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SUB-RIEMANNIAN APPROACH FOR THE FOUCAULT PENDULUM

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Abstract
The well known Foucault pendulum is studied within the formalism of sub-Riemannian geometry on step-2 nilpotent Lie groups. It is shown that in a rotating frame a sub-Riemannian structure can be naturally introduced. Some other physical models such as a falling particle on a rotating planet can be treated in a similar form. Horizontal trajectories are explicitly computed and displayed for the symmetric case.

Key words
Foucault pendulum, sub-Riemannian geometry, nilpotent Lie groups.

1 Introduction

In 1851, the french physicist J.B.L. Foucault suspended a 67 meters, 28 kilograms pendulum from the dome of the Pantheon in Paris, and made the observation that the pendulum's oscillation plane rotated slowly clockwise with respect to the Earth. Since then the experiment has been recognized as a feasible demonstration of the rotation movement of Earth, and has been extensively studied.

In this paper we approach this classical problem as a problem in sub-Riemannian geometry, this approach unveils interesting geometric properties for the trajectories and leads in a natural way to some other physical models such as the classical analog of electrons in certain nanostructures, the so called *quantum dots*.

Generally speaking, sub-Riemannian geometry is the geometry of non-holonomic constraints, see for instance [Vershik, 1991] and [Montgomery, 2002]. This viewpoint is based on the fact that in some dynamical systems, the non-holonomic constraints are encoded by means of a Pfaffian system, that is, a linearly independent set of differential 1-forms representing part of the kinematics. If the smooth vector fields that generate the kernel of the Pfaffian system satisfy a generic condition, (the so-called Hörmander condition), then the distribution of vector fields is called non-holonomic and a Riemannian metric, restricted to the distribution, can be defined and naturally associated to the kinetic energy of the system.

Such a formalism has been used for tackling problems in physics and geometric optimal control theory, see for instance [Brockett, 1993] and [Jurdjevic, 1997]. For dynamical systems described in terms of non-holonomic distributions of vector fields, the structure of the spanned Lie algebra, that is, the Lie algebra obtained by Lie bracketing iteratively the vector fields, determines most of the relevant properties of the system. Non-holonomic distributions are in the opposite side of integrable ones, they provide the main object of study of what is known as Carnot-Caratheodory or sub-Riemannian geometries, see for instance [Montgomery, 2002].

On a connected, simply connected n-dimensional smooth manifold M, a rank-m distribution Δ, with $m < n$, consists of a smooth rank-m sub bundle of the tangent bundle TM. The iteration of the Lie bracket of vector fields in Δ yields the following flag of modules of vector fields

$$\Delta^1 \subset \Delta^2 \subset \cdots \subset \Delta^l \cdots \subset \mathrm{T\,M}$$

where $\Delta^1 = \Delta$ and $\Delta^{i+1} = \Delta^i + [\Delta, \Delta^i]$. The distribution is said to be *non-holonomic* or *bracket generating*, if for each $q \in$ M, there exist a positive integer i for which $\Delta^i_q = \mathrm{T}_q$ M. The first i for which this occurs is called the *degree of non-holonomy of* Δ at q. Let $n_j = n_j(q) = \dim(\Delta^j_q)$, the growth vector of Δ at q is defined as $(n_1, \ldots, n_i)_q$, the distribution is said to be *regular* if the growth vector is independent of the base point. An absolutely continuous curve $t \mapsto q(t)$, is said to be *horizontal*, if $\dot{q}(t) \in \Delta(q(t))$, almost everywhere. The Chow-Rashevski's theorem guarantees that for regular non-holonomic distributions, any two points in M, can be connected by an horizontal curve, see for instance [Montgomery, 2002]. For regular non-holonomic distributions, a sub-Riemannian metric is defined by a smooth varying inner product $q \mapsto \langle \cdot, \cdot \rangle_q$ on the hyperplanes Δ_q.

In the case we are interested on, the distribution is given as $\Delta = \{X_1, \ldots, X_m\}$, and horizontal curves are solutions of the following affine in the controls control system

$$\dot{q} = \sum_{i=1}^{m} u_i \, X_i(q), \tag{1}$$

where the control parameters are measurable and represented by state velocities, $u_i = \dot{q}_i$.
The sub-Riemannian structure is given by a smooth varying inner product $\langle \cdot, \cdot \rangle_q$ defined on the planes $\Delta_q = \text{span}\{X_1(q), \ldots, X_m(q)\}$, by declaring the vector fields as orthonormal, that is, $\langle X_i, X_j \rangle = \delta_{ij}$. This structure is naturally associated to the variational problem of the minimization, in the class $\mathcal{H}$ of horizontal curves $q : [0, T_q] \to$ M, of the functional

$$S = \int_0^{T_q} (T \,-\, V) \, dt, \tag{2}$$

were, T is the kinetic energy and V is the potential energy. The geodesic sub-Riemannian problem can then be approached as the optimal control problem with plant (1) and cost (2).

Apart from this introduction this paper contains four sections, in sections 2 and 3, we derive the dynamic equations for the Foucault pendulum and set the problem as a geodesic sub-Riemannian problem in a step-2 nilpotent Lie group. In section 4, we write explicitly the underlying structure for small oscillations. In section 5 we calculate explicitly integral curves and derive some geometric properties. At the end, in section 6 we derive some conclusions and discuss further research perspectives.

2 Foucault pendulum within the Sub-Riemannian framework.

We consider a pendulum of length ℓ and point mass m oscillating on the Earth surface, taking into account only the rotation movement. An inertial coordinate system (X, Y, Z) is considered, in such a way that Earth's rotation axis coincides with the Z direction. Let $\vec{\omega}$ be the angular velocity of the rotational motion, and let (x, y, z) be the position of the mass measured from a fixed coordinate system with origin located at latitude α on Earth's surface measured from equator, the x direction is taken on a meridian in north-south sense, the y direction on a parallel circle in west-east sense, and the z direction perpendicular to the tangent plane at the intersection of both circles, see figure 1. By taking the origin at the suspension point of the pendulum, the direction cosines are given as $\cos\phi_x = \frac{x}{\ell}$, $\cos\phi_y = \frac{y}{\ell}$ and $\cos\phi_z = -\frac{z}{\ell}$.

For a mass m in a gravitational force field $\vec{F}_G = -\vec{\nabla} V_G$, the trajectories are determined by minimizing the functional

$$S_0 = \int \left(\frac{m}{2} \left| \frac{d\vec{r}}{dt} \right|^2 - V_G \right) dt,$$

subject to the constraints. In our case a vector $\vec{r}$ in the non-inertial system on Earth's surface behaves as

$$\frac{d\vec{r}}{dt} = \dot{\vec{r}} + \vec{\omega} \times \vec{r},$$

where $\dot{\vec{r}} = (\dot{x}, \dot{y}, \dot{z})$ and $\vec{\omega} = (-\omega\cos(\alpha), 0, \omega\sin(\alpha))$, with $\omega \approx 2\pi/24\text{hr} \approx 10^{-4}$ sec^{-1}. Observe that this is a very small angular velocity, compared with the usual pendulum speed of the order of tenths of sec^{-1}.
The kinetic energy is given as follows

$$\frac{m}{2} \left| \frac{d\vec{r}}{dt} \right|^2 = \frac{m}{2} (|\dot{\vec{r}}|^2 + |\vec{\omega} \times \vec{r}|^2 + 2\dot{\vec{r}} \cdot (\vec{\omega} \times \vec{r})\,).$$

The second term leads to a centrifugal force perpendicular to the rotation axis, but together with the central gravitational potential, yields a vertical force of magnitude mg, perpendicular to the tangent plane to the planet surface. Thus, we set

$$-V_G + \frac{m}{2} |\vec{\omega} \times \vec{r}|^2 = -mgz.$$

For the third term, we have the identity

$$\begin{aligned} \dot{\vec{r}} \cdot (\vec{\omega} \times \vec{r}) &= \vec{\omega} \cdot (\vec{r} \times \dot{\vec{r}}) \\ &= \omega_x(y\dot{z} - z\dot{y}) + \omega_y(z\dot{x} - x\dot{z}) + \omega_z(x\dot{y} - y\dot{x}). \end{aligned}$$

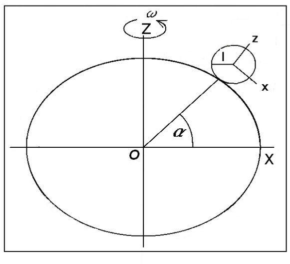

Figure 1. The Pendulum at a latitude α

Finally we add the following holonomic constraint

$$x^2 + y^2 + z^2 - \ell^2 = 0. \tag{3}$$

In consequence the complete functional is written as follows

$$\begin{aligned} S = \int \Big(& (\frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 - 2gz) + \lambda_0 r^2 \\ & + \lambda_1(\dot{\xi} + y\dot{z} - z\dot{y}) + \lambda_2(\dot{\eta} + z\dot{x} - x\dot{z}) \\ & + \lambda_3(\dot{\zeta} + x\dot{y} - y\dot{x}) \Big) \, dt. \end{aligned}$$

where the Lagrange parameter λ_0 has been introduced, set $\lambda_1 = -m\omega\cos\alpha$, $\lambda_2 = 0$ and $\lambda_3 = m\omega\sin\alpha$ and the differentials $\dot{\xi}dt$, $\dot{\eta}dt$ and $\dot{\zeta}dt$. The last differentials do not alter the Euler-Lagrange equations for $\vec{r}$, but are essential for the sub-Riemannian approach. The parameters λ_i can be seen as the Lagrange parameters associated with the following 1-forms

$$\begin{aligned} w_1 &= d\xi + ydz - zdy, \\ w_2 &= d\eta + zdx - xdz, \\ w_3 &= d\zeta + xdy - ydx. \end{aligned}$$

Consider now the six-dimensional manifold M with local coordinates $q = (x, y, z, \xi, \eta, \zeta)$. Associated to the Pfaffian system $\{w_1, w_2, w_3\} \subset \mathrm{T}^*\mathrm{M}$, we have the distribution of smooth vector fields $\Delta = \{X_1, X_2, X_3\} \subset \mathrm{TM}$, where

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x} + y\frac{\partial}{\partial \zeta} - z\frac{\partial}{\partial \eta}, \\ X_2 &= \frac{\partial}{\partial y} + z\frac{\partial}{\partial \xi} - x\frac{\partial}{\partial \zeta}, \\ X_3 &= \frac{\partial}{\partial z} + x\frac{\partial}{\partial \eta} - y\frac{\partial}{\partial \xi}. \end{aligned}$$

The vector fields X_i are dual to the 1-forms w_i and generate a six dimensional step-2 nilpotent Lie algebra $\mathfrak{g}$ with non-zero brackets $[X_i, X_j] = X_{ij}, i < j$, and

$$X_{12} = -2\frac{\partial}{\partial \zeta}, \quad X_{13} = 2\frac{\partial}{\partial \eta}, \quad X_{23} = -2\frac{\partial}{\partial \xi}.$$

Since $\mathrm{T}_q\mathrm{M}$ is six dimensional for all $q \in \mathrm{M}$, the distribution Δ is bracket generating. A sub-Riemannian structure for this problem is given by the pair formed by the distribution Δ and the Euclidean metric given by the kinetic energy, horizontal curves satisfy

$$\dot{q} = \dot{x}X_1(q) + \dot{y}X_2(q) + \dot{z}X_3(q), \tag{4}$$

and horizontal energy minimizers corresponds to extremals of the functional S.

3 Equations of Motion

The Euler-Lagrange equations are the following

$$\begin{aligned} m\ddot{x} + \frac{d}{dt}(\lambda_2 z - \lambda_3 y) &= 2\lambda_0 x - \lambda_2\dot{z} + \lambda_3\dot{y}, \\ m\ddot{y} + \frac{d}{dt}(\lambda_3 x - \lambda_1 z) &= 2\lambda_0 y - \lambda_3\dot{x} + \lambda_1\dot{z}, \\ m\ddot{z} + \frac{d}{dt}(\lambda_1 y - \lambda_2 x) &= 2\lambda_0 z - \lambda_1\dot{y} + \lambda_2\dot{x} - mg, \end{aligned}$$

and $\dot{\lambda}_1 = 0$, $\dot{\lambda}_2 = 0$, $\dot{\lambda}_3 = 0$. The parameters λ_1, λ_2 and λ_3 are therefore constants and the extremals are solutions of the following system

$$m\ddot{x} = 2\lambda_3\dot{y} - 2\lambda_2\dot{z} + 2\lambda_0 x, \tag{5}$$
$$m\ddot{y} = -2\lambda_3\dot{x} + 2\lambda_1\dot{z} + 2\lambda_0 y, \tag{6}$$
$$m\ddot{z} = 2\lambda_2\dot{x} - 2\lambda_1\dot{y} + 2\lambda_0 z - mg, \tag{7}$$

which together with the relations

$$\dot{\xi} = z\dot{y} - y\dot{z}, \tag{8}$$
$$\dot{\eta} = x\dot{z} - z\dot{x}, \tag{9}$$
$$\dot{\zeta} = y\dot{x} - x\dot{y}, \tag{10}$$

yield the equations of motion in $\mathbb{R}^6$.
As we noted above, the coordinates can be selected in such a way that the y component of the angular velocity is zero, i.e. $\lambda_2 = 0$. For a falling particle, $\lambda_0 = 0$, but for the Foucault pendulum a further approximation shall be needed to integrate the equations of motion.
The equations of motion in base space (x, y, z) are gauge invariant. Adding an exact differential $d\phi$ to w leads to the same equations and to the same Lie algebra for the vector fields. Singular trajectories are those curves that satisfy the constraints, as when we solve the equations which result taking $m = 0$ in the functional.

4 Small Oscillations

For small oscillations we take $z = -\ell$ to first order, so that from condition (3), x and y are much smaller, furthermore, equation (7) yields $2\lambda_0 = -mg/\ell$. We neglect finally the terms containing $\dot{z}$ with respect to those having $\dot{x}$ and $\dot{y}$, since the height variation is much smaller than the variation of the position in the horizontal plane.

Since the equations for ξ and η are integrable, we get that the final system is defined on $\mathbb{R}^3$ and is written as follows

$$\ddot{x} = 2\tilde{\omega}\dot{y} - \omega_0^2 x, \quad (11)$$
$$\ddot{y} = -2\tilde{\omega}\dot{x} - \omega_0^2 y, \quad (12)$$
$$\dot{\zeta} = -x\dot{y} + y\dot{x}, \quad (13)$$

with the frequencies $\omega_0 = \sqrt{g/\ell}$ and $\tilde{\omega} = \lambda_3/m$. These equations follow also from the

$$L_0 = \frac{m}{2}(\dot{x}^2+\dot{y}^2) - \frac{mg}{2\ell}(x^2+y^2) + 2\lambda_3(\dot{\zeta}+x\dot{y}-y\dot{x}).$$

From this Lagrangian, and the 1-form associated to equation (13), we can define for this restricted problem a sub-Riemannian structure given by the two dimensional Euclidean metric and the rank two distribution generating the standard Heisenberg algebra.

4.1 The canonical momenta and the Hamiltonian

As customary we consider the canonical momenta

$$p_x = \frac{\partial L_0}{\partial \dot{x}}, \quad p_y = \frac{\partial L_0}{\partial \dot{y}}, \quad p_\zeta = \frac{\partial L_0}{\partial \dot{\zeta}}.$$

It follows that

$$p_x = m\dot{x} - 2\lambda_3 y,$$
$$p_y = m\dot{y} + 2\lambda_3 x,$$
$$p_\zeta = 2\lambda_3$$

and consequently the adjoint system is written as follows

$$\dot{p}_x = -m\omega_0^2 x,$$
$$\dot{p}_y = -m\omega_0^2 y,$$
$$\dot{p}_\zeta = 0.$$

We conclude that $\dot{p}$ is constant along extremals, and furthermore, the first two equations of motion imply

$$\frac{d}{dt}\frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \omega_0^2 x^2 + \omega_0^2 y^2) = 0$$

in consequence, the Hamiltonian is a constant of motion given as follows

$$H = \frac{1}{2m}(p_x+p_\zeta y)^2 + \frac{1}{2m}(p_y-p_z x)^2 + \frac{m}{2}(x^2+y^2)\omega_0^2.$$

4.2 Related problems

The particular case $g/\ell = 0$ coincide with the so-called Heisenberg fly-wheel, see [Montgomery, 2002], but in general this problem is rather a point particle in a two dimensional harmonic oscillator potential with elastic constant equal to mg/ℓ, coupled to a fly-wheel or crank. This problem is also related to a two dimensional analog of a classical (symmetric) *cranking model* for a nucleon in a rotating atomic nucleus. Note that the quantity $m(x^2+y^2)$ in the Hamiltonian has the physical units of a moment of inertia.

A further important physical model is that of an electron, with charge $-e$, in a two dimensional harmonic potential in a perpendicular constant magnetic field B. The electromagnetic vector potential $\vec{A}$ would be given by $eB\,\vec{A}/2c = (\lambda_3 y, -2\lambda_3 x, 0)$, corresponding to the symmetric gauge. The quantum problem is a *quantum dot*, which is a relevant topic in the physics of nanostructures.

5 Horizontal trajectories

In order to integrate the equations of motion, we consider the complex variable $u = x + iy$, which implies $\dot{\zeta} = \text{Im}\,(u\dot{u}^*)$ and

$$\ddot{u} = -i\frac{2\lambda_3}{m}\dot{u} - \frac{g}{\ell}u.$$

By writing $u \propto e^{i\alpha_\pm t}$ we obtain the eigenvalues

$$\alpha_\pm = -\tilde{\omega} \pm \tilde{\omega}_0,$$

with frequencies $\tilde{\omega}_0 = \sqrt{\tilde{\omega}^2 + \omega_0^2}$, $\omega_0 = \sqrt{g/\ell}$ and $\tilde{\omega} = \lambda_3/m$. Here, $\tilde{\omega}$ is equal to the rotation angular speed ω times the sinus of the geographical latitude.

The general solution of the differential equation is the linear combination $u = A_+ e^{i\alpha_+ t} + A_- e^{i\alpha_- t}$, from where we obtain

$$u = e^{-i\tilde{\omega}\,t}(A_+ e^{i\tilde{\omega}_0\,t} + A_- e^{-i\tilde{\omega}_0\,t}),$$
$$\dot{u} = i\,e^{-i\tilde{\omega}\,t}(\alpha_+ A_+ e^{i\tilde{\omega}_0\,t} + \alpha_- A_- e^{-i\tilde{\omega}_0\,t}),$$
$$\zeta = -(\alpha_+|A_+|^2 + \alpha_-|A_-|^2)\,t$$
$$-2\text{Re}\left(A_+ A_-^* \left(e^{2i\,\tilde{\omega}_0\,t} - 1\right)\right).$$

The first two relations yield a rotation given by the slow mode, with frequency $\tilde{\omega}$, of the fast mode motion, with frequency $\tilde{\omega}_0$. Therefore, the trajectory in base space performs a precession with frequency $\omega\sin\alpha$, whereas ζ increases by the same amount after $2\pi/\tilde{\omega}_0$ seconds.

A curve in base space is closed only when the frequencies $\tilde{\omega}$ and $\tilde{\omega}_0$ are congruent modulo 2π, and this occurs only if $\tilde{\omega} = \varpi n$ and $\omega_0 = \varpi m$, where n and m are integer such that the sum of their squares is also the square of an integer, say $n^2 + m^2 = k^2$, so that $\tilde{\omega}_0 = k\varpi$. Set for example $n = a^2 - b^2$, $m = 2ab$ for $a > b$ integers, then $k = a^2 + b^2$. Of course, from the experimental point of view, this may be irrelevant since for the Earth the frequency $\tilde{\omega}$ is much smaller than ω_0. Loops of piecewise continuous horizontal curves allow to define the holonomy group, which in our case is given by translations along the fiber over (x, y). For the just given congruent frequencies, the trajectories close after a time $T = 2\pi/\tilde{\omega}$.

From the equation for u, the conservation of energy is written now as $2H/m = |\dot{u}|^2 + \omega_0^2|u|^2$. We can compute explicitly the products to get

$$\begin{aligned}|u|^2 &= |A_+|^2 + |A_-|^2 + 2\,\mathrm{Re}(A_+A_-^*\,e^{2i\tilde{\omega}_0 t}),\\ |\dot{u}|^2 &= \alpha_+^2|A_+|^2 + \alpha_-^2|A_-|^2 - 2\omega_0^2\,\mathrm{Re}(A_+A_-^*\,e^{2i\tilde{\omega}_0 t}),\\ \frac{2}{m}H &= \alpha_+^2|A_+|^2 + \alpha_-^2|A_-|^2 + \omega_0^2(|A_+|^2 + |A_-|^2).\end{aligned}$$

The initial conditions shall fix the remaining constants for u, and in the trajectories in base space can be written explicitly. In the following three paragraphs we analyze different settings for the pendulum

5.1 Starting from rest with initial conditions on a circle

This corresponds to the original Foucault experimental setting, it consists of an initial position along a circle of radius R, thus $x(0) = R\cos\beta$, $y(0) = R\sin\beta$, for $\beta \in (0, 2\pi)$, and the mass starting from rest, that is, $\dot{x}(0) = 0$ and $\dot{y}(0) = 0$.

We have $A_+ + A_- = R\,e^{i\beta}$ and $A_+\alpha_+ + A_-\alpha_- = 0$, so that,

$$A_\pm = \frac{Re^{i\beta}}{2}\left(1 \pm \frac{\tilde{\omega}}{\tilde{\omega}_0}\right),$$

furthermore,

$$\begin{aligned}\alpha_+^2|A_+|^2 + \alpha_-^2|A_-|^2 &= \frac{R^2\omega_0^4}{2(\tilde{\omega}^2 + \omega_0^2)},\\ A_+A_-^* &= \frac{R^2\omega_0^2}{4(\tilde{\omega}^2 + \omega_0^2)}.\end{aligned}$$

In conclusion the trajectory is given as follows

$$\begin{aligned}x &= R\cos(\tilde{\omega}t - \beta)\cos\tilde{\omega}_0 t + \frac{R\tilde{\omega}}{\tilde{\omega}_0}\sin(\tilde{\omega}t - \beta)\sin\tilde{\omega}_0 t,\\ y &= -R\sin(\tilde{\omega}t - \beta)\cos\tilde{\omega}_0 t + \frac{R\tilde{\omega}}{\tilde{\omega}_0}\cos(\tilde{\omega}t - \beta)\sin\tilde{\omega}_0 t,\\ \zeta &= \frac{R^2\tilde{\omega}\omega_0^2}{\tilde{\omega}_0^2}\left(\frac{t}{2} - \frac{\sin 2\tilde{\omega}_0 t}{4\tilde{\omega}_0}\right).\end{aligned}$$

The expressions for x and y yield a rotation by an angle $\alpha = \tilde{\omega}t - \beta$ of the ellipse given by the vector

$$(R\cos\tilde{\omega}_0\,t, \frac{R\tilde{\omega}}{\tilde{\omega}_0}\sin\tilde{\omega}_0\,t).$$

This vector has initial value $(R, 0)$ and takes the same value at times $t_k = \pi\,k/\tilde{\omega}_0$, for k integer. Since $\tilde{\omega} < \tilde{\omega}_0$, the nearest approach to the origin is at distance $R\tilde{\omega}/\tilde{\omega}_0$. The curves in base space are hypocycloids.

A single to and fro motion occurs after a time $2\pi/\tilde{\omega}_0$ and the difference between the angles α corresponding to these two points is therefore

$$\Delta\alpha = \frac{2\pi\tilde{\omega}}{\tilde{\omega}_0}.$$

This is the phase acquired after a complete oscillation and the horizontal trajectory is lifted above the base space by

$$\Delta\zeta = \frac{R^2\,\pi\,\tilde{\omega}\,\omega_0^2}{\tilde{\omega}_0^3}.$$

The trajectory never reaches the origin and is tangent to a cylinder of radius $R\tilde{\omega}/\tilde{\omega}_0$ at times $\tau_j = (2j + 1)\pi/2\tilde{\omega}_0$, for $j = 0, 1, \ldots$ For general $\tilde{\omega}$ and ω_0 the trajectory in base space never closes, except for certain rational values of their quotients.

5.2 Starting in the origin with non zero velocity

We consider now the case $x(0) = y(0) = 0$, together with $\dot{x}(0) = v_0\cos\beta$ and $\dot{y}(0) = v_0\sin\beta$. In this case we have $u(0) = 0$ and $\dot{u}(0) = v_0e^{i\beta}$ which lead to $A_+ + A_- = 0$ and $A_+\alpha_+ + A_-\alpha_- = -i\,v_0e^{i\beta}$, from where we obtain

$$A_\pm = a_\pm(\sin\beta - i\,\cos\beta), \quad \text{with} \quad a_\pm = \pm\frac{v_0}{2\tilde{\omega}_0},$$

furthermore

$$\begin{aligned}\alpha_+^2|A_+|^2 + \alpha_-^2|A_-|^2 &= \frac{(2\tilde{\omega}^2 + \omega_0^2)v_0^2}{2(\tilde{\omega}^2 + \omega_0^2)},\\ A_+A_-^* &= -\frac{v_0^2}{4(\tilde{\omega}^2 + \omega_0^2)}.\end{aligned}$$

In consequence the trajectory in total space is written as follows

$$
\begin{aligned}
x &= v_0 \cos(\beta - \tilde{\omega} t) \frac{\sin \tilde{\omega}_0 \, t}{\tilde{\omega}_0}, \\
y &= -v_0 \sin(\beta - \tilde{\omega} t) \frac{\sin \tilde{\omega}_0 \, t}{\tilde{\omega}_0}, \\
\zeta &= \frac{v_0^2 \tilde{\omega}}{\tilde{\omega}_0^2} \left(\frac{t}{2} - \frac{\sin 2\tilde{\omega}_0 \, t}{4\tilde{\omega}_0} \right).
\end{aligned}
$$

The trajectories in base space are rhodonea or roulette curves which always close at the origin.

5.3 Starting from an arbitrary point with a given velocity.

Without loss of generality we can assume that $x(0) = x_0$, $y(0) = 0$, and $\dot{x}(0) = v_0 \cos\beta$ and $\dot{y}(0) = v_0 \sin\beta$. In this case we have $u(0) = 1$ and $\dot{u}(0) = v_0 e^{i\beta}$, from where we get

$$
\begin{aligned}
A_+ + A_- &= x_0, \\
A_+ \alpha_+ + A_- \alpha_- &= -i \, v_0 e^{i\beta} \\
2\tilde{\omega}_0 A_\pm &= \mp \alpha_\mp x_0 \mp \, v_0 e^{i\beta + i\pi/2}.
\end{aligned}
$$

We obtain first,

$$
\begin{aligned}
4\tilde{\omega}_0^2 |A_+|^2 &= \alpha_-^2 x_0^2 + v_0^2 + 2\alpha_- x_0 \, v_0 \cos(\beta + \pi/2), \\
4\tilde{\omega}_0^2 |A_-|^2 &= \alpha_+^2 x_0^2 + v_0^2 + 2\alpha_+ x_0 \, v_0 \cos(\beta + \pi/2),
\end{aligned}
$$

and then, $4\tilde{\omega}_0^2 \mathrm{Re}(A_+ A_-^*) = -\alpha_- \alpha_+ x_0^2 + 4\tilde{\omega} x_0 \, v_0 \cos(\beta + \pi/2) - v_0^2$, and $4\tilde{\omega}_0^2 \mathrm{Im}(A_+ A_-^*) = -4\tilde{\omega}_0 x_0 \, v_0 \sin(\beta + \pi/2)$. In these relations we have $\alpha_+ \alpha_- = -\omega_0^2$, $\alpha_+ + \alpha_- = 2\tilde{\omega}_0$ and $\alpha_-^2 + \alpha_+^2 = 4\tilde{\omega}^2 + 2\omega_0^2$. In consequence, trajectories in total space are written as follows

$$
\begin{aligned}
x &= \frac{1}{2\tilde{\omega}_0} (-\alpha_- x_0 \cos \alpha_+ t + \alpha_+ x_0 \cos \alpha_- t \\
&\quad + (v_0 \sin(\alpha_+ t + \beta) - v_0 \sin(\alpha_- t + \beta)), \\
y &= \frac{1}{2\tilde{\omega}_0} (-\alpha_- x_0 \sin \alpha_+ t + \alpha_+ x_0 \sin \alpha_- t \\
&\quad - (v_0 \cos(\alpha_+ t + \beta) - v_0 \cos(\alpha_- t + \beta)), \\
\zeta &= -a_0 t - \frac{a_1}{2\omega_0^2} (\cos 2\tilde{\omega}_0 t - 1) + \frac{a_2}{2\omega_0^2} \sin 2\tilde{\omega}_0 t,
\end{aligned}
$$

where a_0, a_1 and a_2 are the following constants

$$
\begin{aligned}
a_0 &= \alpha_+^2 |A_+|^2 + \alpha_-^2 |A_-|^2, \\
a_1 &= -2\omega_0^2 \mathrm{Re}(A_+ A_-^*), \\
a_2 &= +2\omega_0^2 \mathrm{Im}(A_+ A_-^*).
\end{aligned}
$$

The curves in base space correspond to hypotrochöids, respectively epitrochöids, which are curves generated by rolling without slipping, one circle over another circle.

6 Conclusions and perspectives

We study the classical Foucault's pendulum under the framework of sub-Riemannian geometry. This approach allows to set a differential system for horizontal curves that can be explicitly integrated. This formalism leads also to some other physical models. The calculation of sub-Riemannian spheres and the associated wave fronts, as well as the general non-symmetric Foucault's pendulum are part of our current research and shall be reported elsewhere.

The problem under discussion, although classic, and in some sense standard textbook material, provides under the sub-Riemannian approach, an interesting source of new theoretical and applied problems. For instance, it shall be interesting to establish a connection between the different curves obtained and the other physical models, such as two-level systems, quantum computing. Recently has been reported some interesting applications of the Foucault-like problems satellite formation lying, satellite constellation and space terminal rendezvous, we refer the reader to [Condurache, 2008].

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