STATE ESTIMATION FOR A CLASS OF NONLINEAR DYNAMIC SYSTEMS WITH UNCERTAINTY THROUGH DYNAMIC PROGRAMMING TECHNIQUE

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Abstract
This paper deals with the comparison principle for the first-order ODEs of the Hamilton - Jacobi - Bellman type which describe solutions of the dynamical control system with a special structure, in which the nonlinear terms in the right-hand sides of related differential equations are quadratic in state coordinates. We construct external ellipsoidal estimates of reachable sets of the system assuming that initial system states are unknown but bounded. We apply the techniques of generalized solutions of Hamilton - Jacobi - Bellman inequalities to find the external set-valued estimates of reachable sets as the level sets of a related cost functional.

Key words
Control systems, ellipsoidal estimates, dynamic programming.

1 Introduction

In this paper we study control systems with unknown but bounded uncertainties related to the case of a set-membership description of uncertainty [Bertsekas and Rhodes, 1971; Krasovskii and Subbotin, 1974; Kurzhanski, 1977; Milanese, Norton, Piet-Lahanier and Walter, 1996; Milanese and Vicino, 1991; Schweppe, 1973; Witsenhausen, 1968]. The motivation to consider a set-membership approach is that in traditional formulations the characterization of parameter uncertainties requires assumptions on mean, variances or probability density function of errors. However in many applied areas ranged from engineering problems in physics to economics as well as to biological and ecological modeling it occurs that a stochastic nature of the error sequence is questionable. For instance, in case of limited data or after some nonlinear transformation of the data, the presumed stochastic characterization is not always valid. Hence, as an alternative to a stochastic characterization a so-called bounded-error characterization, also called set-membership approach, has been proposed and intensively developed in the last decades.

The solution of many control and estimation problems under uncertainty involves constructing reachable sets and their analogs. For models with linear dynamics under such set-membership uncertainty there are several constructive approaches which allow finding effective estimates of reachable sets. We note here two of the most developed approaches to research in this area. The first one is based on ellipsoidal calculus [Chernousko, 1994; Kurzhanski and Valyi, 1997; Kurzhanski and Varaiya, 2000; Polyak, Nazin, Durieu and Walter, 2004; Chernousko and Ovseevich, 2004] and the second one uses the interval analysis [Milanese, Norton, Piet-Lahanier and Walter, 1996; Kostousova and Kurzhanski, 1996; Walter and Pronzato, 1997].

However, in many applied problems including physical, ecological or economical applications the models are mostly nonlinear in their parameters (e.g., [Apreutesei, 2009; August, Lu and Koeppl, 2012; Ceccarelli, Di Marco, Garulli and Giannitrapani, 2004; Gusev, 2011]). Then, the set of feasible system states is usually non-convex or even non-connected. Nevertheless, set-membership approaches are able to give guaranteed inner or outer approximations for certain types of nonlinear models. Hence, the key issue in nonlinear set-membership estimation is to find suitable techniques, which are easy to interpret and which produce related bounds for the set of unknown system states without being too computationally demanding. Some approaches to the nonlinear set-membership estimation problems and discrete approximation techniques for differential inclusions through a set-valued analogy of well-known Euler's method were developed in [Dontchev and Lempio, 1992; Panasyuk, 1990; Veliov, 1992; Wolenski, 1990; Baier, Büskens, Chahma and Gerdts, 2007; Chahma, 2003; Häckl, 1996]. Funnel equations for differential inclusions with state constraints were studied in [Kurzhanski and Filippova, 1993], the analogies of funnel equations for impulsive control systems were given in [Filippova, 2004; Filippova, 2005].

In this paper the modified state estimation approaches which use the special quadratic structure of nonlinearity of studied control system and use also the advantages of ellipsoidal calculus [Kurzhanski, 1977; Kurzhanski and Valyi, 1997; Chernousko, 1994] are presented. We develop here techniques related to constructing external set-valued estimates of reachable sets of nonlinear control systems and based on results of the theory of generalized solutions of Hamilton - Jacobi - Bellman equations and inequalities [Bertsekas, 1995; Bertsekas and Rhodes, 1971; Kurzhanski, 2006; Kurzhanski and Varaiya, 2006] and on the comparison method for vector Lyapunov functions [Gurman, 1997; Gusev, 2011].

The paper is organized as follows. After introducing some notations and standard definitions in the next Section 2, the main problem is formulated is Section 3. Ellipsoidal external estimates are developed further in Section 3.1 where differential equations describing parameters of estimating ellipsoids are presented. Section 3.2 contains a numerical example illustrating the theory. The approaches related to estimates of reachable sets in nonlinear case and based on results of the theory of generalized solutions of Hamilton - Jacobi - Bellman inequalities are discussed in Section 3.3. Finally, some concluding remarks are given.

2 Preliminaries

In this section we introduce some notations, standard definitions and necessary techniques related to considered problems.

2.1 Notation and Definitions

We start with the following basic notations. Let R^n be the n–dimensional Euclidean space and $(x, y) = x'y$ be the usual inner product of $x, y \in R^n$ with the prime as a transpose, with $\| x \| = (x'x)^{1/2}$.

Denote $\mathrm{comp}R^n$ to be the variety of all compact subsets $A \subseteq R^n$ and $\mathrm{conv}R^n$ to be the variety of all compact convex subsets $A \subseteq R^n$. We denote as $B(a, r)$ the ball in R^n, $B(a, r) = \{x \in R^n : \| x - a \| \leq r\}$, I is the identity $n \times n$–matrix.

Denote by $E(a, Q)$ the ellipsoid in R^n, $E(a, Q) = \{x \in R^n : (x - a)'Q^{-1}(x - a) \leq 1\}$ with a center $a \in R^n$ and a symmetric positive definite $n \times n$–matrix Q, for any $n \times n$–matrix $M = \{m_{ij}\}$ denote

$$Tr(M) = \sum_{i=1}^{i=n} m_{ii}.$$

Consider the control system described by the ordinary differential equation

$$\dot{x} = f(t, x, u(t)), \ \ t \in [t_0, T] \qquad (1)$$

with function $f : [t_0, T] \times R^n \times R^m \to R^n$ measurable in t and continuous in other variables. Here x stands for the state vector, t stands for time and control $u(\cdot)$ is a measurable function satisfying the constraints

$$u(\cdot) \in U = \{u(\cdot) : u(t) \in U_0, \ \ t \in [t_0, T]\} \qquad (2)$$

where $U_0 \in \mathrm{comp}R^m$.

Let us assume that the initial condition $x(t_0)$ to the system (1) is unknown but bounded

$$x(t_0) = x_0, \ \ x_0 \in X_0 \in \mathrm{comp}R^n. \qquad (3)$$

Let absolutely continuous function

$$x(t) = x(t, u(\cdot), t_0, x_0)$$

be a solution to (1) with initial state x_0 satisfying (3) and with control function $u(t)$ satisfying (2). The differential system (1)–(3) is studied here in the framework of the theory of uncertain dynamical systems (differential inclusions [Aubin and Frankowska, 1990; Deimling,1992; Filippov, 1988]) through the techniques of trajectory tubes [Kurzhanski and Filippova, 1993]:

$$X(\cdot) = X(\cdot;\ t_0, X_0) = \bigcup \{ \ x(\cdot) = x(\cdot, u(\cdot), t_0, x_0) \mid \ x_0 \in X_0, \ u(\cdot) \in U \ \}. \qquad (4)$$

2.2 Dynamic Programming Approach

Let us mention here some important results [Bertsekas, 1995; Bertsekas and Rhodes, 1971; Kurzhanski, 2006; Kurzhanski and Varaiya, 2006; Gusev, 2011] from the optimal control theory.

Consider the control system (1)–(3) and assume in this Section that the function $f(t, x, u)$ in (1) is continuous in all variables and has continuous partial derivatives with respect to x. We also suppose that conditions providing the extendability of solutions to (1)–(3) on the interval $[t_0, T]$ are satisfied.

We denote by $X(t) = X(t;\ t_0, X_0)$ the reachable set of the system (1)–(3) at time t. It is known that the reachable set may be expressed as a level set of a value function for an auxiliary control problem [Kurzhanski, 2006; Kurzhanski and Varaiya, 2006; Gusev, 2011]. The value function for this auxiliary problem is the solution to the following Hamilton - Jacobi - Bellman (HJB) equation

$$V_t(t, x) + \max_{u \in U} \left(V_x, f(t, x, u)\right) = 0. \qquad (5)$$

In the common situation the value function may be not differentiable, in this case a solution to the HJB equation is treated as viscosity or minmax solution [Lions, 1982]. The precise solutions to such equations of the HJB type are rather difficult to calculate. The use of corresponding variational HJB inequalities and related comparison theorems instead makes it possible to obtain approximate estimates of reachable sets (e.g., [Kurzhanski, 2006; Kurzhanski and Varaiya, 2006; Gusev, 2011; Gusev, 2007]).

We will need the following auxiliary result.

Lemma 1 ([Kurzhanski, 2006]). *Assume that there exists a function $\mu(t)$ integrable on $[t_0, T]$ and such that*

$$V_t(t,x) + \max_{u \in U} (V_x, f(t,x,u)) \leq \mu(t). \quad (6)$$

Then the following external estimate of the reachable set $X(t)$ of the system (1)–(3) is true

$$X(t) \subseteq \{ \; x : V(t,x) \; \leq \; \int_{t_0}^{t} \mu(s)ds \; + \max_{x \in X_0} \; V(t_0, x) \; \}, \;\; t_0 \leq t \leq T. \quad (7)$$

Note that, without loss of generality, we may take $\mu(s) = 0$ in (6) [Gusev, 2007].

Instead of (6), we may consider the following inequality of a more general type

$$V_t(t,x) + \max_{u \in U} (V_x, f(t,x,u)) \leq g(t, V(t,x)) \quad (8)$$

where $g(t, V)$ is integrable in $t \in [t_0, T]$ and is continuously differentiable in V.

Consider the following ordinary differential equation

$$\dot{U}(t) = g(t,U), \;\; U(t_0) = U_0, \quad (9)$$

which is called a comparison equation for (1)–(3).

Theorem 1 ([Gurman, 1997; Gusev, 2011]). *Assume that the relations (8) and (9) are fulfilled. Assume also that*

$$\max_{x \in X_0} \; V(t_0, x) \; \leq \; U_0. \quad (10)$$

Then the following upper estimate is valid

$$X(t) \subseteq \{x : V(t,x) \leq U(t)\}, \;\; t_0 \leq t \leq T. \quad (11)$$

3 Problem Statement and Results

One of the main problems of the theory of uncertain systems consists in describing and estimating the trajectory tube $X(\cdot)$ of the nonlinear system (1)- (3). The point of special interest is to find the t – cross-section $X(t)$ of $X(\cdot)$ which is actually the attainability domain (reachable set) of the control system (1)–(3) at the instant t.

It should be noted that the exact description of reachable sets $X(t)$ of a control system is a very difficult problem even in the case of linear dynamics. The estimation theory and related algorithms basing on ideas of construction outer and inner set-valued estimates of reachable sets have been developed in [Kurzhanski and Valyi, 1997; Chernousko, 1994; Kurzhanski and Varaiya, 2000] for linear control systems.

The main problem of this research is to construct external set-valued estimates of reachable sets $X(t)$ for a special class of nonlinear systems (1)–(3). The approach presented here uses the techniques of ellipsoidal calculus together with the techniques of HJB equations for such nonlinear control systems with uncertainty in initial states.

3.1 Ellipsoidal Estimates

In [Filippova, 2009; Filippova, 2010; Filippova, 2012] we presented techniques of constructing the external and internal ellipsoidal estimates of trajectory tubes $X(\cdot, t_0, X_0)$ based on the combination of ellipsoidal calculus [Chernousko, 1994; Kurzhanski and Valyi, 1997] and the techniques of evolution funnel equations [Panasyuk, 1990; Veliov, 1992; Wolenski, 1990; Kurzhanski and Filippova, 1993]. We need to mention here some results related only to the case of upper (external) ellipsoidal estimates of reachable sets.

Considered a nonlinear control system in the form of related differential inclusion [Filippov, 1988] of the following type

$$\dot{x} \in Ax + \tilde{f}(x)d + P(t), \; x_0 \in X_0, \; t_0 \leq t \leq T, \; (12)$$

where $x \in R^n$, $\|x\| \leq K$, $X_0 = E(a_0, Q_0)$, $P(t) = E(\hat{a}, \hat{Q})$, $d, a_0, \hat{a}$ are given n-vectors, a scalar function $\tilde{f}(x)$ has a form $\tilde{f}(x) = x'Bx$, matrices B, Q_0 and $\hat{Q}$ are symmetric and positive definite.

Let k_0^+ be such that the following inclusion holds true

$$E(a_0, Q_0) \subseteq E(a_0, (k_0^+)^2 B^{-1}). \quad (13)$$

We assume that k_0^+ is minimal for which the inclusion (13) is true.

Theorem 2 ([Filippova, 2012]). *The inclusion is true for any $t \in [t_0, T]$*

$$X(t; t_0, X_0) \subseteq E(a^+(t), r^+(t)B^{-1}), \quad (14)$$

where functions $a^+(t)$, $r^+(t)$ are the solutions of the

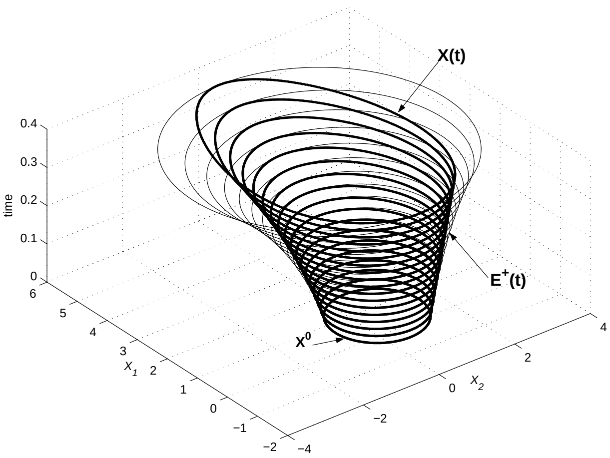

Figure 1. External ellipsoidal estimating tube $E^+(t)$ for $X(t)$.

following system of ordinary differential equations

$$\begin{gathered}\dot{a}^+(t) = Aa^+(t) + ((a^+(t))'Ba^+(t) + \\ r^+(t))d + \hat{a},\ t_0 \le t \le T, \\ \dot{r}^+(t) = \max_{\|l\|=1}\{l^{'}(2r^+(t)(B^{1/2}AB^{-1/2} + \\ 2B^{1/2}d\,(a^+(t))'B^{1/2} + q^{-1}(r^+(t))\times \\ B^{1/2}\hat{Q}B^{1/2})l\} + q(r^+(t))r^+(t), \\ q(r) = ((nr)^{-1}Tr(B\hat{Q}))^{1/2},\end{gathered} \quad (15)$$

with initial condition

$$a^+(t_0) = a_0,\ \ r^+(t_0) = (k_0^+)^2. \quad (16)$$

3.2 Example

Consider the following nonlinear control system in R^2:

$$\begin{aligned}\dot{x}_1 &= 2x_1 + u_1, \\ \dot{x}_2 &= 2x_2 + x_1^2 + x_2^2 + u_2, \\ & x_0 \in X_0,\ 0 \le t \le T.\end{aligned} \quad (17)$$

Here we take $t_0 = 0$, $T = 0.4$, $X_0 = B(0,1)$, $P(t) = B(0,r)$, $r = 0.01$. In this case we have $A = 2I$, $B = I$, $d_1 = 0$, $d_2 = 1$.

The trajectory tube $X(t)$ and its external ellipsoidal tube $E^+(t) = E(a^+(t), Q^+(t))$ found by Theorem 2 are shown as $3D$–graphs in Fig. 1. We see in Fig. 1 that the reachable set $X(t)$ lies inside the ellipsoidal estimate $E^+(t)$ and touches it at some points so that $E^+(t)$ really produces the upper bound for $X(t)$ which is enough accurate in some sense.

3.3 External Estimates by the HJB Inequalities

The solution of problems of state estimation and control synthesis for systems described by ODEs with unknown but bounded disturbances may be reduced to the investigation of first order PDEs of the Hamilton-Jacobi-Bellman (HJB) type and their modifications.

Consider the following HJB inequality

$$V_t(t,x) + \max_{u \in E(\hat{a},\hat{Q})} (V_x, Ax + \tilde{f}(x)d + u) \le 0 \quad (18)$$

with boundary condition

$$V(t_0, x) = \phi(x) \le 0 \quad (19)$$

where $\phi(x)$ is a given continuously differentiable function.

Theorem 3. *Let*

$$\begin{gathered}V(t,x) = \\ (x - a^+(t))'(r^+(t))^{-1}B(x - a^+(t)) - 1\end{gathered} \quad (20)$$

with $a^+(t)$ and $r^+(t)$ defined in (15)-(16). Then $V(t,x)$ satisfies the HJB inequality (18) and the following boundary condition of type (19) is valid

$$V(t_0,x) = (x - a_0)'(k_0^+)^{-2}B(x - a_0) - 1 \le 0 \quad (21)$$

where the number k_0^+ and the matrix B are defined in (13). Moreover, we have also the inclusion

$$X(t) \subseteq \{x : V(t,x) \le 0\},\ \ t_0 \le t \le T. \quad (22)$$

Proof. The proof of this theorem follows directly from Theorem 2 and Theorem 1.

Remark. We observe that Theorem 3 allows us to find the solution of HJB inequality (18)-(19) explicitly. It follows from the special form of the chosen initial function $V(t_0, x)$ (21) and a special type of studied control system (12). In more general cases the use of appropriate approximations gives us the way to establish a similar connection between the techniques of ellipsoidal calculus for dynamic control systems with uncertainties and results based on comparison theorems of theory of Hamilton-Jacobi-Bellman equations and inequalities.

4 Conclusion

Basing on results of ellipsoidal calculus developed for uncertain systems we presented the modified state estimation approach which uses the special nonlinear structure of the control system and is based on related HJB inequalities. We provide comparison results for HJB equations generated by the guaranteed state estimation problem in the case of nonlinear control systems with quadratic nonlinearities. This approach may lead to effective external approximations of the reachable sets of nonlinear control systems using ellipsoidal techniques.

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References

Apreutesei, N. C. (2009). An optimal control problem for a prey-predator system with a general functional response. *Appl. Math. Lett.*, **22**(7), pp. 1062-1065.

Aubin, J. P. and Frankowska, H. (1990). *Set-Valued Analysis*. Birkhauser. Boston.

Baier, R., Büskens, Ch., Chahma, I. A. and Gerdts, M. (2007). Approximation of reachable sets by direct solution methods of optimal control problems. *Optimization Methods and Software*, **22** (3), pp. 433-452.

August, E., Lu, J. and Koeppl, H. (2012). Trajectory Enclosures for Nonlinear Systems with Uncertain Initial Conditions and Parameters. In: *Proc. 2012 American Control Conference*. Fairmont Queen Elizabeth, Montréal, Canada, June 27-29. pp. 1488-1493.

Bertsekas, D. P. (1995). *Dynamic Programming and Optimal Control*. Athena Scientific, v. I, II. Belmont, MA.

Bertsekas, D. P. and Rhodes, I. B. (1971). Recursive state estimation for a set-membership description of uncertainty. *IEEE Transactions on Automatic Control*, **16**, pp. 117-128.

Ceccarelli, N., Di Marco, M., Garulli, A. and Giannitrapani, A. (2004). A set theoretic approach to path planning for mobile robots. In: *Proc. 43rd IEEE Conference on Decision and Control*. Atlantis, Bahamas, December 14-17. pp. 147-152.

Chahma, I. A. (2003). Set-valued discrete approximation of state-constrained differential inclusions. *Bayreuth. Math. Schr.*, **67**, pp. 3–162.

Chernousko, F. L. (1994). *State Estimation for Dynamic Systems*. CRC Press. Boca Raton.

Chernousko, F. L. and Ovseevich, A. I. (2004). Properties of the optimal ellipsoids approximating the reachable sets of uncertain systems. *J. Optimization Theory Appl.*, **120**(2), pp. 223–246.

Deimling, K. (1992). *Multivalued Differential Equations*. De Gruyter. Berlin.

Dontchev, A. L. and Lempio, F. (1992). Difference methods for differential inclusions: a survey. *SIAM Review*, **34**, pp. 263–294.

Filippov, A. F. (1988). *Differential Equations with Discontinuous Right-Hand Sides*. Kluwer Academic Publishers. Dordrecht.

Filippova, T. F. (2004). Sensitivity problems for impulsive differential inclusions. In: *Proc. 6th WSEAS International Conference on Applied Mathematics*. Corfu Island, Greece, August 17-19. pp. 1–6.

Filippova, T. F. (2005). Set-valued solutions to impulsive differential inclusions. *Mathematical and Computer Modelling of Dynamical Systems*, **11** (2), pp. 149–158.

Filippova, T. F. (2009). Estimates of trajectory tubes in control problems under uncertainty. In: *Proc. 4th International Conf. "Physics and Control - 2009", PhysCon2009*. Italy, Catania, Sept. 1-4. pp. 1–6.

Filippova, T. F. (2010). Trajectory tubes of nonlinear differential inclusions and state estimation problems. *J. of Concrete and Applicable Mathematics, Eudoxus Press, LLC*, **8**, pp. 454–469.

Filippova, T. F. (2012). Set-valued dynamics in problems of mathematical theory of control processes. *International Journal of Modern Physics B (IJMPB)*, **26** (25), 2012, pp. 1–8.

Gurman, V. I. (1997). *The Extension Principle in Problems of Control*. Fizmatlit. Moscow.

Gusev, M. I. (2011). On estimates of reachable sets of multidimensional control systems. In: *Preprints of the 18th IFAC World Congress*. Milano, Italy, August 28 - September 2. pp. 7842–7847.

Gusev, M. I. and Kurzhanski, A. B. (2007). On the Hamiltonian techniques for designing nonlinear observers under set-membership uncertainty. In: *Preprints of the 7-th IFAC Symposium on Nonlinear Control Systems*. Pretoria, South Africa, August 21-24. pp. 343–348.

Häckl, G. (1996). *Reachable sets, control sets and their computation*. Augsburger Mathematisch-Naturwissenschaftliche Schriften, **7**. PhD Thesis, University of Augsburg, Augsburg.

Kostousova, E. K. and Kurzhanski, A. B. (1996) Theoretical framework and approximation techniques for parallel computation in set-membership state estimation. In: *Proc. Symposium on Modelling Analysis and Simulation*. Lille, France, July 9-12. **2**, pp. 849–854.

Krasovskii, N. N. and Subbotin, A. I. (1974). *Positional Differential Games*. Nauka. Moscow.

Kurzhanski, A. B. (1977). *Control and Observation under Conditions of Uncertainty*. Nauka. Moscow.

Kurzhanski, A. B. (2006). Comparison principle for equations of the Hamilton-Jacobi Type. *Proceedings of the Steklov Institute of Mathematics Suppl. 1*, pp. 185–195.

Kurzhanski, A. B. and Filippova, T. F. (1993). On the theory of trajectory tubes a mathematical formalism for uncertain dynamics, viability and control. In *Advances in Nonlinear Dynamics and Control: a Report from Russia, Progress in Systems and Control Theory, (ed. A.B. Kurzhanski)*, Birkhauser, Boston. **17**. pp. 22–188.

Kurzhanski, A. B. and Valyi, I. (1997). *Ellipsoidal Calculus for Estimation and Control*. Birkhauser. Boston.

Kurzhanski, A. B. and Varaiya, P. (2000). Ellipsoidal techniques for reachability analysis. In: *Lecture Notes in Computer Sciences*, Springer-Verlag, **1790**, pp. 202–214.

Kurzhanski, A. B. and Varaiya, P. (2006). A comparison principle for equations of the Hamilton-Jacobi type in set-membership filtering. Commun. Inf. Syst., **6** (3), pp. 179–192.

Lions, P. L. (1982). *Generalized Solutions of Hamilton-Jacobi Equations*. Research Notes in Mathematics, Vol. 69, Pitman Advanced Publishing Program. Boston, MA.

Milanese, M., Norton, J. P., Piet-Lahanier, H. and Walter, E. (Eds.) (1996). *Bounding Approaches to System Identication*. Plenum Press. New York.

Milanese, M. and Vicino, A. (1991). Estimation theory for nonlinear models and set membership uncertainty. *Automatica*, **27** (2), pp. 403–408.

Panasyuk, A. I. (1990). Equations of attainable set dynamics. Part 1: Integral funnel equations. *J. Optimiz. Theory Appl.*, **64**, pp. 349–366.

Polyak, B. T., Nazin, S. A., Durieu, C. and Walter, E. (2004). Ellipsoidal parameter or state estimation under model uncertainty. *Automatica J., IFAC*, **40**, pp. 1171–1179.

Schweppe, F. (1973). *Uncertain Dynamic Systems*. Prentice-Hall, Englewood Cliffs. New Jersey.

Veliov, V. (1992). Second-order discrete approximation to linear differential inclusions. *SIAM J. Numer. Anal.*, **2**, pp. 439–451.

Walter, E. and Pronzato, L. (1997). *Identification of parametric models from experimental data*. Springer-Verlag, Heidelberg.

Witsenhausen, H. S. (1968). Sets of possible states of linear systems given perturbed observations. *IEEE Transactions on Automatic Control*, **13** (5), pp. 556–558.

Wolenski, P. R. (1990). The exponential formula for the reachable set of a Lipschitz differential inclusion. *SIAM J. Control Optimization*, **28**, pp. 1148–1161.