Ginzburg-Landau equation. The evolution of $|u(x,t)|$ is shown for the system length (*a*) $L = 12.63$ and (*b*) $L = 13.25$

Poincaré secant upon the number of iteration n of the SD–method when the UPSTS of the length $p = 1$ is studied. One can see clearly that the iteration process of SD–method converges to the value corresponding to the unstable time-periodical spatio-temporal state of the system. Fig. 3 shows the distribution $\rho(x,t)$ corresponding to the USTSs with different periods T detected by means of SD-method.

3 Detection of UPSTS in CGLE

To show the universality of the proposed approach we also report the results of detecting the UPSTSs for the one-dimensional complex Ginzburg-Landau equation (CGLE) (Aranson and Kramer, 2002). The CGLE is a fundamental model for the pattern formation and turbulence description.

We have considered one-dimensional CGLE

$$\frac{\partial u}{\partial t} = u - (1 - i\alpha)|u|^2 u + (1 + i\beta)\frac{\partial^2 u}{\partial x^2} \qquad (9)$$

with periodical boundary conditions $u(L,t) = u(0,t)$. All calculations were performed for a fixed system parameters $\alpha = \beta = 4$ and random initial conditions.

The system length L has been chosen as the control parameter. In our study we examined two values of the control parameter: $L_1 = 12.63$ and $L_2 = 13.25$. For both these values L CGLE demonstrates the spatiotemporal chaotic regime. The corresponding dynamics of CGLE are shown in Fig. 4 for the lengths $L = 12.63$ and $L = 13.25$. One can see easily that the second case is characterized by more complex irregular spatio-temporal chaotic dynamics. Indeed, in the first case ($L = 12.63$) the chaotic dynamics is characterized by only one positive Lyapunov exponent $\Lambda_1 = 0.04$, while the second chaotic regime ($L = 13.25$) is characterized by two positive Lyapunov exponents $\Lambda_1 = 0.10$ and $\Lambda_2 = 0.07$.

Applying the modified SD-method to the spatial extended CGLE we can find the demanded unstable periodical spatio-temporal states as well as for the fluid model of Pierce diode. We have constructed the vector (3) of the auxiliary low dimensional system as

$$\mathbf{y}(t) = (u(x_1,t), \dots, u(x_m,t))^T, \qquad (10)$$

where m is the dimension of the auxiliary system vector, $x_i = iL/m$, $i = \overline{1,m}$.

In contrast to the fluid model of Pierce diode the dimension m of the auxiliary vector $\mathbf{y}(t)$ is unknown for CGLE. Therefore, we have to try to find UPSTSs by means of the SD-method (8) for the different values of the dimension m starting from the minimal dimension value $m = 3$. If the required UPSTS is not found for the selected value of the auxiliary vector dimension m^*, the SD-method procedure should be repeated for the greater dimension value $m = m^* + 1$.

For the system length $L = 12.63$ the dimension of the auxiliary system $m = 3$ is found to be adequate for the correct UPSTSs detection. As it was mentioned above the system behavior is characterized by one positive Lyapunov exponent. For the more complicated case $L = 13.25$ (when the behavior of CGLE is characterized by two positive Lyapunov exponents) the dimension of the auxiliary vector should be taken as $m = 4$ for UPSTSs to be detected successfully.

Fig. 5 shows the evolution of the profiles $|u(x,t)|$ corresponding to the unstable periodic spatio-temporal states with the different periods T detected by means of SD-method for the system length $L = 12.63$, when the dimension of the auxiliary vector (10) has been chosen as $n = 3$. The analogous evolution of the profiles $|u(x,t)|$ corresponding to the unstable periodic spatio-temporal states with the different lengths p and periods T is shown in Fig. 6 for $L = 13.25$ and $m = 4$.

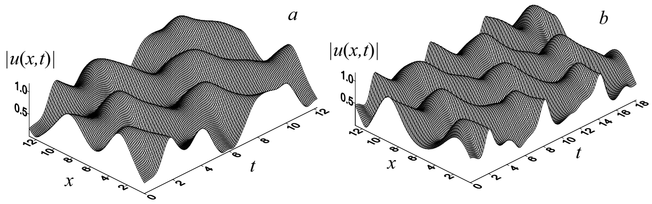

Figure 5. The evolution of the module $|u(x,t)|$ corresponding to the UPSTSs with the following lengths p and periods T: (*a*) $p = 1$, $T = 12.1$; (*b*) $p = 2$, $T = 18.1$ for $L = 12.63$. The dimension of the vector $\mathbf{y}(t)$ has been chosen as $m = 3$.

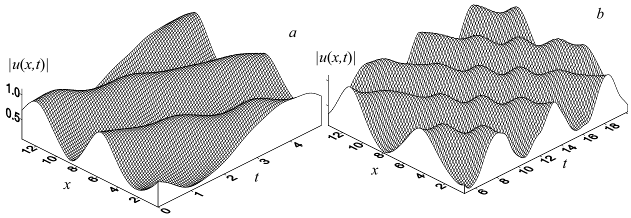

Figure 6. The evolution of $|u(x,t)|$ corresponding to UPSTSs with (*a*) $p = 1, T = 4.95$ and (*b*) $p = 3, T = 20.2$ for $L = 13.25$. The dimension of the auxiliary vector $\mathbf{y}(t)$ has been chosen as $m = 4$

4 Conclusion

We have proposed the method of the detection of the UPSTSs of spatial extended chaotic systems being the extension of the well known SD-method. The effectiveness of this method is illustrated by the consideration of the fluid model of Pierce diode and CGLE.

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