DISPLACEMENT IN SPACE OF THE EQUILIBRIA OF UNSTABLE DISSIPATIVE SYSTEMS

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Abstract

Piecewise linear systems based on unstable dissipative systems (UDS) in $\mathbf{R}^3$ consist of one of the two possible conditions regarding their eigenvalues. The UDS of the type I, present a λ_1 real negative eigenvalue and two $\lambda_{2,3}$ complex conjugated values with real positive parts. Since the trajectories formed by these systems are unbounded due to their stability, two or more subsystems need to be considered in order to restrain the resulting orbit generating self-sustained oscillations. To do so, here the intrinsic properties of the systems along with the location in space in which the equilibrium is located are described, in order to design switching control laws that bound the resulting trajectories.

Key words

Nonlinear dynamics, unstable dissipative systems, chaos theory, multi-scroll attractors.

1 Introduction

Theories and applications regarding Piecewise Linear ("PWL") systems have been widely used, ranging from vibrational analysis, dry-friction oscillators to voltage-stepping neural networks and abruptly changes in chemical process [Yu, 2013; Natsiavas, 1998; Zheng, Tonnelier and Martinez, 2009; Christophersen, 2007]. In the last decades, new interesting areas of application regarding multi-scroll chaotic systems have come to light. For example in the process of masking information in chaotic communications through transmission channels [Gámez-Guzmán *et. al.*, 2009], in the encryption of fingerprint images [Han *et. al.*, 2007] and cellular neural networks [Yalçın, 2007; Yalçın, Suykens and Vandewalle, 2005].

The tendency on generating and designing systems that result in multiple scrolls by the interaction of PWL systems, has opened many fields by means of different methods and technics, for example nonsmooth nonlinear functions, such as hysteresis [Lü *et.al.*, 2004; Deng and Lü, 2007], saturation [Lü, Guanrong Chen and Yu, 2004; Sánchez-López *et. al.*, 2010], threshold and step functions [Lü *et. al.*, 2008; Yalçin *et. al.*, 2002; Yu *et. al.*, 2005; Campos-Cantón *et. al.*, 2008; Campos-Cantón *et. al.*, 2010].

In [Ontañón-García *et. al.*, 2014] and in the same spirit of [Campos-Cantón *et. al.*, 2008; Campos-Cantón *et. al.*, 2010; Campos-Cantón, Femat and Guanrong Chen, 2012], a unified method to design systems using the Unstable Dissipative Systems (UDS) theory with a family of hyperchaotic attractors was proposed. The idea is to consider the properties of the saddle-focus equilibria PWL systems dividing them in two UDS categories, regarding on the eigenvalues that their linear part presents. Due to the intrinsic unstable dynamics that these systems present they are incapable of generating bounded attractors. Therefore, in order to generate bounded trajectories, at least two identical affine subsystems with combined unstable one-spiral trajectories located in the space are required to contain the trajectory. The approach only considered the case of the displacement of the equilibria along the x axis. However, the location can be designed towards any region of space.

In this work, different displacements along different axes in space will be considered in order to reaffirm the UDS theory described in [Ontañón-García *et. al.*, 2014]. The article is presented as follows: Section 2 contains all the definitions and considerations of the UDS theory. In Section 3 a method for generating UDS of the type I equilibria along the axes is presented. In Section 4 the method is extended to the space. And finally conclusions are drawn in the last Section.

2 Unstable dissipative system theory

Consider the class of affine linear system given by the following equation:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}, \quad (1)$$

where $\mathbf{x} = [x_1, x_2, x_3]^T \in \mathbf{R}^3$ represents the state vector, $\mathbf{B} = [b_1, b_2, b_3]^T \in \mathbf{R}^3$ stands for a real vector, $\mathbf{A} = [a_{ij}] \in \mathbf{R}^{3\times 3}$ with $i, j = 1, 2, 3$, is a linear nonsingular matrix which determines the dynamics of the system and catalogs the system on its UDS type. The matrix $\mathbf{A}$ has to present a stable manifold E^s and an unstable one E^u. Two types of unstable dissipative systems exist, however only the type I will be considered here. This type is defined in the following way:

Definition 2.1. *It is said that the system* (1) *is of the UDS type I if* $\sum_{i=1}^{3} \lambda_i < 0$*, where* $\lambda_i, i = 1, 2, 3$*, are the eigenvalues of the matrix* $\mathbf{A}$*, and one* λ_1 *is real negative* $(Re\{\lambda_1\} < 0$ *and* $Im\{\lambda_1\} = 0)$*, and two* $\lambda_{2,3}$ *are complex conjugated with positive real part* $(Re\{\lambda_{2,3}\} > 0$ *and* $Im\{\lambda_{2,3}\} \neq 0)$*.*

Notice that systems with these particular characteristics present equilibrium of the saddle-focus type at $\mathbf{x}^* = -\mathbf{A}^{-1}\mathbf{B}$ and any initial condition $\mathbf{x}_0 \in \mathbf{R}^3$ will be affected by its dynamics. Therefore if a system with ordered eigenvalues according to their real part $\lambda_1 \leq \lambda_2 \leq \lambda_3$ satisfies Definition 2.1, then it has a stable eigendirection given by $E^s = span\{\mathbf{v}^1\} \subset \mathbf{R}^3$, where $\mathbf{v}^1 = [v^1_{x_1}, v^1_{x_2}, v^1_{x_3}] \in \mathbf{R}^3$ is the corresponding eigenvector of the negative real part of the real eigenvalue λ_1. This stable manifold is represented by the following space linear equation:

$$\frac{x_1 - \alpha_{x_1}}{v^1_{x_1}} = \frac{x_2 - \alpha_{x_2}}{v^1_{x_2}} = \frac{x_3 - \alpha_{x_3}}{v^1_{x_3}} \quad (2)$$

where $\alpha_{x_1}, \alpha_{x_2}, \alpha_{x_3} \in \mathbf{R}$ are the coordinates of the corresponding equilibrium $\mathbf{x}^*$. Any trajectory with initial condition in E^s will be attracted towards the equilibrium. On the other hand, an unstable spiral eigendirection given by $E^u = span\{\mathbf{v}^2, \mathbf{v}^3\} \subset \mathbf{R}^3$, in consequence of the real positive part of the complex conjugated, where $\mathbf{v}^j = [v^j_{x_1}, v^j_{x_2}, v^j_{x_3}] \in \mathbf{C}^3$ are the corresponding eigenvectors of the complex conjugate eigenvalues λ_j with $j = 2, 3$. This manifold can be represented by a characteristic plane obtained from the real and complex conjugate values of the eigenvectors in the following way:

$$\varphi = [\varphi_{x_1}, \varphi_{x_2}, \varphi_{x_3}] = Re\{\mathbf{v}^j\} \times Imag\{\mathbf{v}^j\}; \quad (3)$$

where $\times$ corresponds to the cross product of the eigenvectors, therefore, the plane equation of the unstable manifold will be given as:

$$\varphi_{x_1}(x_1 - \alpha_{x_1}) + \varphi_{x_2}(x_2 - \alpha_{x_2}) + \varphi_{x_3}(x_3 - \alpha_{x_3}) = 0. \quad (4)$$

Any trajectory with initial condition in the unstable plane manifold, will be driven in spirals away from the equilibrium on the plane.

A system given by Eq. (1) under Definition 2.1, cannot produce bounded oscillations autonomously due to its instability. To attend this matter it is required to design a commuted system dividing the space at least in the domains of two subsystems in order to trap the trajectory. This commuted system takes the following form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}(\mathbf{x}),$$
$$\mathbf{B}(\mathbf{x}) = \begin{cases} \mathbf{B}_1, \text{ if } \mathbf{x} \in \mathcal{D}_1; \\ \mathbf{B}_2, \text{ if } \mathbf{x} \in \mathcal{D}_2; \end{cases} \quad (5)$$

where $\mathbf{B}_{1,2} \in \mathbf{R}^3$ and the domains $\mathcal{D}_{1,2}$ are such that $\mathbf{R}^3 = \mathcal{D}_1 \cup \mathcal{D}_2$. Thus, the equilibrium of the system given by Eq. (5) result in $\mathbf{x}^*_i = -\mathbf{A}^{-1}\mathbf{B}_i$, with $i = 1, 2$. Therefore, $\mathbf{B}_i$ vectors correspond to each subsystem, which together ensure bounded trajectories and the stability of a class of switched dynamical systems in $\mathbf{R}^3$. The idea is the following: After any initial condition inside the basin of attraction is given, and assuming that it is in the domain $\mathcal{D}_1$ for a subsystem given by $(\mathbf{A}, \mathbf{B}_1)$, the evolution of the system due to its instability will tend to take the trajectory to infinity. Later, when the trajectory crosses from its initial domain in $\mathcal{D}_1$ to the adjacent one $\mathcal{D}_2$, the subsystem given by $(\mathbf{A}, \mathbf{B}_2)$ will trapped the trajectory for some time until the process is inverted and repeated continuously, resulting in a bounded trajectory. The challenge here lies in the location of the equilibria in space, since the trajectory once it begins to grow spinning along the unstable manifold, at the moment of crossing the commutation surface must match the stable manifold direction in order to be trapped and bounded.

3 Displacement of the equilibria along the axis of UDS type I

In order to describe the displacement of the equilibria along the axes, their position with respect to $\mathbf{x}^*_i = -\mathbf{A}^{-1}\mathbf{B}_i$ with $i = 1, 2$ will be considered defining the commuting values of $\mathbf{B}(\mathbf{x})$. The resulting displaced equilibria will take the form:

$$\mathbf{x}^* = \begin{bmatrix} \alpha_{x_1} \\ \alpha_{x_2} \\ \alpha_{x_3} \end{bmatrix}; \quad (6)$$

with $\alpha_{x_1}, \alpha_{x_2}, \alpha_{x_3}$ corresponding to the distance from the origin to the x_1, x_2 and x_3 axes, respectively. Thus, if a specific position in space is desired, the following equation needs to be solved:

$$\mathbf{B}(\mathbf{x}) = -\mathbf{A}\mathbf{x}^*. \tag{7}$$

Consider first the case of a displacement along the x_1 axis as described next.

3.1 Displacement along the x_1 axis

A displacement in the x_1 axis can be easily implemented considering a matrix $\mathbf{A}$ that satisfies Definition 2.1. Based on the well known case of the jerky equation in $\mathbf{R}^3$, here, the matrix $\mathbf{A}$ takes the following form:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -1 & -1 \end{bmatrix}. \tag{8}$$

The characteristic polynomial of matrix $\mathbf{A}$ given by (8) takes the form $\lambda^3 + \lambda^2 + \lambda + 2$ and by calculating its roots, the set of eigenvalues of the system results in $\Lambda = \{-1.3532, 0.1766 \pm 1.2028i\}$, which satisfies Definition 2.1 and ensures that the system is of the UDS type I. The eigenvectors of the system result in $\mathbf{v}^1 = [-0.4021, 0.5441, -0.7363]^T$ and $\mathbf{v}^{2,3} = [0.4433 \pm 0.1331i, -0.0818 \pm 0.5571i, -0.6845]^T$, generating stable and unstable manifolds, respectively, and required to yield bounded trajectories as a consequence of the stretching and folding behavior. Now, in order to determine the displacement of the equilibria along x_1, consider that $\mathbf{x}^*$ takes the following form:

$$\mathbf{x}^* = \begin{bmatrix} \alpha_{x_1} \\ 0 \\ 0 \end{bmatrix}. \tag{9}$$

Therefore by solving (7) the affine vector $\mathbf{B}(\mathbf{x})$ will be defined as follows:

$$\mathbf{B}(\mathbf{x}) = \begin{bmatrix} 0 \\ 0 \\ 2\alpha_{x_1} \end{bmatrix}. \tag{10}$$

Now considering a displacement of $\alpha_{x_1} = 1/2$, a commutation law must be defined for each $\mathcal{D}_i$ in order to trap the unstable trajectory with at least two subsystems. This can be fulfilled given the following commutation law:

$$\mathbf{B}(\mathbf{x}) = \begin{cases} \mathbf{B}_1 = [0, 0, 1]^T, & \text{if } x_1 \geq 0; \\ \mathbf{B}_2 = [0, 0, -1]^T & \text{otherwise.} \end{cases} \tag{11}$$

The equilibria of the system (5) with matrix $\mathbf{A}$ and the vector $\mathbf{B}_i$ defined in (11) result in: $\mathbf{x}_1^* = [1/2, 0, 0]^T$ and $\mathbf{x}_2^* = [-1/2, 0, 0]^T$.

With this commutation law, the system results in a 2 scroll attractor which oscillates around the two displaced equilibrium points (see Figure 1). The space linear equations of E^s of the equilibria $\mathbf{x}_1^*$ and $\mathbf{x}_2^*$ result in the following:

$$\frac{x_1 \pm 1/2}{-0.4021} = \frac{x_2}{0.5441} = \frac{x_3}{-0.7363}; \tag{12}$$

and the planes of E^u of the equilibria $\mathbf{x}_1^*$ and $\mathbf{x}_2^*$ result in:

$$0.3813(x_1 \pm 1/2) - 0.0911x_2 + 0.2578x_3 = 0. \tag{13}$$

In order to understand more the dynamics of the system, the stable and unstable manifolds E^s and E^u, are plotted along with the projection of the attractor onto the (x_1, x_2) and (x_1, x_3) planes in Figures 1a) and Figure 1b), respectively. The initial condition considered for the system is $x_0 = [1, 0, 0]^T$. Notice that the number of equilibrium points equals the number of scrolls in the attractor and similarly, the direction in which the orbits are oscillating around the equilibria matches the unstable plane of the manifold E^u, which is represented with a blue line in the corresponding projection of the attractor. However, the trajectory once it grows and crosses the commutation surface in the plane $x_1 = 0$, is forced towards the equilibria affected by the stable manifold E^s, marked with a red line in Figure 1. The dynamic of the system, in order to be trapped, must force the trajectory to escape from one domain $\mathcal{D}_1$ through the unstable plane E^u. At the moment it crosses the commutation surface, the trajectory must be attracted through E^s in $\mathcal{D}_2$ in such a way that the trajectory doesn't escapes to infinity. The process must be reversed in the same way but in opposite direction at the moment of return.

If one considers the proper matrix $\mathbf{A}$ and vector $\mathbf{B}(\mathbf{x})$ the displacement can be obtained along different axes. For example in the x_2 axis as commented next.

3.2 Displacement along the x_2 axis

In order to generate a displacement along x_2, consider first a matrix that satisfies Definition 2.1. For example the one described below:

$$\mathbf{A} = \begin{bmatrix} -2 & 1 & 3 \\ 5 & -5 & -5 \\ -1 & -2.8 & -1 \end{bmatrix}. \tag{14}$$

Therefore, the resulting eigenvalue set is $\Lambda = \{-8.3004, 0.1502 \pm 1.8631i\}$, which ensures that the

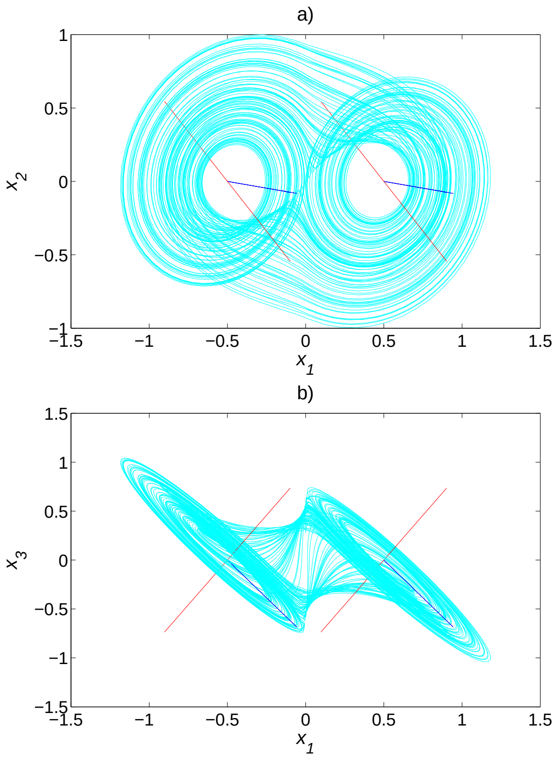

Figure 1. Projection of the attractor of the system (5) with $\mathbf{A}$ from (8), $\mathbf{B}(\mathbf{x})$ from (11) onto the: a) (x_1, x_2) plane; (x_1, x_3) plane. Marked with a red line the stable manifold E^s given by (12) and with a blue line a section of the unstable plane manifold E^u given by (13).

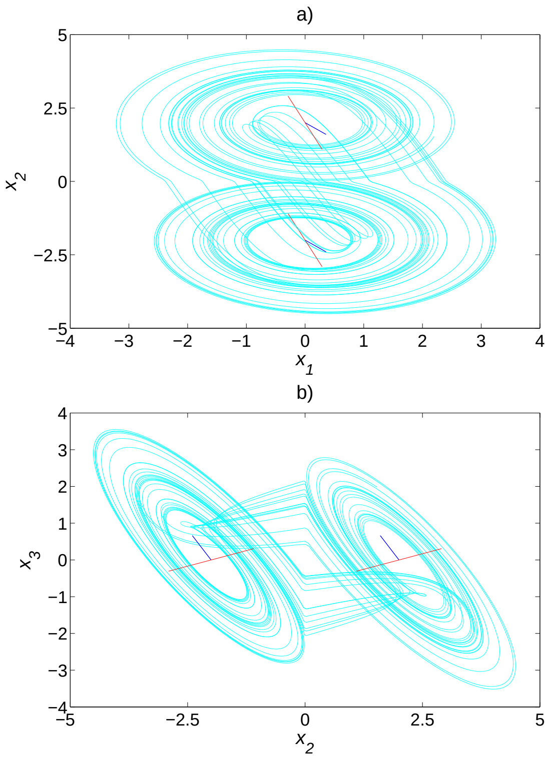

Figure 2. Projection of the attractor of the system (5) with $\mathbf{A}$ from (14), $\mathbf{B}(\mathbf{x})$ from (17) onto the: a) (x_1, x_2) plane; (x_2, x_3) plane. Marked with a red line the stable manifold E^s given by (18) and with a blue line a section of the unstable plane manifold E^u given by (19).

system is of the UDS type I. The eigenvectors are $\mathbf{v}^1 = [-0.2903, 0.9061, 0.3077]^T$ and $\mathbf{v}^{2,3} = [0.3565 \mp 0.4407i, -0.3994 \mp 0.2834i, 0.6624]^T$. In order to generate a displacement along the corresponding axis, consider that the equilibria takes the form:

$$\mathbf{x}^* = \begin{bmatrix} 0 \\ \alpha_{x_2} \\ 0 \end{bmatrix}, \tag{15}$$

and by solving (7) it will result in the following affine vector:

$$\mathbf{B}(\mathbf{x}) = \begin{bmatrix} -1\alpha_{x_2} \\ 5\alpha_{x_2} \\ 2.8\alpha_{x_2} \end{bmatrix}; \tag{16}$$

if a displacement $\alpha_{x_2} = 2$ is considered, the following commutation law will be defined:

$$\mathbf{B}(\mathbf{x}) = \begin{cases} \mathbf{B}_1 = [-2, 10, 5.6]^T, & \text{if } x_2 \geq 0; \\ \mathbf{B}_2 = [2, -10, -5.6]^T & \text{otherwise.} \end{cases} \tag{17}$$

The equilibria of the system (5) with matrix $\mathbf{A}$ and the vector $\mathbf{B}$ defined by the commutation law given by (17) are: $\mathbf{x}_1^* = [0, 2, 0]^T$ and $\mathbf{x}_2^* = [0, -2, 0]^T$. The space linear equations of E^s of the equilibria $\mathbf{x}_1^*$ and $\mathbf{x}_2^*$ result in the following:

$$\frac{x_1}{-0.2903} = \frac{x_2 \pm 2}{0.9061} = \frac{x_3}{0.3077}; \tag{18}$$

and the plane of E^u of the equilibria $\mathbf{x}_1^*$ and $\mathbf{x}_2^*$ result in:

$$-0.1877x_1 + 0.2919(x_2 \pm 2) + 0.2771x_3 = 0. \tag{19}$$

This can be appreciated in more detail from the projection of the attractor of the system onto the (x_1, x_2) and (x_2, x_3) planes, given in Figure 2a) and 2b), respectively. In this case the displacement occurs only in the desired x_2 axis and the system solution results also in a two scroll attractor. Similar characteristics as the x_1 case result from their eigenvectors and the direction in which the trajectories travels in space.

3.3 Displacement along the x_3 axis

Consequently, consider a displacement along x_3. The matrix $\mathbf{A}$ will take the form described below:

$$\mathbf{A} = \begin{bmatrix} -1 & -1 & -1.5 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \quad (20)$$

Therefore, the resulting eigenvalue set is $\Lambda = \{-1.2972, 0.1486 \pm 0.6028i\}$, which ensures that the system is of the UDS type I. The eigenvectors $\mathbf{v}^1 = [-0.3847, 0.7310, -0.5636]^T$ and $\mathbf{v}^{2,3} = [-0.7545, 0.0376 \pm 0.3441i, 0.5527 \pm 0.0738i]^T$. In order to generate a displacement along the corresponding axis, consider that the equilibria take the form:

$$\mathbf{x}^* = \begin{bmatrix} 0 \\ 0 \\ \alpha_{x_3} \end{bmatrix}, \quad (21)$$

and by solving (7) it will result in the following affine vector:

$$\mathbf{B}(\mathbf{x}) = \begin{bmatrix} 1.5\alpha_{x_3} \\ -1\alpha_{x_3} \\ 0 \end{bmatrix}; \quad (22)$$

considering a displacement of $\alpha_{x_3} = 10$, then a commutation law that bounds the trajectory will be given as follows:

$$\mathbf{B}(\mathbf{x}) = \begin{cases} \mathbf{B}_1 = [15, -10, 0]^T, \text{ if } x_3 \geq 0; \\ \mathbf{B}_2 = [-15, 10, 0]^T, \text{ otherwise.} \end{cases} \quad (23)$$

The equilibria of the system (5) with matrix $\mathbf{A}$ and the vector $\mathbf{B}$ considering the commutation law given by (23) are: $\mathbf{x}_1^* = [0, 0, 10]^T$ and $\mathbf{x}_2^* = [0, 0, -10]^T$. The space linear equations of E^s of the equilibria $\mathbf{x}_1^*$ and $\mathbf{x}_2^*$ result in the following:

$$\frac{x_1}{-0.3847} = \frac{x_2}{0.7310} = \frac{x_3 \pm 10}{-0.5636}; \quad (24)$$

and the planes of E^u of the equilibria $\mathbf{x}_1^*$ and $\mathbf{x}_2^*$ result in:

$$-0.1874x_1 + 0.0559x_2 - 0.2596(x_3 \pm 10) = 0. \quad (25)$$

This can be appreciated in more detail from the projection of the attractor of the system onto the (x_1, x_3) and (x_2, x_3) planes, given in Figure 3a) and 3b), respectively. Here, the displacement is in the x_3 axis and the number of scrolls also matches the equilibria.

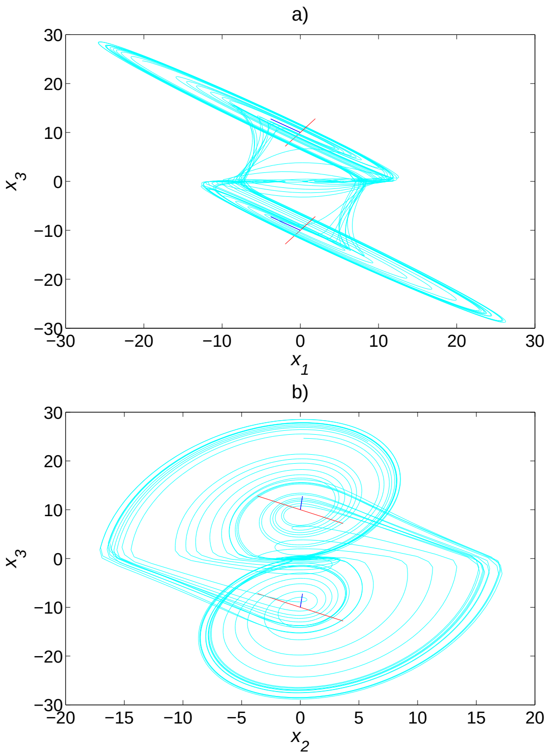

Figure 3. Projection of the attractor of the system (5) with $\mathbf{A}$ from (20), $\mathbf{B}(\mathbf{x})$ from (23) onto the: a) (x_1, x_3) plane; (x_2, x_3) plane. Marked with a red line the stable manifold E^s given by (24) and with a blue line a section of the unstable plane manifold E^u given by (25).

4 Displacement along the space

Following the same technique described above the displacement can be considered not only through the axes, but also in any region of the space. For example, in order to locate the equilibria in symmetric positions among the space, take the example of the matrix given in Eq. (14). The equilibria will result in the following position:

$$\mathbf{x}^* = \begin{bmatrix} \alpha_{x_1} \\ \alpha_{x_2} \\ \alpha_{x_3} \end{bmatrix}. \quad (26)$$

By solving the Eq. (7), the following affine vector will result:

$$\mathbf{B}(\mathbf{x}) = \begin{bmatrix} 2\alpha_{x_1} - 1\alpha_{x_2} - 3\alpha_{x_3} \\ -5\alpha_{x_1} + 5\alpha_{x_2} + 5\alpha_{x_3} \\ \alpha_{x_1} + 2.8\alpha_{x_2} + \alpha_{x_3} \end{bmatrix}. \quad (27)$$

For the sake of simplicity, the displacement values will be set as $\alpha_{x_1} = \alpha_{x_2} = \alpha_{x_3} = 1$. The commutation surface will be considered along the $x_1 + x_2 = 0$ plane, taking the following form:

$$\mathbf{B}(\mathbf{x}) = \begin{cases} \mathbf{B}_1 = [-2, 5, 4.8]^T, & \text{if } x_1 + x_2 \geq 0; \\ \mathbf{B}_1 = [2, -5, -4.8]^T, & \text{otherwise.} \end{cases} \tag{28}$$

By doing so, the equilibria of the system (5) with matrix $\mathbf{A}$ from Eq. (14) and the vector $\mathbf{B}$ defined in the commutation law given by (28) are: $\mathbf{x}_1^* = [1, 1, 1]^T$ and $\mathbf{x}_2^* = [-1, -1, -1]^T$. In this case, the space linear equations of E^s of the equilibria $\mathbf{x}_1^*$ and $\mathbf{x}_2^*$ result in the following:

$$\frac{x_1 \pm 1}{-0.2903} = \frac{x_2 \pm 1}{0.9061} = \frac{x_3 \pm 1}{0.3077}; \tag{29}$$

and the planes of E^u of the equilibria $\mathbf{x}_1^*$ and $\mathbf{x}_2^*$ result in:

$$-0.1877(x_1 \pm 1) + 0.2919(x_2 \pm 1) + 0.2771(x_3 \pm 1) = 0. \tag{30}$$

This is depicted in the projection of the attractor of the system onto the (x_1, x_2), (x_1, x_3) and (x_2, x_3) planes, given in Figure 4a), 4b) and 4c), respectively.

5 Conclusion

PWL systems that results in multi-scroll attractors have been implemented in several applications, and now a days, the possibility of generating trajectories that oscillate according to a pre-designed commuted system widens the possibilities of generating more complex trajectories. Considering the theory of attractors based on UDS, in which linear unstable system with specific eigenvalues are located along the space in order to trap the unstable trajectories. Here, a method to displace the equilibria not only among the x_1 axis, but in any region of space was presented. The idea is to present a matrix that satisfies the Definition 2.1 of UDS type I. Their eigenvectors must force the trajectory to oscillate along the equilibria whose desired position can be specified, and then adjust the affine vector by solving the linear system. One of the simple cases of forming an attractor comes from commutation laws that result in symmetric equilibria. An important feature of this type of systems, is that for every equilibrium of the UDS type I introduced, among with a corresponding commutation law, the number of scrolls presented in the attractor match the number of equilibria. Application of this theory can be extended to encryption and secure communication methods. The results can be reported elsewhere.

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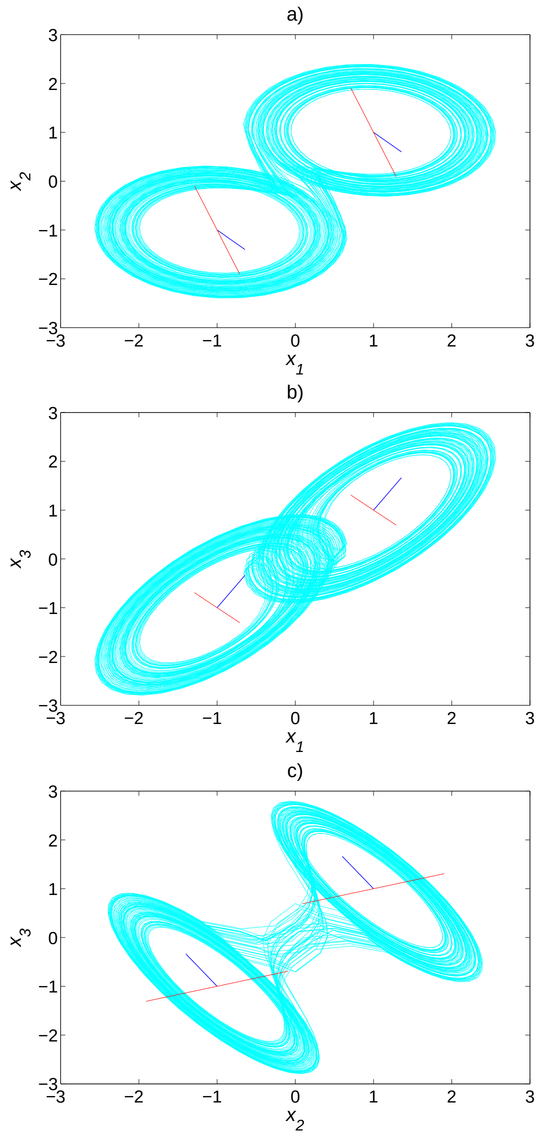

Figure 4. Projection of the attractor of the system (5) with $\mathbf{A}$ from (14), $\mathbf{B}(\mathbf{x})$ from (28) onto the: a) (x_1, x_2) plane; b) (x_1, x_3); c) (x_2, x_3) plane. Marked with a red line the stable manifold E^s given by (29) and with a blue line a section of the unstable plane manifold E^u given by (30).

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