Practically stable observer-based synchronization of discrete-time chaotic systems over the limited-band communication channel

Alexander L. Fradkov, Boris Andrievsky and Alexey Andrievsky

***Abstract*— Solution to the problem of observer-based synchronization over the limited bandwidth communication link for a class of discrete-time chaotic systems is presented. The result is demonstrated on the synchronization of chaotic Henon systems via a channel with limited capacity. It is shown that the proposed synchronization system is practically stable in the sense that the limit synchronization error can be made depending only on the number of digits in the computer.**

Key words: Chaotic behavior, Communication constraints, Synchronization, Practical stability

I. INTRODUCTION

Chaotic synchronization has attracted the attention of researchers since the 1980s [1]–[4] and is still an area of active research [5]–[9]. Recently information-theoretic concepts were applied to analyze and quantify synchronization [10]–[14]. In [11], [12] mutual information measures were introduced for evaluating the degree of chaotic synchronization. In [10], [13] the methods of symbolic dynamics were used to relate synchronization precision to capacity of the information channel and to the entropy of the drive system. Baptista and Kurths [14] introduced the concept of a chaotic channel as a medium formed by a network of chaotic systems that enables information from a source to pass from one system (transmitter) to another system (receiver). They characterized a chaotic channel by the mutual information (difference between the sum of the positive Lyapunov exponents corresponding to the synchronization manifold and the sum of positive exponents corresponding to the transverse manifold). However, in existing papers limit possibilities for the precision of controlled synchronization have not been analyzed.

Recently the limitations of control under constraints imposed by a finite capacity information channel have been investigated in detail in the control theoretic literature, see [15] and references therein. It was shown that stabilization under information constraints is possible if and only if the capacity of the information channel exceeds the entropy production of the system at the equilibrium [16]–[18]. In [19], [20] a general statement was proposed, claiming that the difference between the entropies of the open loop and the closed loop systems cannot exceed the information introduced by the controller, including the transmission rate of the information channel.

However, results of the mentioned works on control system analysis and design under information constraints do not apply to synchronization systems since in a synchronization problem trajectories in the phase space converge to a set (a manifold) rather than to a point, i.e. in the general case the problem cannot be reduced to simple stabilization. The problem is still more complicated for nonlinear systems, for incomplete state measurements and in the presence of uncertainty. Specifically, almost nothing is known about limit possibilities of estimation and control under information constraints for the partial stabilization, or set stabilization problem. Such a problem arises if one needs to stabilize a limit cycle or a chaotic attractor, which is important for the control of oscillatory modes in engineering systems [5], [21], [22]. However, analytical performance estimates of chaotic control systems are known only for a few cases, even without information constraints, see, e.g. [23], [24]; their development requires a sophisticated mathematical apparatus.

Observer-based synchronization systems are used in the case of incomplete measurements, when all phase variables are not available for measurement and coupling. Such systems are well studied without information constraints [25]–[27]. Observer-based synchronization of continuous-time chaotic systems under information constraints is studied in [28], [29], where limit possibilities are established. Similar results for *adaptive* synchronization were recently obtained in [30]. The papers [28]–[30] deal with synchronization of continuous-time chaotic systems over the digital communication link with finite capacity. The overall system can be naturally viewed as a *hybrid* one, i.e., system described by a coupling between continuous and discrete dynamics [31]. In the mentioned works the sampling rate was considered as a design parameter and the *one-step-memory* coder was used. Based on these conditions, it was established in [28]–[30] that the *binary* coding procedure minimizes the bit-per-second data rate over the channel, and also the ratio between the optimal sample time and the upper bound of the limit synchronization error was found.

The present paper is devoted to the synchronization problem of the *discrete-time* chaotic systems. In this case, the sampling time is not considered as a system parameter, and the *bit-per-step* rate is used as a measure of the channel capacity. Besides, the overall system for the discrete case is not a hybrid one. Therefore, the errors of modeling continuous-time systems by difference equations do not occur. This makes possible to use the *full-order* coders,

A. L. Fradkov and B. Andrievsky are with the Institute for Problems of Mechanical Engineering, Russian Academy of Sciences, 61, V.O. Bolshoy Ave., 199178, Saint Petersburg, Russia. {bandri,alf}@control.ipme.ru

A. Andrievsky is with the Control Systems Department, Baltic State Technical University, Saint Petersburg, Russia alexeyandrievsky@mail.ru

Corresponding author: Prof. Boris Andrievsky.

ensuring decay of the synchronization error asymptotically. This result complies with the statement, that that if the capacity of the channel is larger than the Kolmogorov–Sinai entropy of the driving system, then the synchronization error can be made arbitrarily small [10].

The paper is organized as follows. General form of the system model and problem statement are presented in Sec. II. Coding procedure is described in Sec. III. The example of synchronization of discrete-time chaotic *Henon systems* over the limited-band communication link is given in Sec. IV. Concluding remarks are given in Sec. V.

II. Description of observed-based synchronization system

Consider the n-dimensional discrete-time unidirectionally coupled drive–response systems. A block-diagram for implementing drive–response synchronization of two unidirectionally coupled systems via a discrete communication channel is shown in Fig. 1. To simplify exposition we will consider drive (master, entraining) system in so-called *Lurie form*: right-hand sides are split into a linear part and a nonlinearity vector depending only on the measured output. Then the drive system is modeled as follows:

$$x_{k+1} = Ax_k + B\varphi(y_k),\ \ y_k = Cx_k, \tag{1}$$

where $k \in \mathbb{Z}$ is discrete time, $k = 0, 1, \ldots$; $x_k \in \mathbb{R}^n$ is the vector of state variables, y_k is the scalar output (coupling) variable, A is an $(n \times n)$-matrix, B is $(n \times 1)$-matrix C is $(1 \times n)$-matrix, $\varphi(y)$ is a continuous nonlinearity. We assume that all the trajectories of the system (1) belong to a bounded set Ω (e.g. attractor of a chaotic system). Such an assumption is typical for chaotic systems.

The response (slave, entrained) system is described as a *nonlinear observer*

$$\hat{x}_{k+1} = A\hat{x}_k + B\varphi(y_k) + L\varepsilon_k,\quad \varepsilon_k = y_k - \hat{y}_k,\ \ \hat{y}_k = C\hat{x}_k, \tag{2}$$

where L is the vector of the observer parameters (gain). Apparently, the dynamics of the state error vector $e_k = x_k - \hat{x}_k$ is described by a linear equation

$$e_{k+1} = A_L e_k, \tag{3}$$

where $A_L = A - LC$.

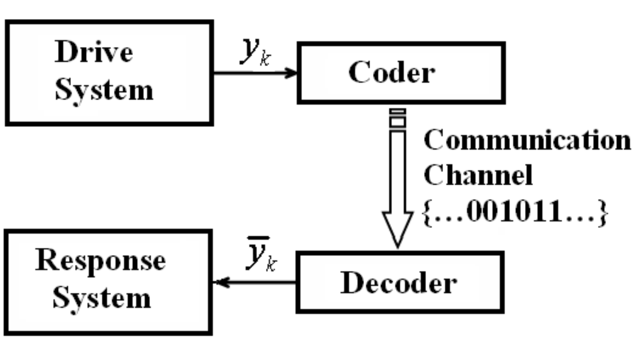

Fig. 1. Block-diagram for drive–response synchronization using a discrete communication channel.

As is known from control theory [32], if the pair (A, C) is observable, i.e. if $\operatorname{rank}[C^{\mathrm{T}}, A^{\mathrm{T}}C^{\mathrm{T}}, \ldots, (A^{\mathrm{T}})^{n-1}C^{\mathrm{T}}] = n$, then there exists L providing the matrix A_L with any given eigenvalues. Particularly, if all eigenvalues of A_L lie inside the unit circle on the complex plane, the system (3) is asymptotically stable [1] and $e_k \to 0$ as $k \to \infty$. The gain vector L may be found using standard pole locus technique, or applying H_2/H_∞ optimization procedure. Therefore, in the absence of measurement and transmission errors the synchronization error e_k decays to zero.

Now let us take into account transmission errors. We assume that the observation signal y_k is coded with symbols from a finite alphabet $\mathscr{S} = \{s_1, s_2, \ldots s_\aleph\}$ at time instants $k = 0, 1, 2, \ldots$. Let the coded symbol $\bar{y}_k$ be transmitted over a digital communication channel with a finite capacity. To simplify the analysis, we assume that the observations are not corrupted by observation noise; transmissions delay and transmission channel distortions may be neglected. Therefore, it is assumed that the coded symbols are available at the receiver side at the same time instant k, as they are generated by the coder.

Since the signal $\bar{y}_k$, instead of the system output y_k, is available at the receiver end, the response system is described by the following equation

$$\hat{x}_{k+1} = A\hat{x}_k + B\varphi(\bar{y}_k) + L(\bar{y}_k - \hat{y}_k),\ \ \hat{y}_k = C\hat{x}_k \tag{4}$$

instead of (2).

Due to a presence of the *transmission error*

$$\delta_{y,k} = y_k - \bar{y}_k, \tag{5}$$

equation (3) does not describe the estimation error dynamics, and should be replaced with the following error model:

$$e_{k+1} = A_L e_k + B\big(\varphi(y_k) - \varphi(y_k + \delta_{y,k})\big) - L\delta_{y,k}. \tag{6}$$

Similarly to [29] it may be shown that the limit synchronization error $\overline{\lim_{k\to\infty}}\|e_k\|$ satisfies inequality $\overline{\lim_{k\to\infty}}\|e_k\| \le C_e^+\Delta$ for some C_e^+, where Δ is the upper bound of the absolute transmission error.

Our goal is to demonstrate, that the full-order (of order n) time-varying coder makes possible to ensure asymptotically arbitrarily small synchronization error if the channel capacity is sufficiently large. This result comply with those obtained for a linear case in [33]–[36]. The idea is to use the “zooming” strategy to increase coder accuracy as the estimation error decreases, and, at the same time, to prevent coder saturation at the beginning of the process [16], [31], [37], [38]. Since the quantizer range decreases from step to step, the *prediction* procedure is used at the coder/decoder pair. To accomplish this procedure, the full-order coder is used, where the state estimation algorithm (4) is realized.

[1] The finite n-step transient time may be obtained if all eigenvalues of A_L are zeros.

III. Coding procedure

At first, consider the memoryless (static) coder with uniform discretization and constant range. For a given real number $M>0$ and natural number $\nu \in \mathbb{Z}_+$ define a uniform scaled coder to be a discretized map $q_{\nu,M} : \mathbb{R} \to \mathbb{R}$ as follows. Introduce the *range interval* $\mathscr{I} = [-M, M]$ of length $2M$ and the *discretization interval* of length $\delta = M/\nu$. Therefore, 2ν is the number of different levels of the quantizer output in the interval $\mathscr{I}$. Define the coder function $q_{\nu,M}(y)$ as

$$q_{\nu,M}(y) = \min\left(\left\langle \delta^{-1}|y|\right\rangle \delta, M\right) \cdot \mathrm{sign}(y), \tag{7}$$

where $\langle\cdot\rangle$ rounds the argument to the nearest integers towards infinity, $\mathrm{sign}(\cdot)$ is the signum function:

$$\mathrm{sign}(y) = \begin{cases} 1 & \text{if } \ y \geq 0, \\ -1 & \text{otherwise.} \end{cases}$$

For illustration, the plot of function $q_{\nu,M}(y)$ for $\nu=3$, $M=1$ is given in Fig. 2.

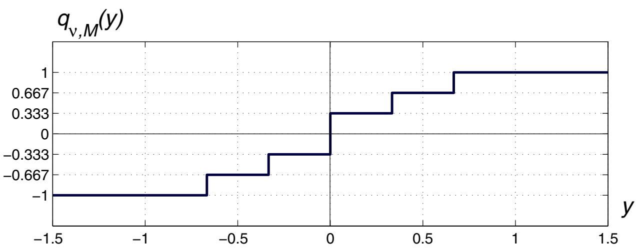

Fig. 2. Plot of the quantizer function (7) $q_{\nu,M}(y)$, $\nu=3$, $M=1$.

Notice that the interval $\mathscr{I}$ is equally split into 2ν parts. Therefore, the cardinality of the mapping $q_{\nu,M}$ image is equal to 2ν, and each codeword symbol contains $R = 1+\log_2 \nu$ bits of information.[1] Thus, the discretized output of the considered coder is found as $\bar{y} = q_{\nu,M}(y)$. We assume that the coder and decoder make decisions based on the same information.

Expression (7) describes a simple ("primitive") static coder. More sophisticated encoding schemes utilize time-varying coders with memory, see, e.g., [16], [31], [37], [38]. The underlying idea for coders of this kind is to reduce the range parameter M, replacing the symmetric range interval $\mathscr{I}$ by the interval $\mathscr{Y}_{k+1}$, covering some area around the predicted value for the $(k+1)$th observation y_{k+1}, $y_{k+1} \in \mathscr{Y}_{k+1}$. If the length of $\mathscr{Y}_{k+1}$ is small compared with the full range of possible measured output values y, then there is an opportunity to reduce the range parameter M and, consequently, to decrease the coding interval δ preserving the bit-rate of transmission. To realize this scheme, memory should be introduced into the coder. Using such a "zooming" strategy it is possible to increase coder accuracy in the steady-state mode, and, at the same time, to prevent coder saturation at the beginning of the process. This means that the quantizer range M is updated during the time and a time-varying quantizer (with different values of M for each instant, $M = M_k$) is used. The values of M_k may be precomputed (the *time-based* zooming), or, alternatively, current quantized measurements may be used at each step to update M_k (the *event-based* zooming).

In this paper we use an n-order coder with time-based zooming. To describe it, introduce the *central number* c_k, $k = 0,1,2,\ldots$ as the predicted value for the kth observation $\tilde{y}_k$, generated by observer (3) based on the $(k-1)$th step. At step k the coder compares the current measured output y_k with the number c_k, forming the deviation signal $\partial y_k = y_k - c_k$. Then this signal is discretized with a given ν and $M = M_k$ according to (7). The output signal

$$\bar{\partial} y_k = q_{\nu,M_k}(\partial y_k) \tag{8}$$

is represented as an R-bit information symbol from the coding alphabet $\mathscr{S}$ and transmitted over the communication channel to the decoder. Then the central number c_{k+1} and the range parameter M_k are renewed. We use the following recurrence equation for the range parameter M_k:

$$M_k = \max(\rho M_{k-1}, \varepsilon), \qquad k = 1,2,\ldots, \tag{9}$$

with some initial condition $M_0 > 0$. In (9), $0 < \rho \leq 1$ is the decay parameter, $\varepsilon > 0$ stands for the limit value of M_k. The initial value M_0 should be large enough to capture all the region of possible initial values of y_0. Expression (9) describes exponentially vanishing sequence of M_k, bounded below by ε.

The equations (7), (8), (9) describe the coder algorithm. The same algorithm is realized by the decoder. Namely, the decoder calculates the variables $\tilde{c}_k$, $\tilde{M}_k$ based on received codeword flow similarly to c_k, M_k.

IV. Example. Synchronization of chaotic Henon systems

Let us apply the above results to synchronization of two chaotic Henon systems coupled via a channel with limited capacity.

1) System equations: Consider the following chaotic *Henon system* [21], [41]:

$$\begin{cases} \xi_{1,k+1} = 1 - a\xi_{1,k}^2 + \xi_{2,k}, \\ \xi_{2,k+1} = bx_{1,k}, \end{cases} \tag{10}$$

where $\xi = [\xi_1, \xi_2]^{\mathrm{T}} \in \mathbb{R}^2$ is the state vector, a, b are system parameters (in the sequel, $a=1.4$, $b=0.3$ are taken).

After the state-space transform, Eqs. (10) may be rewritten in the following form:

$$\begin{cases} x_{1,k+1} = x_{2,k}, \\ x_{2,k+1} = bx_{1,k} - ax_{2,k}^2 + 1, \end{cases} \tag{11}$$

where $x = [x_1, x_2]^{\mathrm{T}} \in \mathbb{R}^2$ is the state vector. Assume that $y(t) = x_{2,k}$ is the system output signal to be transmitted over the

[1] The information content of the codeword depends on the probability distribution of the source [39]. Since we are interested in the guaranteed bounds, we evaluate the information content of the codeword as $R = 1+\log_2 \nu$ bits, which is an upper bound on the information content of a codeword, realized if the source has a uniform distribution over the interval $[-M, M]$. In other words, we use a *combinatorial* definition of information [40], rather than a *probabilistic* one.

communication channel. The vector of initial conditions $x_0 = x(0)$ is assumed to be unknown at the receiver end

2) Observer design: Since the drive system (11) has a Lurie form (1), let us take the response system in the form of observer (4) and apply the coding procedure (8), (9).

For the considered case, in (4) the matrices A, B, C and the function $\varphi(\cdot)$ are as follows:

$$A=\begin{bmatrix}0 & 1\\ b & 0\end{bmatrix}, \quad B=\begin{bmatrix}0\\ 1\end{bmatrix}, \quad C=[0,1], \quad \varphi(z)=1-az^2. \qquad (12)$$

The explicit form of the slave system (4) for the considered drive system (11) is as follows:

$$\begin{cases}\hat{x}_{1,k+1} = \hat{x}_{2,k} + l_1(\bar{y}_k - \hat{y}_k), \quad \hat{y}_k = x_{2,k},\\ \hat{x}_{2,k+1} = b\hat{x}_{1,k} - a\bar{y}_k^2 + 1 + l_2(\bar{y}_k - \hat{y}_k),\end{cases} \qquad (13)$$

where $\bar{y}$ is the output signal of the decoder.

The characteristic polynomial $A_L(\lambda) = \det(\lambda \mathbf{I}_2 - A_L)$ (where $\lambda \in \mathbb{C}$, $\mathbf{I}_2$ is (2×2)-identity matrix) of the matrix

$$A_L = A - LC = \begin{bmatrix}0 & 1-l_1\\ b & -l_2\end{bmatrix}$$

has a form $A_L(\lambda) = \lambda^2 + \lambda l_2 - b + bl_1$. Let the desired eigenvalues $\lambda_{1,2} \in \mathbb{C}$ of the matrix A_L be given. Then the desired characteristic polynomial $A_L^*(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)$ $=\lambda^2+d_1\lambda+d_2$, where $d_1=-(\lambda_1+\lambda_2)$, $\lambda_2=\lambda_1\lambda_2$. Therefore, one obtains the following expressions for the observer (13) gains: $l_1=(\lambda_1\lambda_2+b)b^{-1}$, $l_2=-\lambda_1-\lambda_2$.

3) Simulation results: The system (8), (9), (10), (13) was studied numerically for the following parameter values and initial conditions:

$$a=1.4, \quad b=0.3, \quad l_1=1.0, \quad l_2=-1.2\cdot 10^{-3},$$
$$\rho=0.8; \quad x_{1,0}=0.75, \quad x_{2,0}=-0.5, \quad \hat{x}_{1,0}=\hat{x}_{2,0}=0.$$

The number of quantizer levels $\nu = 3$ was chosen. This number corresponds the channel bit-rate $R = 1+\log_2(\nu) = 2.585$ bit per step. The minimal bound for the decay parameter ρ in (9), $\rho_{\min}\approx 0.75$, was also found. For the less values of ν and ρ the synchronization process failed. This result confirms the general statement, claiming that the difference between the entropies of the open loop and the closed loop systems cannot exceed the information introduced by the controller, including the transmission rate of the information channel [16].

Different values of the parameter ε in (9) were taken to evaluate the minimal possible limit synchronization error $\overline{\lim_{k\to\infty}}\|e_k\|$, obtained by means of available computer platform and simulation environment. [1] During the simulations was found that the minimal admissible ε is $\varepsilon_{\min} = 2\cdot 10^{-15}$.

Some simulation results are presented in Figs. 3–6.

Simulation results show that the limit synchronization error can be made close to the maximum achievable accuracy of the given computer (*computer epsilon*, see [42]) depending only on the number of digits in the computer.

[1] The 32-bit AMD Athlon™ processor and MATLAB software with the floating point relative accuracy (*computer epsilon*) $\varepsilon_c = 2.2204\cdot 10^{-16}$ were used.

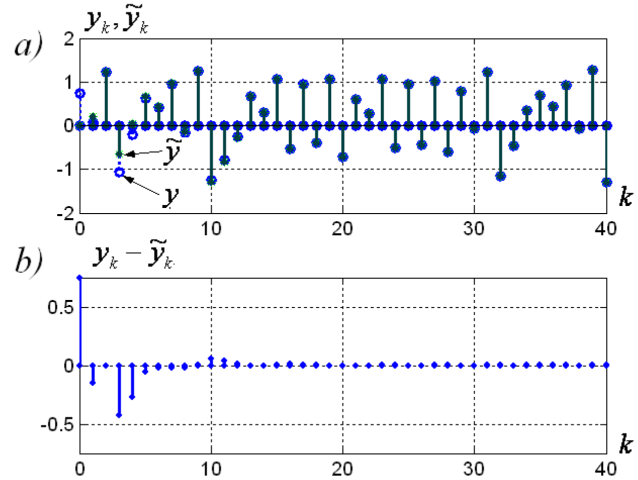

Fig. 3. Time histories of the drive and response systems outputs (*a*) and of the synchronization error (*b*).

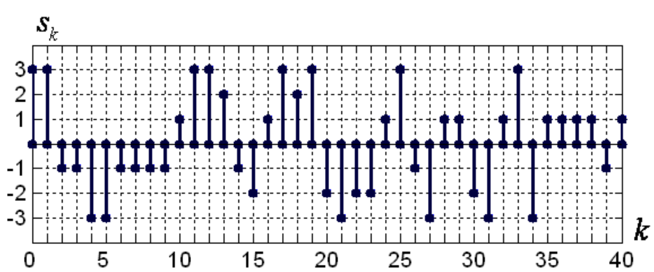

Fig. 4. Sequence of the codewords, transmitted over the channel, ($\mathscr{S} = \{-1,-2,-1,1,2,3\}$).

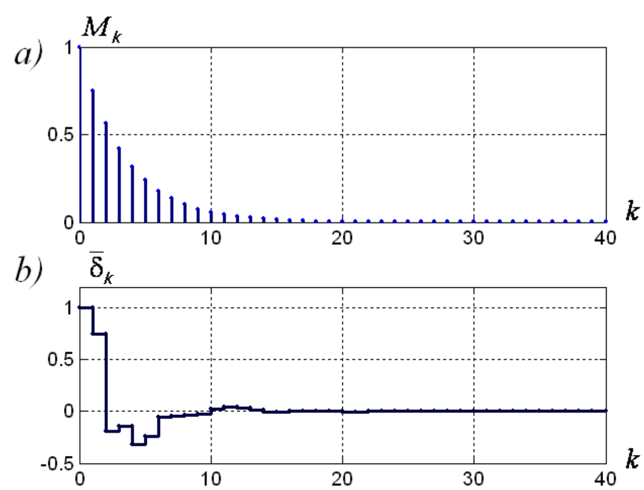

Fig. 5. Time histories of the quantizer range M_k (*a*) and the quantizer output (*b*).

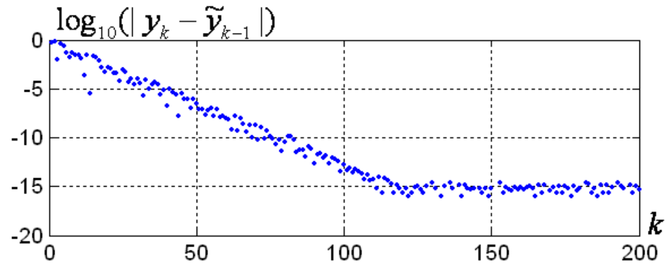

Fig. 6. Logarithmically scaled synchronization error.

Such a phenomenon can be called *practical synchronization*, by analogy with practical stability [43]: the limit absolute value of synchronization error decreases unlimitedly if the accuracy of the computation increases unlimitedly.

V. Conclusions

We have studied synchronization over the limited bandwidth communication link for a class of discrete-time chaotic systems. Solution to the problem of observer-based synchronization over the limited bandwidth communication link for a class of discrete-time chaotic systems is presented. The result is demonstrated on the synchronization of chaotic Henon systems via a channel with limited capacity.

It is shown that if a real computer is used for computation, then an effect of *practical synchronization* rather then asymptotical synchronization is observed. Namely, an absolute value of synchronization error is bounded by a value Δ rather then tends to zero. If transmission rate is sufficiently large, the value of Δ depends solely on the number of digits of the computer and it is close to the maximum achievable accuracy of the given computer (computer epsilon, see [42]). If the accuracy of the computation increases unlimitedly, then the limit absolute value of synchronization error decreases unlimitedly.

Acknowledgements

The work was supported by the Russian Foundation for Basic Research (projects RFBR 05-01-00869, 06-08-01386) and Scientific Program of RAS No 22 (project 1.8).

References

[1] E. N. Dudnik, Y. I. Kuznetsov, I. I. Minakova, and Y. M. Romanovskii, *Vestn. Mosk. Gos. Univ., Ser. 3. Fiz. Astron.*, vol. 24, no. 4, pp. 84–87, 1983.

[2] H. Fujisaka and T. Yamada, *Prog. Theor. Phys.*, vol. 69, no. 1, pp. 32–47, 1983.

[3] V. S. Afraimovich, N. N. Verichev, and M. I. Rabinovich, *Radiophys. Quant. Elect.*, vol. 29, pp. 795–803, 1987.

[4] L. M. Pecora and T. L. Carroll, *Phys. Rev. Lett.*, vol. 64, p. 821, 1990.

[5] A. L. Fradkov and A. Y. Pogromsky, *Introduction to control of oscillations and chaos*. Singapore: World Scientific Publishers, 1998.

[6] L. M. Pecora and T. L. Carroll, "Master stability functions for synchronized coupled systems," *Phys. Rev. Lett.*, vol. 80, p. 2109, 1998.

[7] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences*. Cambridge University Press, Cambridge, 2001.

[8] S. Boccaletti, J. Kurths, G. Osipov, D. L. Valladares, and C. Zhou, "The synchronization of chaotic systems," *Physics Reports*, vol. 366, no. 1–2, pp. 1–101, 2002.

[9] U. S. Freitas, E. E. N. Macau, and C. Grebogi, "Using geometric control and chaotic synchronization to estimate an unknown model parameter," *Phys. Rev. E*, vol. 71, no. 4, p. 047203, 2005.

[10] T. Stojanovski, L. Kočarev, and R. Harris, *IEEE Trans. Circuits Syst. I*, vol. 44, p. 1014, 1997.

[11] M. Paluš, V. Komárek, Z. Hrnčir, and K. Štěrbová, "Synchronization as adjustment of information rates: Detection from bivariate time series," *Phys. Rev. E*, vol. 63, p. 046211, 2001.

[12] A. Shabunin, V. Demidov, V. Astakhov, and V. Anishchenko, "Information theoretic approach to quantify complete and phase synchronization of chaos," *Phys. Rev. E*, vol. 65, p. 056215, 2002.

[13] S. D. Pethel, N. J. Corron, Q. R. Underwood, and K. Myneni, "Information flow in chaos synchronization: Fundamental tradeoffs in precision, delay, and anticipation," *Phys. Rev. Lett.*, vol. 90, p. 254101, 2003.

[14] M. S. Baptista and J. Kurths, "Chaotic channel," *Phys. Rev. E*, vol. 72, p. 045202(R), 2005.

[15] G. N. Nair, F. Fagnani, S. Zampieri, and R. Evans, "Feedback control under data rate constraints: an overview," *Proc. IEEE*, 2007, (in press).

[16] G. N. Nair and R. J. Evans, "Exponential stabilisability of finite-dimensional linear systems with limited data rates," *Automatica*, vol. 39, pp. 585–593, 2003.

[17] ——, "Stabilizability of stochastic linear systems with finite feedback data rates," *SIAM J. Control Optim*, vol. 43, no. 2, pp. 413–436, 2004.

[18] G. N. Nair, R. J. Evans, I. Mareels, and W. Moran, "Topological feedback entropy and nonlinear stabilization," *IEEE Trans. Automat. Contr.*, vol. 49, no. 9, pp. 1585–1597, Sept. 2004.

[19] H. Touchette and S. Lloyd, *Phys. Rev. Lett.*, vol. 84, p. 1156, 2000.

[20] ——, "Information-theoretic approach to the study of control systems," *Physica A – Statistical Mechanics and its Applications*, vol. 331, no. 1–2, pp. 140–172, 2004.

[21] B. R. Andrievsky and A. L. Fradkov, "Control of chaos: I, Methods," *Autom. Remote Control*, vol. 64, no. 5, pp. 673–713, 2003.

[22] A. L. Fradkov and R. J. Evans, "Control of chaos: Methods and applications in engineering," *Annual Reviews in Control*, vol. 29, no. 1, pp. 33–56, 2005.

[23] A. L. Fradkov and S. M. Khryaschev, "How much control needs control of chaos," in *Proc. 5th EUROMECH Nonlinear Dynamics Conference (ENOC 2005)*, Aug. 2005, pp. 1295–1302.

[24] S. M. Khryashchev, "Estimation of control time for systems with chaotic behavior: Part i," *Autom. Remote Control*, vol. 64, no. 10, pp. 1566–1579, 2004.

[25] O. Morgül and E. Solak, "Observer based synchronization of chaotic systems," *Phys. Rev. E*, vol. 54, no. 5, pp. 4803–4811, 1996.

[26] H. Nijmeijer and I. M. Y. Mareels, "An observer looks at synchronization," *IEEE Trans. on Circuits and Systems I*, vol. 44, no. 10, pp. 882–890, 1997.

[27] H. Nijmeijer, "A dynamical control view on synchronization," *Physica D*, vol. 154, no. 3–4, pp. 219–228, 2001.

[28] B. Andrievsky, A. L. Fradkov, and R. J. Evans, "Analysis of a chaotic synchronisation system under information constraints," in *Proc. 1st IFAC Conf. Analysis and Control of Chaotic Systems (Chaos 06)*, June 2006.

[29] A. L. Fradkov, B. Andrievsky, and R. J. Evans, "Chaotic observer-based synchronization under information constraints," *Physical Review E*, vol. 73, p. 066209, 2006.

[30] ——, "Adaptive observer-based synchronization of chaotic systems in presence of information constraints," in *Proc. 1st IFAC Conf. Analysis and Control of Chaotic Systems (Chaos 06)*, June 2006.

[31] D. Liberzon, "Hybrid feedback stabilization of systems with quantized signals," *Automatica*, vol. 39, pp. 1543–1554, 2003.

[32] T. Glad and L. Ljung, *Control Theory (multivariable and nonlinear methods)*. Taylor and Francis, 2000.

[33] G. N. Nair and R. J. Evans, "State estimation via a capacity-limited communication channel," in *Proc. 36th IEEE Conference on Decision and Control*, vol. WM09. San Diego, Califomia USA: IEEE, Dec. 1997, pp. 866–871.

[34] ——, "State estimation under bit-rate constraints," in *Proc. 37th IEEE Conference on Decision and Control*, vol. WA09. Tampa, Florida USA: IEEE, Dec. 1998, pp. 251–256.

[35] ——, "Structural results for finite bit-rate state estimation," in *Proc. IDC'99*. IEEE, 1999, pp. 47–52.

[36] A. V. Savkin and I. R. Petersen, "Set-valued state estimation via a limited capacity communication channel," *IEEE Trans. Automat. Contr.*, vol. 48, no. 4, pp. 676–680, Apr. 2003.

[37] R. W. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Trans. Automat. Contr.*, vol. 45, no. 7, pp. 1279–1289, 2000.
[38] S. Tatikonda and S. Mitter, "Control over noisy channels," *IEEE Trans. Automat. Contr.*, vol. 49, no. 7, pp. 1196–1201, July 2004.
[39] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. New York, Chichester, Brisbane, Toronto, Singapore: John Wiley & Sons, Inc., 1991.
[40] A. N. Kolmogorov, "Three approaches to the quantitive definition of information," *Problems of Information Transmission*, vol. 1, pp. 1–17, 1965.
[41] F. Moon, *Chaotic and Fractal Dynamics. An Introduction for Applied Scientists and Engineers*. Wiley & Sons, 1992.
[42] G. E. Forsythe, M. A. Malcolm, and C. B. Moler, *Computer Methods for Mathematical Computations*. Prentice-Hall, 1977.
[43] L. Moreau and D. Aeyels, "Practical stability and stabilization," *IEEE Trans. Automat. Contr.*, vol. 45, no. 8, pp. 1554–1558, 2000.