SOME AMAZING PHENOMENA IN STABILITY OF NONLINEAR DYNAMICAL SYSTEMS

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Abstract

Nonlinear dynamical systems possess some properties which at the first sight look unexpected and even surprising (see, e.g., [1,2]). In this paper a collection of new qualitative results relating to stability of such systems is presented. In particular, for periodic oscillations of systems with non-monotonic elastic forces, new stability and instability effects are found. For some systems with uncertain bounded terms, necessary and sufficient stability conditions are obtained; the surprising feature is that they are independent upon arbitrary time-varying delays in the uncertain terms. It is shown that the known mathematical model of a swing – a pendulum with a periodic length – is incorrect. An unknown feature relating to regions of parametrical resonances is found.

Key words

Nonlinear systems, stability, periodic oscillations, delay function, swing.

1 Systems with non-monotonic elastic forces

Forced and parametric oscillations of a system with one degree of freedom are usually described by the equation

$$\ddot{x} + \mu\dot{x} + f(x) = p\sin\omega t \qquad (1)$$

and

$$\ddot{x} + \mu\dot{x} + (1 + p\sin\omega t)f(x) = 0\,, \qquad (2)$$

respectively.

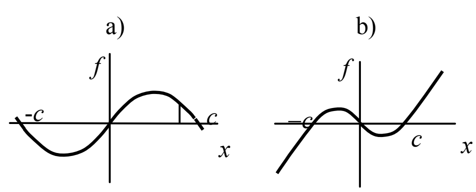

Figure 1

We assume that the value μ is small and the elastic characteristic $f(x)$ is non-monotonic and softening (Figure 1,a) or hardening (Figure 1,b). In Figure 2 some mechanical models with such characteristics are presented.

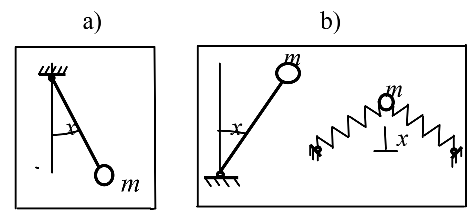

Figure 2

The universally adopted notions on the behavior of amplitude-frequency characteristics $A_i(\omega)$ of hamonic oscillations $x_i(t), i = 1{,}2$ in systems (1), (2) and their stability are mainly based on an analysis of specific models via numerical and approximate analytical methods. A qualitative analysis of such systems enabled us to obtain rigorous general results and find out some unknown features.

If a periodic solution $x(t)$ of equation (1) or (2) with $\mu = 0$ is stable (unstable), then for small μ, the corresponding solution $x(t,\mu)$ is asymptotically stable (unstable). So, in what follows we assume $\mu = 0$.

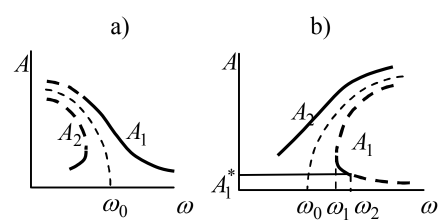

Figure 3

It can be shown that the amplitude-frequency characteristics $A_i(\omega), i = 1{,}2$ in systems with non-monotonic elastic forces are qualitatively the same as that in the case of monotonic $f(x)$ (Figure 3). However, some new features relating to stability of the family $x_1(t,\omega)$ are found.

Consider first forced oscillations. As is known, in the case of monotonic $f(x)$, the solutions $x_1(t,\omega)$ in a system with softening nonlinearity are stable for all ω . If $f(x)$ is non-monotonic, a break of stability occurs as the amplitude A_1 increases (the dotted line in Figure 3,a). It should be emphasized that the instability begins with an amplitude $A_1^* < c$ (in particular, for a pendulum, $A_1^* < \pi$). So, the above effect may exist in a system which has no unstable equilibrium positions. Moreover, it can be shown that for any system with softening non-monotonic characteristic, an exciting force $p(\omega t)$ can be found such that the periodic oscillations $A_1(\omega)$ are unstable for some ω .

In a system with monotonic hardening nonlinearity the solutions $x_1(t,\omega)$, corresponding to the lower branch of $A_1(\omega)$, are stable for all ω . It is found [3] that in the non-monotonic case, $A_1(\omega)$ is stable for some $\omega \in (\omega_1, \omega_2)$ and unstable for $\omega \in (\omega_2, \infty)$ (Figure 3,b). Note that the value $A_1(\omega_2) < c$; it follows that stable oscillations exist even when a system has no stable equilibrium positions, i.e., the elastic force is completely repulsive.

In particular, oscillations of a pendulum about the upper equilibrium point are unstable for small amplitudes. As the amplitude increases (due to either a decrease in oscillation frequency or an increase in the exciting force), the oscillations become stable beginning with some $A_1^* \in (\pi/2, \pi)$. Thus, stable oscillations proceed at a completely repulsive part of the elastic force.

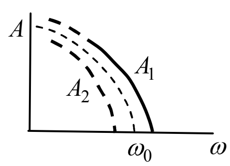

Figure 4

The amplitude-frequency characteristics $A_i(\omega)$, $i = 1,2$ of main ($T = 4\pi/\omega$) parametric oscillations in systems with softening nonlinearity are shown in Figure 4. Analogously to the case of forced oscillations, for non-monotonic $f(x)$, the solutions $x_1(t)$ with the amplitudes $A_1 \in (A_1^*, c)$ are unstable (while for monotonic $f(x)$, the branch $A_1(\omega)$ is stable for all ω).

Note that, to our knowledge, the presence of instability interval on the branch $A_1(\omega)$ (Figures 3,a and 4) is not known so far (though such systems, especially a pendulum, were widely treated in the literature). The matter is that stability is usually analyzed via perturbation methods with p being a small parameter (quasi-conservative approach). For sufficiently small p, any solution $x_1(t,p)$ is stable (precisely this fact is established by perturbation methods). However, the stability breaks for some $p = p_*(\omega)$; it can be shown that $p_*(\omega) \to 0$ as $\omega \to 0$. So, for a fixed p , there always is an interval $(0, \omega_*)$ for which the solutions $x_1(t,p)$ are unstable.

Therefore, for a system with a non-monotonic elastic force, stability conditions, obtained by perturbation methods, should be taken very cautiously.

Analogously, the presence of the stability interval (ω_1, ω_2) on the branch $A_2(\omega)$ (Figure 3,b) in a system with a repulsive force $f(x)$ cannot be found by perturbation methods, because here the generating system ($p = 0$) has no a periodic solution.

The break of stability in systems (1) and (2) with softening nonlinearity is explained as follows. The stability of a periodic solution $x(t)$ is determined by the corresponding variational equation

$$\ddot{y} + a(t)y = 0 \tag{3}$$

where $a(t) = a(t + T/2)$ when $f(x) = -f(-x)$. For a periodic solutions $x_1(t)$ with small amplitude A_1 , $a(t) > 0$ and equation (3) belongs to the zero stability region of Hill equation (3) [4]. As A_1 increases, $a(t)$ becomes sign-varying and for some $A_1 = A_1^*$, equation (3) passes to the zero instability region. Meanwhile, in a system with monotonic $f(x)$, $a(t) > 0$ for all A_1.

On contrary, in system (1) with non-monotonic hardening nonlinearity, $a(t) < 0$ for small A_1, so the corresponding solutions $x_1(t)$ are unstable. As A_1 increases, $a(t)$ becomes sign-varying and stability begins with some $A_1 = A_1^*$.

2. **Mathematical model of a swing**

Swinging is realized by squatting and raising; as a result, the distance l between the center of gravity and the suspension point changes (Figure 5,a). Under a periodic motion, the function $l(t)$ is also periodic, so the swing is usually modeled with a pendulum having a periodic length (Figure 5,b). The corresponding equation is

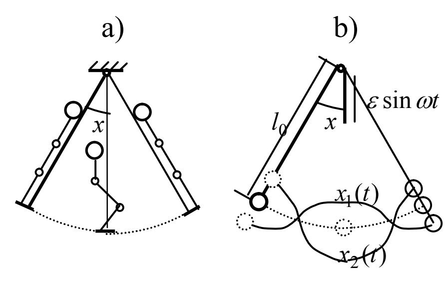

Figure 5

$$[l^2(\omega t)\dot{x}]^{\cdot} + \mu\dot{x} + gl(\omega t)\sin x = 0 \,. \tag{4}$$

In a qualitative analysis of equation (4) [5] it was assumed that

$$l(t) = l(-t) = l(t + T_l)\ ,\ T_l = 2\pi / \omega\ ,$$
$$\dot{l}(t) \ge 0 \quad \text{for} \quad t \in (0, T_l / 2)\ .$$

The last condition means that the length changes monotonically between the minimal and maximal values $l(0)$ and $l(\pi/\omega)$. No restrictions on oscillations amplitude and the variation of the pendulum length were imposed.

The main parametric oscillations ($T = 4\pi/\omega$) were studied. There exist two such families, $x_1(t,\omega)$ and $x_2(t,\omega)$; their amplitude-frequency characteristics $A_i(\omega)$ are qualitatively the same as that in Figure 4; $A_i(\omega) \to \pi$ as $\omega \to 0$. The corresponding trajectories of the pendulum are shown in Figure 5,b. As is seen, for the solution $x_1(t,\omega)$ ($x_2(t,\omega)$) the length $l(t)$ reaches its minimal (maximal) value when the pendulum passes the equilibrium position.

Stability analysis showed that the solutions $x_1(t,\omega)$ are certainly stable for as long as the amplitudes do not exceed $\pi/2$. For some $A_1 = A_*^1 \in (\pi/2, \pi)$, the stability breaks, so the oscillations with amplitudes close to π are unstable. The solutions $x_2(t,\omega)$) are unstable for all ω.

The above results, being applied to a swing, come to the following contradiction. As is seen from Figure 5,a, the center of the gravity takes its lower position when the swing passes the equilibrium point. So, oscillations of a swing correspond to the solutions $x_2(t)$). However, as is mentioned above, such solutions are unstable and, therefore, not physically realizable.

The obtained contradiction shows that the parametrically excited pendulum cannot serve as a mathematical model of a swing. The case is that the behavior of a man on a swing (and, therefore, the length of the corresponding pendulum) is determined not by the time but the current position and velocity of the swing. Therefore, a swing should be simulated with a self-oscillatory pendulum whose length depends on the angle coordinate and velocity ($l = l(x, \dot{x})$).

Two such models of a swing were considered [5]. For the first one, the length increases, for the second decreases as the swing moves to the equilibrium position. Numerical calculations showed that each model has a unique limit cycle, corresponding to a periodic oscillation of the swing. Thus, it is also possible to raise when a swing moves down and squat when it moves up (however, in this case the oscillation amplitudes are smaller).

3. Stability regions of parametric oscillations

The following results relate to stability of parametric oscillations in Hamiltonian systems. They are described by the equation

$$J\dot{x} = H(\omega t, \mu) x \tag{5}$$

where $H(\omega t, \mu) = H(\omega t + 2\pi, \mu)$ is a symmetric and J is a skew-symmetric matrix of order $2n$, ω is a frequency of parametric excitation, the parameter μ characterizes its intensity ($H(\omega t, \mu)$ increases in μ , $H(\omega t, 0) = H_0$ is a constant matrix).

The basis of the contemporary theory of such equations is laid by Krein, Gel'fand, Lidskii, Yakubovich and other researchers. Unfortunately, the proofs of many theorems are very laborious and use quite complex mathematical tools [4]. A new approach to the theory, developed in [6], enabled us to substantially simplify the proofs of main theorems and obtain some new qualitative results on stability of equation (5).

The stability regions are usually plotted in the plane μ, ω (Figure 6 where the instability regions are shaded). The boundaries ω_i^- and ω_i^+, $i = 1,2,...$ are calculated using numerical or approximate analytical methods. Then it is accepted that the regions between these boundaries correspond to unstable solutions while for the rest values of the parameters μ and ω, equation (5) is stable. However, such approach is not justified (in principle, it is not excluded that there are "islands" of instability within the stability regions, and vice versa). It is proved [6] that the first case is impossible, i.e., the regions between neighboring instability boundaries, emanating from different points ω^i , are completely stable. On contrary, it was found that within instability regions some stability ones may exist (in Figure 6 such a region emerges from the point K).

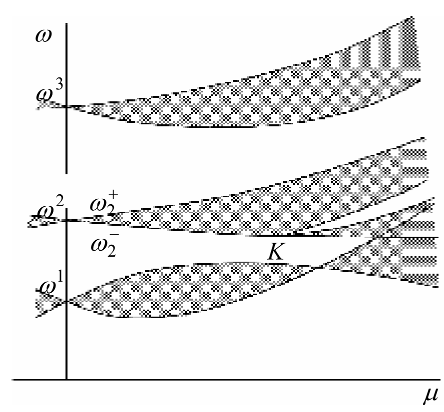

Figure 6

As is known [4], for any μ , ω , corresponding to a boundary of a stability region, equation (5) has an indefinite Floquet multiplier on the unit circle; generically, its multiplicity $r = 2$. The stability region indicated above may arise only if equation (5), corresponding to the point K, has an indefinite multiplier of an order $r \geq 3$. Thus, such effect may appear in systems of an order $n \geq 3$ only.

4. Exponential stability of uncertain nonlinear systems with delay

Consider the equation

$$\dot{x}(t) + A(t)x(t) = f(x(t-\tau(t),t), \qquad (6),$$

$$\|f(x,t)\| \leq k\|x\|$$

where $x \in R^n$, $\tau(t) \in [0, H < \infty]$ is a delay function, $\|\cdot\|$ means an Euclidean norm.

Equation (6) is called exponentially stable if for any $f(x,t)$ and $\tau(t)$, satisfying the above conditions, and a given initial function $x_0(t) = x(t)$ for $t \in [-H, 0]$, the corresponding solution $x(t)$ exponentially tends to zero.

Let $W(t,s)$ $\left(W(t,t) = I_{2n}\right)$ be the transfer matrix for equation $\dot{x}(t) + A(t)x(t) = 0$. We put

$$v(t) = \int_0^t \|W(t,s)\| ds, \quad v_\infty = \lim v(t) \text{ for } t \to \infty. \qquad (7)$$

It is proved [7] that for exponential stability of system (5), it is sufficient that

$$k < 1/v_\infty. \qquad (8)$$

In the following cases this condition is also necessary.

1. The matrix $W(t,s)$ is nonnegative for $t > s$.
2. The matrix A in (6) is constant and symmetric.

Note that the first condition is, in particular, fulfilled when the off-diagonal elements of the matrix $A(t)$ are non-negative.

In the both cases the precise critical value of the parameter k equals $k_* = 1/v_\infty$. It is interesting to note that the stability breaks for $f(x) \equiv k_* x$, i.e., the “most destabilizing” unknown term is linear. Another surprising feature is that the value k_* is invariant with the delay function $\tau(t)$.

In systems with one nonlinearity, $f = b\varphi(\sigma(t-\tau(t),t)$, $\sigma = (c,x)$ where b and c are vectors; the scalar function $\varphi(\sigma,t)$ satisfies the inequality

$$|\varphi(\sigma,t)| \leq k|\sigma|.$$

The goal is to find necessary and sufficient stability conditions expressed in the value k and the matrix $A(t)$. The problem is reduced to the Volterra equation

$$\sigma(t) = f(t) + \int_0^t w(t,s)\varphi(\sigma(s-\tau(s)),s)\,ds \qquad (9)$$

where $w(t,s)$ is the transfer function.

Note that in the absence of the delay, this problem (called the Lur’e problem) is classical. In spite of great number of the corresponding papers, the problem remains unsolved even in the case of constant A (only sufficient stability conditions are obtained).

At the first sight, the presence of an unknown delay function $\tau(t)$ should substantially complicate the problem. Actually, this enabled us to obtain a precise solution of the problem [8]. Namely, inequality (8) (where) $\|W(t,s)\|$ is replaced by $|w(t,s)|$), provides the required necessary and sufficient stability condition for system (9).

Note that if $w(t,s) \geq 0$ for $t > s$, the stability breaks for $\varphi(\sigma,t) \equiv k_*\sigma$ and any $\tau(t)$. If $w(t,s)$ is sign-varying, the destabilizing function $\varphi(\sigma,t)$ changes from $k_*\sigma$ to $-k_*\sigma$, and vice versa, while $\tau(t)$ is a saw-like function (so that the function $t-\tau(t)$ is piece-wise constant) [8]. Thus, in the case of sign-varying $w(t,s)$, condition (8) is not invariant with $\tau(t)$.

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