CYBERNETICS AND PHYSICS, VOL. 2, NO. 3, 2013 , 159–164

A PROPOSAL OF PROBLEMS ABOUT STABLE HURWITZ POLYNOMIALS

Baltazar Aguirre-Hernández
Department of Mathematics
Universidad Autónoma Metropolitana – I
Mexico
bahe@xanum.uam.mx

Carlos Arturo Loredo-Villalobos
SEPI – UPIICSA
Instituto Politécnico Nacional
Department of Mathematics
Universidad Autónoma Metropolitana – I
Mexico
calv@xanum.uam.mx

Faustino Ricardo García-Sosa
SEPI – UPIICSA
Instituto Politécnico Nacional
Mexico
frgarcia@ipn.mx

Abstract
We say that a real polynomial is a Hurwitz polynomial if all of its roots have negative real part. The importance of the Hurwitz polynomials can be appreciated in the study of stability of a linear systems: if the characteristic polynomial is Hurwitz then the system is stable. In this paper we explain the main criteria about Hurwitz polynomials and we pose some open problems about them.

Key words
Hurwitz polynomials, stable systems, principal diagonal minors.

1 Introduction

Maxwell raised in 1868 the problem of deciding if a polynomial has all its roots with negative real part ([Maxwell, 1868]). Regardless Maxwell, the Austrian engineer A. Stodola also raise the question to A. Hurwitz. The problem is interesting because if the characteristic polynomial of a linear system is a Hurwitz polynomial then the system is stable. At first the researchers investigated criteria for deciding if a polynomial is Hurwitz polynomial and several criteria were found, among which can mention the Routh-Hurwitz criterion ([Hurwitz, 1895]), the Hermite- Biehler's theorem ([Hermite, 1856]), the Liènard-Chipar conditions ([Liènard and Chipart, 1914]), the stability test ([Battacharayya *et al.* 1995]), the Mihailov's theorem ([Loredo-Villalobos, 2012],[Mihailov, 1938]) or Routh's algorithm ([Routh, 1877]). Later researchers also investigated interesting properties about these polynomials, for instance, the relation between the Hadamard product and Hurwitz was studied and several result were found ([Garloff and Wagner, 1996],[Loredo-Villalobos and Aguirre-Hernández, 2011], [Loredo-Villalobos and Aguirre-Hernández, 2012], [Loredo-Villalobos, 2012]).

It is worth mentioning that topological and geometric approaches have also used in the study of the set of Hurwitz polynomials (see [Aguirre-Hernández *et al.*, 2009], [Aguirre-Hernández *et al.*, 2012], [Aguirre-Hernández *el al.*, 2012]).A substantial amount of information about these polynomials and issues can be found in [Battacharayya *et al.* 1995], [Gantmacher, 1959], [Lancaster and Tismenetsky, 1985], [Loredo-Villalobos, 2012] and [Zabczyk, 1992]. Recent information about Hurwitz polynomials can be consulted [Rahman and Schmeiser, 2002] y [Fisk, 2008]. In this paper we will discuss some of these criteria with the aim of developing some open research problems.

A short version of this paper was presented in Physcon 2013 congress (see [Aguirre-Hernández *et al.*, 2013]).

2 Necessary Conditions

We begin this section by setting the main definition.

Definition. A polynomial with real coefficients $f(t) = b_0t^n + b_1t^{n-1} + \cdots + b_{n-1}t + b_n$ is Hurwitz, if all its roots have negative real part. These polynomials also are named stable Hurwitz polynomials or stable polynomials.

Example 1. The polynomial $g(t) = t^2 + 5t + 6$ is a polynomial Hurwitz because its roots are $t = -2, -3$.

Example 2. The polynomial $h(t) = t^2 + 25$ is not a Hurwitz polynomial because its roots are $t = -5i, 5i$.

The following theorem establishes a necessary condition for a polynomial is a Hurwitz polynomial.

Theorem 1. If $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ is a Hurwitz polynomial and $\xi \in \mathcal{C}$ then one of the following conditions is hold:

a) If $\mathrm{Re}(\xi) > 0$, $|p(\xi)| > |p(-\xi)|$.
b) If $\mathrm{Re}(\xi) = 0$, $|p(\xi)| = |p(-\xi)|$.
c) If $\mathrm{Re}(\xi) < 0$, $|p(\xi)| < |p(-\xi)|$.

Proof. We can write $p(x)$ as

$$p(x) = a_n(x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_n)$$

with $\alpha_i \in \mathbb{C}^-$. We have two cases:
Case 1. Suppose that $\alpha_k \in \mathbb{R}^-$ and $\xi = a + ib$. If $\mathrm{Re}(\xi) > 0$ then $(a - \alpha_k)^2 + b^2 > (a + \alpha_k)^2 + b^2$, this imply that $|\xi - \alpha_k|^2 > |-\xi - \alpha_k|^2$, from where $|\xi - \alpha_k| > |-\xi - \alpha_k|$ and $|p(\xi)| > |p(-\xi)|$. For the remaining b) and c) items we follow a similar reasoning.
Case 2. Pair of conjugate roots. Now we suppose that

$$\alpha_r = \gamma + i\delta, \qquad \alpha_s = \gamma - i\delta, \qquad \gamma < 0,\, \delta > 0$$

If $\mathrm{Re}(\xi) > 0$ then

$$(a-\gamma)^2 > (a+\gamma)^2 \text{ and } (a-\gamma)^4 > (a+\gamma)^4$$

this imply that

$$(a-\gamma)^4 + (a-\gamma)^2[(b-\delta)^2 + (b+\delta)^2] > \\ (a+\gamma)^4 + (a+\gamma)^2[(b-\delta)^2 + (b+\delta)^2]$$

namely

$$|\xi - \alpha_r|^2|\xi - \alpha_s|^2 > |-\xi - \alpha_r|^2|-\xi - \alpha_s|^2$$

Therefore

$$|\xi - \alpha_r||\xi - \alpha_s| > |-\xi - \alpha_r||-\xi - \alpha_s|$$

from where $|p(\xi)| > |p(-\xi)|$. Remaining items can prove similarly.

Problem 1. An open problem is to determine which other assumptions must be added to the previous theorem to have necessary and sufficient conditions.

Theorem 2. If $p(x)$ is a Hurwitz polynomial then $p'(x)$ is a Hurwitz polynomial.

Proof. Let $w_1, w_2, \ldots, w_n$ roots of $p(t)$. Since $p(t)$ is Hurwitz then $\mathrm{Re}\, w_i < 0\ \forall i\, (1 \le i \le n)$. Let t such that $\mathrm{Re}\, t \ge 0$. If $\mathrm{Re}\,(-w_i) > 0$ then $\mathrm{Re}\,(t - w_i) = \mathrm{Re}\, t + \mathrm{Re}\,(-w_i) > 0$. From where

$$\mathrm{Re}\left(\frac{1}{t - w_i}\right) > 0$$

$$\Rightarrow \mathrm{Re}\left(\sum_{i=1}^{n} \frac{1}{t - w_i}\right) > 0$$

then

$$\frac{p'(t)}{p(t)} = \sum_{i=1}^{n} \frac{1}{t - w_i} \neq 0$$

Therefore $p'(t) \neq 0\ \forall t\, (\mathrm{Re}\, t \ge 0)$, from where the roots of $p'(t)$ belong to $\mathbb{C}^-$.

Problem 2. What conditions should be added to $p'(x)$ for it implies the stability of p?

Remark 1. Theorem 2 let us verify if a polynomial is unstable checking the instability of a smaller degree polynomial ($p'(x)$).

Remark 2. If we solve the problem 2, it let us verify if a polynomial is Hurwitz checking the stability of $p'(x)$ and an additional condition. The criteria would be different to the test explained in section 7.

3 Routh-Hurwitz Theorem

Next we set up the Routh-Hurwitz criterion . There are several tests of this theorem. To see some of them please refer to the references [Battacharayya *et al.* 1995], [Lancaster and Tismenetsky, 1985] and [Loredo-Villalobos, 2012].

Theorem 3 (Routh-Hurwitz criterion). Given a polynomial with real coefficients $f(t) = b_0t^n + b_1t^{n-1} + \cdots + b_{n-1}t + b_n$ we define the Hurwitz matrix associated with this polynomial

$$H(f) = \begin{bmatrix} b_1 & b_0 & 0 & \cdots & 0 \\ b_3 & b_2 & b_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{2n-3} & b_{2n-4} & b_{2n-5} & \cdots & b_{n-2} \\ b_{2n-1} & b_{2n-2} & b_{n-3} & \cdots & b_n \end{bmatrix}$$

where $b_k = 0$ if $k > n$.
to such polynomial has all of his roots with negative real part , it is necessary and sufficient that it satisfies

$$b_0\Delta_1 > 0, \Delta_2 > 0, b_0\Delta_3 > 0, \Delta_4 > 0, \\ \ldots, \begin{cases} b_0\Delta_n > 0, & \text{if } n \text{ is odd} \\ \Delta_n > 0, & \text{if } n \text{ is even} \end{cases}$$

where Δ_i are principal diagonal minors of the matrix of Hurwitz, i.e.

$$\Delta_1 = \det(b_1),$$
$$\Delta_2 = \det\begin{pmatrix} b_1 & b_0 \\ b_3 & b_2 \end{pmatrix},$$
$$\vdots$$
$$\Delta_n = \det H(f)$$

In case $b_0 = 1$, the condition simply said that the lower main diagonals must be positive, i.e. $\Delta_1 > 0, \Delta_2 > 0, \Delta_3 > 0, \ldots, \Delta_n > 0$.

Example 3. Consider the polynomial $p(t) = t^3+5t^2+3t + 7$. The Hurwitz matrix corresponding to the polynomial p is the matrix

$$H(p) = \begin{pmatrix} 5 & 1 & 0 \\ 7 & 3 & 5 \\ 0 & 0 & 7 \end{pmatrix}$$

so we have to the lower main diagonals are $\Delta_1 = 5 > 0$, $\Delta_2 = 8 > 0$, $\Delta_3 = 56 > 0$, then we can claim that the polynomial $p(t)$ is a Hurwitz polynomial.

4 Lienard-Chipart's Conditions

In verifying whether a polynomial of degree n is a polynomial Hurwitz or not using the Routh-Hurwitz criterion, we see that we will have to compute n determinants and check if they have positive sign. If the degree is so large then we will have to make a good amount of operations. For this reason it is desirable that one could work with criteria to reduce operations. This objective is satisfied by following theorem, which may be considered as an improvement Criteria Routh-Hurwitz.

Theorem 4 (Lienard-Chipart's conditions). The polynomial $f(t) = b_0t^n + b_1t^{n-1} + \cdots + b_{n-1}t + b_n$ ($b_0 > 0$) is Hurwitz if and only if satisfies any of the following conditions:

1) $b_n > 0, b_{n-2} > 0, b_{n-4} > 0, ...;$
$\Delta_1 > 0, \Delta_3 > 0, \Delta_5 > 0, ...$

2) $b_n > 0, b_{n-2} > 0, b_{n-4} > 0, ...;$
$\Delta_2 > 0, \Delta_4 > 0, \Delta_6 > 0, ...$

3) $b_n > 0, b_{n-1} > 0, b_{n-3} > 0, ...;$
$\Delta_1 > 0, \Delta_3 > 0, \Delta_5 > 0, ...$

4) $b_n > 0, b_{n-1} > 0, b_{n-3} > 0, ...;$
$\Delta_2 > 0, \Delta_4 > 0, \Delta_6 > 0, ...$

Example 4. Consider the polynomial $q(t) = t^3+8t^2+2t + 9$. The Hurwitz matrix corresponding to the polynomial q is the array

$$H(q) = \begin{pmatrix} 8 & 1 & 0 \\ 9 & 2 & 8 \\ 0 & 0 & 9 \end{pmatrix}$$

First let's look at that all the coefficients of the polynomial are positive , then we can use the 2) or 3) of the conditions of Lienard-Chipart: as $\Delta_2 = 7 > 0$ then $q(t)$ is a Hurwitz polynomial.

Problem 3. Does Lienard-Chipart's criterion have the lowest number of principal minors needed or could be improved?

Remark 3. The answer to this question could offer a computational advantage.

5 Phase theorem

The following is the theorem of the phase, also known as the theorem of Mihailov . The Mikhailov criterion gives a necessary and sufficient condition for the asymptotic stability of a linear differential equation of order n.

Theorem 5 (Mihailov criterion). The real polynomial $p(t) = a_0t^n + a_1t^{n-1} + ... + a_n$ is Hurwitz if and only if the argument of $p(i\omega)$, $\arg(p(i\omega))$, is a function of ω and strictly increasing continuous on $(-\infty, \infty)$. In addition, the net increase of the argument of $-\infty$ to ∞ is $n\pi$, i.e.

$$\arg[p(i\infty)] - \arg[p(-i\infty)] = n\pi \tag{1}$$

Proof. By the fundamental algebra theorem we can write

$$p(t) = a_n(t - a_1 - ib_1)(t - a_2 - ib_2)\cdots(t - a_nib_n)$$

then

$$p(i\omega) = a_n[-a_1 + i(w - b_1)]\cdots[-a_n + i(w - b_n)]$$

and

$$\begin{aligned}\arg p(i\omega) &= \arg(a_n) + \arg[-a_1 + i(\omega - b_1)] + \\ &\quad \cdots + \arg[-a_n + i(\omega - b_n)] \\ &= \arg(a_n) + \arctan\left(\frac{\omega - b_1}{-a_1}\right) + \\ &\quad \cdots + \arctan\left(\frac{\omega - b_n}{-a_n}\right)\end{aligned}$$

It is easy to verify that $\arg p(i\omega)$ is an increasing function in ω. On the other hand

$$\lim_{\omega\to+\infty} \arg p(i\omega) = \arg(a_n) + \frac{n\pi}{2}$$

and

$$\lim_{\omega\to-\infty} \arg p(i\omega) = \arg(a_n) - \frac{n\pi}{2}$$

therefore

$$\arg[p(+i\infty)] - \arg[p(-i\infty)] = n\pi$$

The proof of the necessary condition can be consulted in [Battacharayya *et al.* 1995].

Problem 4. What will be the roots of $P(t)$ if the curve $p(i\omega)$ not intersects and what is the meaning into applications?

Remark 4. It would be interesting the meaning of this property in electrical circuits or mechanical systems where the Hurwitz polynomials appear.

Problem 5. Do a description of the analytical functions $f(t)$ that satify $\arg(f(i\omega))$ are strictly increasing functions?

Remark 5. This is a problem of mathematical interest.

6 Hermite-Biehler's Theorem

The theorem of Hermite-Biehler is one of the most useful criteria to determine the stability of a real polynomial. Spell it out for the following definitions are necessary, for more details see [Gantmacher, 1959]. Consider the real polynomial

$$p(x) = a_0 + a_1x + a_2x^2 \cdots + a_nx^n$$

We can write p the following way

$$p(x) = (a_0 + a_2x^2 + \cdots) + x(a_1 + a_3x + \cdots) \quad (2)$$

Evaluanting in $i\omega$, we have

$$p(i\omega) = (a_0 - a_2\omega + \cdots) + i\omega(a_1 - a_3\omega^2 + \cdots) \quad (3)$$

We define

$$p^{even}(x) = a_0 + a_2x^2 + a_4x^4 + \cdots \quad (4)$$
$$p^{odd}(x) = a_1 + a_3x^2 + a_5x^4 + \cdots \quad (5)$$
$$p^{e}(\omega) = a_0 - a_2\omega^2 + a_4\omega^4 - \cdots \quad (6)$$
$$p^{o}(\omega) = a_1 - a_3\omega^2 + a_5\omega^4 - \cdots \quad (7)$$

A pair of polynomials u, v are said to be a couple positive coefficients, if the principal of u and v have the same sign and the roots μ_i u and ν_i of v are various, real and negative and satisfies any of the following two properties of interlaced:

$$\nu_m < \mu_m < \nu_{m-1} < \cdots < \nu_1 < \mu_1 < 0, \ (m = k) \quad (8)$$

$$\mu_m < \nu_{m-1} < \mu_{m-1} \cdots < \nu_1 < \mu_1 < 0, \ (m = k+1) \quad (9)$$

where $m = \deg(u)$ y $k = \deg(v)$. Any polynomial that satisfies one of these partnerships has only real zeros. Note that the polynomials given in (6) and (7) form a couple positive and that the expression (2) can be written and the way

$$p(x) = f(x^2) + xg(x^2)$$

where f and g are like (4) and (5).

Theorem 6 (Hermite-Biehler). A polynomial $p(x) = f(x^2) + xg(x^2)$ with real coefficients is Hurwitz if and only if f and g are a couple positive.

Hermite-Biehler's theorem can be also found in [Battacharayya *et al.* 1995].

7 Stability Test

From a computational viewpoint it is interesting to have a method that let us to verify the stability of a n-degree polynomial by means the stability of a $n-1$ degree polynomial. In this section we expose a such method.

Definition. Given the polynomial $P(t) = a_nt^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0$, if $a_{n-1} \neq 0$, we define

$$Q(t) = a_{n-1}t^{n-1} + \left(a_{n-2} - \frac{a_n}{a_{n-1}}a_{n-3}\right)t^{n-2} + a_{n-3}t^{n-3} + \left(a_{n-4} - \frac{a_n}{a_{n-1}}a_{n-5}\right)t^{n-4} + \cdots \quad (10)$$

Theorem 7. If $P(t)$ has all of its positive coefficients, then $P(t)$ is Hurwitz if and only if $Q(t)$ is Hurwitz.

The previous theorem shows how to check if a polynomial P(t) is Hurwitz through successive reduction of your grade. This result allows us to provide an algorithm to check if a polynomial is Hurwitz or not.

Algorithm.

1) Do $P^{(0)}(t) = P(t)$.
2) Verify that all coefficients of $P^{(i)}(t)$ are positive.

3) Build $P^{(i+1)}(t) = Q(t)$ using equation 1
4) Return to (2). If the polynomial does not satisfy (2) stop the process and then $P(t)$ is not Hurwitz. In another case to continue the process until reach $P^{(n-2)}(t)$ which is grade 2 and then $P(t)$ is Hurwitz.

Example 6. We verify if the polynomial

$$q(t) = t^5 + 5t^4 + 10t^3 + 10t^2 + 5t + 1$$

is Hurwitz. We take

$$P^{(0)}(t) = t^5 + 5t^4 + 10t^3 + 10t^2 + 5t + 1$$

then we build $P^{(1)}(t)$:

$$\begin{aligned} P^{(1)}(t) &= 5t^4 + \left(10 - \frac{1}{5}10\right)t^3 + 10t^2 \\ &\quad + \left(5 - \frac{1}{5}1\right)t + 1 \\ &= 5t^4 + 8t^3 + 10t^2 + \frac{24}{5}t + 1 \end{aligned}$$

Observe that the coefficients of $P^{(1)}(t)$ are positive, so the step 2) is hold, then we build $P^{(2)}(t)$:

$$\begin{aligned} P^{(2)}(t) &= 8t^3 + \left(10 - \frac{5}{8}\left(\frac{24}{5}\right)\right)t^2 + \frac{24}{5}t \\ &\quad + \left(1 - \frac{5}{8}0\right) \\ &= 8t^3 + 7t^2 + \frac{24}{5}t + 1 \end{aligned}$$

We see that $P^{(2)}(t)$ satisfies the step 2) of the algorithm, then following we build $P^{(3)}(t)$

$$\begin{aligned} P^{(3)}(t) &= 7t^2 + \left(\frac{24}{5} - \frac{8}{7}(1)\right)t + 1 \\ &= 7t^2 + \frac{128}{75}t + 1 \end{aligned}$$

As $P^{(3)}(x)$ is of degree 2 and all its coefficients are positive then, by the step 4) of the algorithm, we can conclude that $q(x)$ is Hurwitz.

Problem 6. What of the mentioned criteria will be the most efficient from a computational point of view?

Remark 6. Obviously this problem has a computational interest.

Problem 7. Describe the functions that can be approximate by Hurwitz polynomials.

Remark 7. This problem is in the domain of the Mathematical Analysis, but it could be researched if can be used in applications. It is important to mention that the Routh-Hurwitz criterion and the stability test are equivalent criteria.

8 Importance of the Hurwitz Polynomials in Control Theory

We consider the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} u$$

We look for a control $u = -kc^T x$ with $k \geq 0$ and $c \in \mathbb{R}^n$, then the controlled system is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_n - kc_n & -a_{n-1} - kc_{n-1} & \cdots & -a_1 - kc_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

The characteristic polynomial is

$$p(t) = t^n + a_1 t^{n-1} + \cdots + a_n + k(c_1 t^{n-1} + \cdots + c_n)$$

Then, to obtain a stabilizing control for all $k \geq 0$ we need that $p(t)$ is Hurwitz for all $k \geq 0$ and then we could use the theorems explained in this paper.

Final remark. Some of the open problems have a practical interest, but most of them are theoretical. We wonder if all open problems have utility in applications, but this is also a topic of future research.

9 Conclusions

In this paper we presented some criteria for deciding if a polynomial is a Hurwitz polynomial and we showed some open problems about these polynomials, in order to motivate the researcher continue with the study of them.

Acknowledgments. The second author would like to thank to CONACyT for support through postdoctoral scholarship program.

References

Aguirre-Hernández, B., Frías-Armenta, M.E., and Verduzco, F. (2009). Smooth trivial vector bundle structure of the space of Hurwitz polynomials. *Automatica*, **45**, pp. 2864–2868.

Aguirre-Hernández, B., Cisneros-Molina, J.L., and Frías-Armenta M.E. (2012). Polynomials in control theory parametrized by their roots. *International Journal of Mathematics and Mathematical Sciences*. vol. 2012. Article ID 595076, 19 pages. doi: 1155/2012/595076.

Aguirre-Hernández, B., Frías-Armenta, M.E and Verduzco, F. (2012). On differential structures of polynomial space in control theory. *Journal of Systems Science and Systems Engineering*, **21**(3), pp. 372–382. doi:10.1007/511518-012-5197-y.

Aguirre-Hernández, B., Loredo-Villalobos, C.A., and García-Sosa, F.R. (2013). Open problems on Hurwitz polynomials. *Proceed. Physcon 2013*.

Barnett, S., and Cameron, R. (1985) *Introduction to Mathematical Control Theory* Clarendon Press, Oxford.

Bhattacharayya, S.P., Chapellat, H., and Keel, L.H. (1995). *Robust Control. The Parametric Approach* Prentice-Hall, Upper Saddle River, NJ.

Fisk, S. (2008). *Polynomials, roots and interlacing*. E-book, http://arxiv.org/abs/math/0612833v2.

Gantmacher, F.R. (1959). *The Theory of Matrices.* Chelsea Publ. Co., New York.

Garloff, J. and Wagner, D. (1996). Hadamard product of stable polynomials are stable. *Journal of Math. Analysis and Applications*, **202**, pp. 797–809.

Hermite, C. (1856). Sur le nombre des racines dune èquation algèbrique comprise entre des limites donnés. *J. Reine Angew. Math.*, **53**, pp. 39–51.

Hurwitz, A. (1895). Uber die bedingungen, unter welchen eine gleichung nur wurzlen mit negativen reellen teilen besitzt. *Math. Ann.*, **46**, pp. 273–284.

Lancaster, P., and Tismenetsky M. (1985). *The Theory of Matrices with applications, second edition*, Academic Press, Orlando.

Liènard A., and Chipart A. (1914). Sur le signe de la partie rèelle des racines d'une èquation algèbraique, *J. Math. Pures Appl.* ***(6),*** Vol. 10, pp. 291-346.

Loredo-Villalobos, C.A. (2012). *Factorización de Hadamard para polinomios Hurwitz.* Ph.D. Thesis. Universidad Autónoma Metropolitana, Unidad Iztapalapa.

Loredo-Villalobos, C.A., and & Aguirre-Hernández, B. (2011). Necessary conditions for Hadamard factorizations of Hurwitz polynomials. *Automatica*, **47**(7), pp.1409-1413.

Loredo-Villalobos, C.A., and & Aguirre-Hernández, B. (2012). Hadamard factorization of stable polynomials. *Advances in Mathematical and Computational Methods. I. Kotsireas, R. Melnik and B. West, Eds. AIP Procced.* **1368**, pp. 251–254.

Maxwell, J. C. (1868). On governors. *Proc. of the Royal Society.*

Mikhailov, A.V. (1938). *Avtomat. i Telemekh*, **3**, pp. 27–81.

Rahman, Q.J., and Schmeiser, G. (2002). *Analytic theory of polynomials*. Clarendon Press.

Routh, E. J. (1877). *Treatise on the Stability of a Given State of Motion.* McMillan and Co., London.

Uspensky, J. V. (1990). *Teoría de Ecuaciones.* Limusa. México, D. F.

Zabczyk, J. (1992). *Mathematical Control Theory: An Introduction*. Birkhäuser, Boston.