THEORY OF NOISE-INDUCED INTERMITTENCY IN BISTABLE DYNAMICAL SYSTEMS

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Abstract
Theory of intermittency taking place in bistable dynamical systems subjected to additional noise influence have been proposed. The main characteristic of intermittency namely the residence time distribution for both coexisting regimes has been obtained analytically and numerically. The proposed theory has been applied to bistable energy model and erbium-doped fiber laser with two coexisting periodic orbits.

Key words
Multistablity, Stochastic systems, Synchronization, Chaos, Nonlinear dynamics, Numerical methods

1 Introduction

Intermittency is an ubiquitous phenomenon in nonlinear science [Berge et al., 1984]. It is observed in different systems including the physical, physiological and biological ones (see, e.g., [Kim et al., 1998; Perez Velazquez and et al., 1999; Kiss and Hudson, 2001; Boccaletti et al., 2002; Cabrera and Milton, 2002; Hramov et al., 2006b; Sitnikova et al., 2012]). It manifests itself as alternation of the episodes of periodic and chaotic regimes [Manneville and Pomeau, 1979] or different forms of the chaotic motion [Grebogi et al., 1987]. It can also be observed near the boundaries of the different synchronous regimes demonstrating the interchange of the phases of synchronous and asynchronous behavior (see, e.g. [Pikovsky et al., 1997; Boccaletti and Valladares, 2000; Hramov and Koronovskii, 2005; Hramov et al., 2006a; Moskalenko et al., 2011; Hramov et al., 2014]).

Several types of the intermittent behavior are traditionally distinguished, among which there are type I-III [Berge et al., 1984; Dubois et al., 1983], on-off [Heagy et al., 1994], eyelet [Pikovsky et al., 1997; Boccaletti et al., 2002] and ring [Hramov et al., 2006a] intermittencies or their common coexistence [Hramov et al., 2013; Moskalenko et al., 2014; Koronovskii et al., 2016]. Each of types of intermittency mentioned above is characterized by its mechanism and its own statistical characteristics. One can say that such characteristics allows to unambiguously define the type of intermittency realized in the system.

Recently the concept of intermittency has been extended to multistable systems. In such case the alternation between coexisting periodic or chaotic regimes regardless of the form of motion realized in the systems can also be observed [Pisarchik et al., 2012; Sevilla-Escoboza et al., 2015]. At that, the switches between coexisting regimes can be induced by noise. Therefore, the system under study demonstrates the so-called noise-induced intermittency or noise-induced attractor hopping [Arecchi et al., 1985; Wiesenfeld and Hadley, 1989; Kraut and Feudel, 2002; Pisarchik et al., 2011; Hramov et al., 2016].

Despite of a great interest to the problem of noise-induced intermittency (see, e.g. [Lai and Grebogi, 1995; Pisarchik and Pinto-Robledo, 2002; Hramov et al., 2016]) there is a number of questions demanding consideration and discussion. One of such problems consists in the fact that there is no appropriate theory allowing to obtain the characteristics of noised-induced intermittency even in the case of two different regime coexistence. In the present paper the theory of noise-induced intermittency in bistable dynamical systems would be proposed. As would be shown bellow, the residence time distributions for every coexisting regime should obey the exponential law.

2 Theory of noised-induced intermittency

The standard bistable system capable to demonstrate noised-induced intermittency is bistable energy model

$$\frac{dx}{dt} = -\frac{dU(x)}{dx} + \xi(t), \tag{1}$$

where $\xi(t)$ is zero mean δ-correlated Gaussian noise $[\langle\xi_n\rangle = 0, \langle\xi_n\xi_m\rangle = D\delta(n-m)]$, D is its variance,

$$U(x) = \frac{x^4}{4} - \frac{x^2}{2} + bx \tag{2}$$

is the dimensionless energy function with two local minima, b is the parameter of symmetry [Pisarchik et al., 2014; Moreno-Bote et al., 2007].

The differential equation (1) with stochastic term $\xi(t)$ results in the stochastic differential equation

$$dX = \frac{dU(x)}{dx}dt + dW, \tag{3}$$

(where $X(t)$ is a stochastic process, $W(t)$ is a one-dimensional Winner process) which is equivalent to the Fokker-Plank equation

$$\frac{\partial\rho_X(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[\frac{dU(x)}{dx}\rho_X(x,t)\right] + \frac{D}{2}\frac{\partial^2\rho_X(x,t)}{\partial x^2} \tag{4}$$

written for probability density $\rho(x,t)$ of the stochastic process $X(t)$.

Since in the regime of intermittency the coordinate of the system state stays for a long time in the vicinity of the one local minimum one can assume that the solution of the equation (4) should be found in the form of the metastable distribution decaying slowly for a long period of time, i.e.

$$\rho(x,t) = A(t)r(x), \tag{5}$$

(where $r(x)$ is the stationary probability density obtained from the solution of equation (4) in stationary case, $A(t)$ is a coefficient decreasing slowly with the time increases) which results in the following relation for the residence time distributions for both coexisting regimes

$$p_{1,2}(t) = \frac{1}{T_{1,2}}\exp\left(-\frac{t}{T_{1,2}}\right). \tag{6}$$

where

$$T_{1,2} = \frac{P_{1,2}}{kr(x^*)}, \tag{7}$$

x^* is a boundary point being located on the equal distances from the local minima of $U(x)$, $P_{1,2}$ are the probabilities of location of the representation point in the vicinity of the first or second local minimum, k is a normalization factor.

In other words, in the regime of noise-induced intermittency the residence time distributions should obey the exponential law (6).

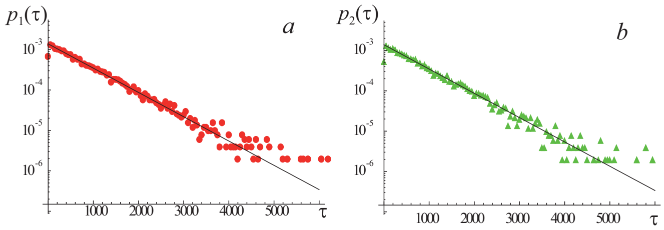

Figure 1. Residence time distributions for two coexisting regimes in bistable energy model (1) for $b = 0$. The results of numerical calculations are marked by points. Theoretical approximations by the regularity (6) are shown by solid lines. The parameters of approximations are the following: (*a*) $T_1 = 722$, (*b*) $T_2 = 721$

3 Numerical verifications of the proposed theory

To confirm the results of theoretical predictions we analyze numerically the behavior of two different systems capable to demonstrate the regime of noise-induced intermittency. As the first example we consider the same energy model (1) with the same potential function (2) and characteristics of noise with its variance $D = 0.1$. In Fig. 1 the statistical distributions of the residence times corresponding to two coexisting regimes in symmetrical case ($b = 0$) are shown. Fig. 1,*a* corresponds to the first coexisting regime (the first local minimum of the potential function), Fig. 1,*b* refers to the second one. The results of numerical simulation are marked by points for both coexisting regimes, their theoretical approximations by the exponential laws (6) with the parameters indicated in the caption are shown by solid lines. It is clearly seen the excellent agreement between the results of numerical calculations and theoretical approximations that testifies the validity of the proposed theory.

As the second example we consider the dynamics of the erbium-doped fiber laser which is known to demonstrate the noised-induced intermittency [Pisarchik et al., 2005; Hramov et al., 2016]. The system under study is given by

$$\begin{aligned}\frac{dx}{dt} &= \frac{2L}{T_r}x\left\{r_w\alpha_0\left[y\left(\xi_1-\xi_2\right)-1\right]-\alpha_{th}\right\}+P_{sp}, \\ \frac{dy}{dt} &= -\frac{\sigma_{12}r_wx}{\pi r_0^2}(y\xi_1-1)-\frac{x}{\tau}+P_{pump},\end{aligned} \tag{8}$$

where x is the intracavity laser power, $y = \frac{1}{n_0L}\int_0^L N_2(z)dz$ is the averaged (over the active fiber length L) population of the upper lasing level, N_2 is the upper level population at the z coordinate, n_0 is the refractive index of a "cold" erbium-doped fiber core, ξ_1 and ξ_2 are parameters defined by the relationship between cross sections of ground state absorption (σ_{12}), return stimulated transition (σ_{21}), and exited state absorption (σ_{23}). T_r is the photon intracavity round-trip time, α_0 is the small-signal absorption of the erbium fiber at the laser wavelength, α_{th} accounts for the intracavity losses on the threshold, τ is the lifetime of erbium ions in the excited state, r_0 is the fiber core radius, w_0 is the radius of the fundamental fiber mode, and r_w is the factor that conveys the match between the laser fundamental mode and erbium-doped core volumes inside the active fiber. The spontaneous emission into the fundamental laser mode is derived as

$$P_{sp} = y\frac{10^{-3}}{\tau T_r}\left(\frac{\lambda_g}{w_0}\right)^2 \frac{r_0^2\alpha_0 L}{4\pi^2\sigma_{12}}, \quad (9)$$

where λ_g is the laser wavelength. The pump power is expressed as

$$P_{pump} = P_p \frac{1-\exp\left[-\alpha_0\beta L\left(1-y\right)\right]}{N_0\pi r_0^2 L}, \quad (10)$$

where P_p is the pump power at the fiber entrance and β is a dimensionless coefficient. In analogous with the previous works the control parameter values have been selected as follows: $L = 0.88$ m, $T_r = 8.7$ ns, $r_w = 0.308$, $\alpha_0 = 40$ m^{-1}, $\xi_1 = 2$, $\xi_2 = 0.4$, $\alpha_{th} = 3.92 \times 10^{-2}$, $\sigma_{12} = 2.3 \times 10^{-17}$ m^2, $r_0 = 2.7 \times 10^{-6}$ m, $\tau = 10^{-2}$ s, $\lambda_g = 1.65 \times 10^{-6}$ m, $w_0 = 3.5 \times 10^{-6}$ m, $\beta = 0.5$, and $N_0 = 5.4 \times 10^{25}$ m^{-3}, that correspond to the real experimental conditions.

Under the harmonic and random modulations

$$P_p = p\left[1 - m_d \sin\left(2\pi f_d t\right) + \eta G(\zeta, f_n)\right], \quad (11)$$

(where p is the pump power, $m_d = 0.95$ and $f_d = 80$ kHz are the driving frequency and amplitude, respectively, η is the noise amplitude and $G(\zeta, f_n)$ is the zero-mean noise function of a random number $\zeta \in [-1, 1]$ and noise low-pass cut-off frequency $f_n = 30$ kHz) the system (8) demonstrates the noised-induced intermittency with up to four coexisting regimes A_i ($i = 1, 3, 4, 5$) with frequencies $f_i = f_d/i$ being observed [Pisarchik et al., 2012; Hramov et al., 2016].

If the noise intensity η is small enough the system (8) demonstrated the coexistence of two different regimes that allows us to apply the theory proposed in Section 2 to the system under study. In Fig. 2 the numerically obtained distributions of the residence times corresponding to the regimes A_1 and A_3 for the value of the noise intensity $\eta = 0.11$ and their theoretical approximations

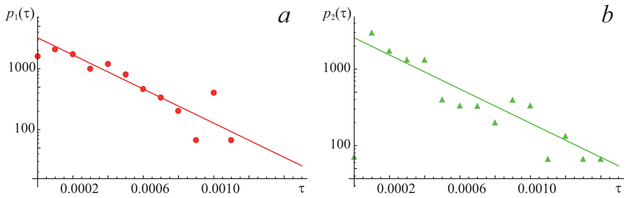

Figure 2. Residence time distributions for two coexisting regimes (a — A_1, b — A_3) in erbium-doped fiber laser (8) for the noise intensity $\eta = 0.11$ (points) and their analytical approximations by the regularity (6) (solid lines). The parameters of approximations are the following: (a) $T_1 = 3.1 \times 10^{-4}$, (b) $T_2 = 3.9 \times 10^{-4}$

by the regularities (6) are presented. It is clearly seen a good agreement between the theoretically and numerically obtained results for both considered regimes. So, one can conclude that for small value of the noise intensity the residence time distributions corresponding to two coexisting regimes in erbium-doped fiber laser obey the exponential law.

4 Conclusion

In the present paper the theory of noise-induced intermittency in bistable dynamical systems has been proposed. We have shown that the residence time distributions for every coexisting regime obey the exponential law. The main results have been illustrated using the examples of bistable energy model and erbium-doped fiber laser.

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