STABILITY ANALYSIS OF NONSTATIONARY MECHANICAL SYSTEMS WITH TIME DELAY VIA AVERAGING METHOD

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Abstract

A linear mechanical system with constant matrix of dissipative forces and continuous and bounded matrices of positional forces is studied. It is assumed that there are a large positive multiplier at the vector of dissipative forces and a constant delay in positional forces. The case is considered where the associated delay-free averaged system is asymptotically stable. With the aid of the Lyapunov direct method and the averaging method, conditions are derived under which neither delay nor time-varying perturbations with zero mean values disturb the asymptotic stability. The developed approach is used in the problem of monoaxial stabilization of a rigid body.

Key words

Mechanical systems; nonstationary perturbations; delay; averaging method, Lyapunov–Krasovskii functional, rigid body.

1 Introduction

The averaging method is a powerful tool for the investigation of dynamics of time-varying systems [Bogoliubov and Mitropolsky, 1961; Grebennikov, 1986; Khalil, 2002]. It allows us to determine conditions under which conclusions on the properties of original nonstationary systems can be obtained via analysis of the associated time-invariant averaged systems. This method is especially effectively used for the stability investigation (see, for instance, [Grebennikov, 1986; Khapaev, 1993; Peuteman and Aeyels, 1999; Aleksandrov and Efimov, 2022]).

There are different approaches to solving stability problem for nonstationary systems on the basis of averaging. One of such approaches is the Lyapunov direct method. In particular, in the papers [Mitropolsky and Martynyuk, 1980; Bodunov and Kotchenko, 1988; Sosnitskii, 2010], special constructions of Lyapunov functions were proposed for linear systems whose matrices contain small parameters and time-varying perturbations with zero mean values. On the other hand, in [Fridman and Zhang, 2020], an original technique based on the artificial introducing delays in the considered systems and constructing Lyapunov–Krasovskii functionals was used. This technique was applied for both delay-free and time-delay linear systems.

It should be noted that the approaches developed in [Mitropolsky and Martynyuk, 1980; Bodunov and Kotchenko, 1988; Sosnitskii, 2010; Fridman and Zhang, 2020] permit us not only to derive stability conditions but also to deduce estimates for admissible values of small parameters under which stability of associate averaged systems implies that for the perturbed systems.

However, it is worth mentioning that the results of [Mitropolsky and Martynyuk, 1980; Bodunov and Kotchenko, 1988; Sosnitskii, 2010; Fridman and Zhang, 2020] were obtained for delay-free linear systems with periodic or almost periodic right-hand sides. The goal of the present contribution is an extension of approaches from [Mitropolsky and Martynyuk, 1980; Bodunov and Kotchenko, 1988; Sosnitskii, 2010] to some types of time-delay systems with nonstationary perturbations that are not necessarily periodic or almost periodic.

We consider a linear mechanical system with a large positive parameter as a multiplier at the constant matrix of dissipative forces and with nonstationary positional forces containing time-delay terms. It is assumed that the associated delay-free averaged system is asymptotically stable. With the aid of a special construction of Lyapunov–Krasovskii functional, conditions ensuring asymptotic stability for the original system are derived.

Moreover, the proposed approach is applied to the problem of monoaxial stabilization of a rigid body.

2 Statement of the Problem

Let the motions of a mechanical system be modeled by the equations

$$A\ddot{x}(t) + hB\dot{x}(t) + C(t)x(t) + D(t)x(t-\tau) = 0. \quad (1)$$

Here $x(t), \dot{x}(t) \in \mathbb{R}^n$ are vectors of generalized coordinates and velocities, respectively, A is a constant symmetric and positive definite matrix of inertial characteristics of the system, B is a constant symmetric and positive definite matrix of dissipative forces, matrix functions $C(t)$ and $D(t)$ are continuous and bounded for $t \in [0, +\infty)$, h is a positive parameter, τ is a constant positive delay. The term $D(t)x(t-\tau)$ can be interpreted as a result of application of a control with delay in the feedback law [Fridman, 2014; Andreev and Peregudova, 2021; Khac, Vlasov and Pyrkin, 2022].

Assume that initial functions for solutions of (1) belong to the space $C^1([-\tau, 0], \mathbb{R}^n)$ of continuously differentiable functions $\varphi(\xi) : [-\tau, 0] \to \mathbb{R}^n$ with the uniform norm

$$\|\varphi\|_\tau = \sup_{\xi \in [-\tau, 0]} (\|\varphi(\xi)\| + \|\dot{\varphi}(\xi)\|),$$

where $\|\cdot\|$ is the Euclidean norm of a vector. Denote by x_t the restriction of a solution $x(t)$ to the segment $[t-\tau, t]$, i.e., $x_t : \xi \mapsto x(t+\xi)$ for $\xi \in [-\tau, 0]$.

Assumption 1. *Let*

$$\frac{1}{T}\int_t^{t+T} C(s)ds \to \bar{C}, \qquad \frac{1}{T}\int_t^{t+T} D(s)ds \to \bar{D}$$

as $T \to +\infty$ uniformly with respect to $t \geq 0$.

Thus, $C(t) = \bar{C} + \widetilde{C}(t)$, $D(t) = \bar{D} + \widetilde{D}(t)$, where the matrices $\widetilde{C}(t)$, $\widetilde{D}(t)$ have zero mean values.

Remark 1. Under Assumption 1, the associated averaged system for (1) is

$$A\ddot{x}(t) + hB\dot{x}(t) + \bar{C}x(t) + \bar{D}x(t-\tau) = 0.$$

Assumption 2. *The matrix $\bar{C} + \bar{D}$ is positive definite.*

Remark 2. It is well-known [Merkin, 1997] that if Assumption 2 is fulfilled, then the corresponding delay-free system

$$A\ddot{x}(t) + hB\dot{x}(t) + (\bar{C} + \bar{D})x(t) = 0 \qquad (2)$$

is asymptotically stable.

We look for conditions guaranteeing that neither delay nor time-varying perturbations disturb the stability. Using the Lyapunov direct method, we will show that, for any given $\tau > 0$, one can choose sufficiently large value of the parameter h for which the system (1) is asymptotically stable.

Furthermore, we will apply the obtained result to the problem of monoaxial stabilization of a rigid body.

Remark 3. It is worth noticing that, for the stability analysis of the system (1), one can try to use the approach developed in [Fridman and Zhang, 2020]. However, in [Fridman and Zhang, 2020], more conservative constraints on nonstationary perturbations were imposed. In addition, the stability conditions derived in [Fridman and Zhang, 2020] are formulated in terms of feasibility of systems of linear matrix inequalities of high dimension. Such conditions can be effectively verified for given numerical values of system parameters, but it is problematic to obtain with their help explicit analytical dependencies for admissible values of the parameters.

3 Stability Conditions

We will use the approaches for the construction of Lyapunov functions and Lyapunov–Krasovskii functionals developed in [Antonchik, 1983; Efimov and Aleksandrov, 2021; Aleksandrov, 1996; Aleksandrov, Aleksandrova and Zhabko, 2013].

Theorem 1. *Let Assumptions 1 and 2 be fulfilled. Then, for any delay $\tau > 0$ there exists $h_0 > 0$ such that if $h \geq h_0$, then the system (1) is asymptotically stable.*

Proof. First, according to [Antonchik, 1983], a Lyapunov function for (2) can be chosen as follows:

$$V_1(x, \dot{x}) = \frac{1}{2}\dot{x}^\top A\dot{x} + \frac{h}{2}x^\top Bx + x^\top A\dot{x}.$$

Differentiating this function along the solutions of (1) we obtain

$$\dot{V}_1 = -h\dot{x}^\top(t)B\dot{x}(t) + \dot{x}^\top(t)A\dot{x}(t)$$

$$-\dot{x}^\top(t)(C(t)x(t) + D(t)x(t-\tau)) - x^\top(t)\widetilde{C}(t)x(t)$$

$$-x^\top(t)(\bar{C} + \bar{D})x(t) + x^\top(t)\bar{D}(x(t) - x(t-\tau))$$

$$-x^\top(t)\widetilde{D}(t)x(t-\tau) \leq -(ha_1 - a_2)\|\dot{x}(t)\|^2$$

$$-a_3\|x(t)\|^2 + a_4\|\dot{x}(t)\|\|x(t)\| + a_5\|\dot{x}(t)\|\|x(t-\tau)\|$$

$$-x^\top(t)\widetilde{C}(t)x(t) + x^\top(t)\bar{D}(x(t) - x(t-\tau))$$

$$-x^\top(t)\widetilde{D}(t)x(t-\tau).$$

Here a_1, a_2, a_3, a_4, a_5 are positive constants.

Next, using the approach proposed in [Efimov and Aleksandrov, 2021], construct a Lyapunov–Krasovskii functional in the form

$$V_2(x_t) = V_1(x(t),\dot{x}(t)) - x^\top(t)\int_{t-\tau}^{t}\widetilde{D}(\xi+\tau)x(\xi)d\xi$$

$$-x^\top(t)\bar{D}\int_{t-\tau}^{t}x(\xi)d\xi + \int_{t-\tau}^{t}(\lambda+\beta(\xi-t+\tau)\|x(\xi)\|^2 d\xi,$$

where λ and β are positive parameters.

Differentiating this functional along the solutions of (1) we arrive at the estimate

$$\dot{V}_2 \le -(ha_1-a_2)\|\dot{x}(t)\|^2 - (a_3-\lambda-\beta\tau)\|x(t)\|^2$$

$$-\lambda\|x(t-\tau)\|^2 + \|\dot{x}(t)\|(a_4\|x(t)\| + a_5\|x(t-\tau)\|)$$

$$+a_6\|\dot{x}(t)\|\int_{t-\tau}^{t}\|x(\xi)\|d\xi - \beta\int_{t-\tau}^{t}\|x(\xi)\|^2 d\xi$$

$$-x^\top(t)\widetilde{C}(t)x(t) - x^\top(t)\widetilde{D}(t+\tau)x(t),$$

where $a_6 = \text{const} > 0$.

Finally, according to the approach developed in [Aleksandrov, 1996; Aleksandrov, Aleksandrova and Zhabko, 2013], define a nonstationary functional by the formula

$$V_3(t,x_t) = V_2(x_t) + x^\top(t)\int_0^t e^{\varepsilon(u-t)}\widetilde{C}(u)du\, x(t)$$

$$+x^\top(t)\int_0^{t+\tau} e^{\varepsilon(u-t-\tau)}\widetilde{D}(u)du\, x(t).$$

Here ε is a positive parameter.

The functional $V_3(t,x_t)$ and its derivative along the solutions of (1) satisfy the inequalities

$$b_1\|\dot{x}(t)\|^2 + hb_2\|x(t)\|^2 - b_3\|x(t)\|\|\dot{x}(t)\|$$

$$-b_4\sqrt{\tau}\|x(t)\|\left(\int_{t-\tau}^{t}\|x(\xi)\|^2 d\xi\right)^{1/2} + \lambda\int_{t-\tau}^{t}\|x(\xi)\|^2 d\xi$$

$$-\frac{b_5}{\varepsilon}\|x(t)\|^2 \le V_3(t,x_t) \le b_6\|\dot{x}(t)\|^2 + hb_7\|x(t)\|^2$$

$$+(\lambda+\beta\tau)\int_{t-\tau}^{t}\|x(\xi)\|^2 d\xi + b_3\|x(t)\|\|\dot{x}(t)\|$$

$$+\frac{b_5}{\varepsilon}\|x(t)\|^2 + b_4\sqrt{\tau}\|x(t)\|\left(\int_{t-\tau}^{t}\|x(\xi)\|^2 d\xi\right)^{1/2},$$

$$+\frac{b_8}{\varepsilon}\|x(t)\|\|\dot{x}(t)\| + \varepsilon\left\|\int_0^t e^{\varepsilon(u-t)}\widetilde{C}(u)du\right\|\|x(t)\|^2$$

$$+\varepsilon\left\|\int_0^{t+\tau} e^{\varepsilon(u-t-\tau)}\widetilde{D}(u)du\right\|\|x(t)\|^2,$$

where $b_j > 0$, $j = 1,\ldots,8$.

It is known (see [Bogoliubov and Mitropolsky, 1961]), that

$$\varepsilon\left\|\int_0^t e^{\varepsilon(u-t)}\widetilde{C}(u)du\right\| \to 0 \quad \text{as } \varepsilon\to 0,$$

$$\varepsilon\left\|\int_0^{t+\tau} e^{\varepsilon(u-t-\tau)}\widetilde{D}(u)du\right\| \to 0 \quad \text{as } \varepsilon\to 0$$

uniformly with respect to $t \ge 0$. Hence, one can choose and fix the tuning parameters $\varepsilon, \lambda, \beta$ such that $\lambda+\beta\tau < a_3/4$ and

$$\varepsilon\left\|\int_0^t e^{\varepsilon(u-t)}\widetilde{C}(u)du\right\| \le \frac{a_3}{8},$$

$$\varepsilon\left\|\int_0^{t+\tau} e^{\varepsilon(u-t-\tau)}\widetilde{D}(u)du\right\| \le \frac{a_3}{8}$$

for all $t \ge 0$. Then

$$\dot{V}_3 \le -(ha_1-a_2)\|\dot{x}(t)\|^2 - \frac{a_3}{2}\|x(t)\|^2 - \lambda\|x(t-\tau)\|^2$$

$$+\|\dot{x}(t)\|(a_4\|x(t)\| + a_5\|x(t-\tau)\|) + \frac{b_8}{\varepsilon}\|x(t)\|\|\dot{x}(t)\|$$

$$+a_6\sqrt{\tau}\|\dot{x}(t)\|\left(\int_{t-\tau}^{t}\|x(\xi)\|^2 d\xi\right)^{1/2} - \beta\int_{t-\tau}^{t}\|x(\xi)\|^2 d\xi.$$

Applying the Sylvester criterion, we obtain that if

$$h > \frac{1}{b_2}\left(\frac{b_5}{\varepsilon} + \frac{b_3^2}{4b_1} + \frac{\tau b_4^2}{4\lambda}\right),$$

$$h > \frac{1}{a_1}\left(a_2 + \frac{1}{2a_3}\left(\frac{b_8}{\varepsilon} + a_4\right)^2 + \frac{a_5^2}{4\lambda} + \frac{\tau a_6^2}{4\beta}\right),$$

then there exist positive numbers c_1, c_2, c_3 such that

$$c_1\left(\|\dot{x}(t)\|^2 + \|x(t)\|^2 + \int_{t-\tau}^{t}\|x(\xi)\|^2 d\xi\right) \le V_3(t,x_t)$$

$$\le c_2\left(\|\dot{x}(t)\|^2 + \|x(t)\|^2 + \int_{t-\tau}^{t}\|x(\xi)\|^2 d\xi\right),$$

4 Monoaxial Stabilization of a Rigid Body

Next, we will show that the developed approach can be applied to problems of the attitude control of rigid bodies.

Consider a rigid body rotating around its mass center O with angular velocity ω. Let the axes $Oxyz$ be principal central axes of inertia of the body. The Euler equations modeling the attitude motion of the body under the action of a control torque M have the form

$$J\dot{\omega}(t) + \omega(t) \times (J\omega(t)) = M, \tag{3}$$

where $J = \mathrm{diag}\{A_1, A_2, A_3\}$ is a body inertia tensor in the axes $Oxyz$, see [Beletsky, 1965].

Let unit vectors s and r be given. The vector s is constant in the inertial space, whereas the vector r is constant in the body-fixed frame. Then vector s rotates with respect to the system $Oxyz$ with the angular velocity $-\omega$. Hence,

$$\dot{s}(t) = -\omega(t) \times s(t). \tag{4}$$

As a result, we obtain the system consisting of the Euler dynamic equations (3) and the Poisson kinematic equations (4).

Consider the problem of monoaxial stabilization of the body [Zubov, 1978]: it is required to design a control torque M ensuring that system (3), (4) admits the asymptotically stable equilibrium position

$$\omega = 0, \qquad s = r. \tag{5}$$

In [Zubov, 1978], it was proved that the torque M can be chosen as the sum of dissipative component M_1 and restoring one M_2: $M = M_1 + M_2$, where

$$M_1 = -hB\omega, \quad M_2 = -as \times r,$$

h and a are positive coefficients, B is a constant symmetric positive definite matrix.

In the paper [Aleksandrov and Tikhonov, 2018], with the aid of the averaging method, the impact of time-varying perturbations with zero mean values on the stability of the equilibrium position (5) was studied. In this section, along with nonstationary perturbations, we will take into account delay in control and disturbed torques.

Let the Euler equations be of the form

$$J\dot{\omega}(t) + \omega(t) \times (J\omega(t)) = -hB\omega(t) - \bar{c}s(t) \times r$$

$$-\bar{d}s(t-\tau)\times r + \widetilde{C}(t)(s(t)-r) + \widetilde{D}(t)(s(t-\tau)-r). \tag{6}$$

Here $h, \bar{c}, \bar{d}$ are constant coefficients with $h > 0$, B is a constant symmetric positive definite matrix, matrices $\widetilde{C}(t)$ and $\widetilde{D}(t)$ are continuous and bounded for $t \in [0, +\infty)$, $\tau > 0$ is a constant delay.

Assumption 3. *Let*

$$\frac{1}{T}\int_t^{t+T} \widetilde{C}(s)ds \to 0, \qquad \frac{1}{T}\int_t^{t+T} \widetilde{D}(s)ds \to 0$$

as $T \to +\infty$ uniformly with respect to $t \geq 0$.

Remark 4. Thus, the disturbed torques admit zero mean values. It is well-known (see [Beletsky, 1965; Giri and Sinha, 2017; Akulenko, Leshchenko and Chernous'ko, 1986]), that perturbations of a such type are often used in models of satellites moving in circular or elliptic orbits.

Assumption 4. *The inequality $\bar{c} + \bar{d} > 0$ is valid.*

Theorem 2. *Let Assumptions 3 and 4 be fulfilled. Then, for any delay $\tau > 0$ there exists $h_0 > 0$ such that if $h \geq h_0$, then the equilibrium position (5) of the system (4), (6) is asymptotically stable.*

Proof. In a similar way as in the proof of Theorem 1, we sequentially construct a Lyapunov function and Lyapunov–Krasovskii functionals by the following formulae:

$$V_1(\omega, s) = \frac{1}{2}\omega^\top J\omega + \frac{h}{2}\|s - r\|^2 + (s \times r)^\top B^{-1} J\omega,$$

$$V_2(\omega(t), s_t) = V_1(\omega(t), s(t))$$

$$+\bar{d}(s(t) \times r)^\top B^{-1}\left(r \times \int_{t-\tau}^{t} s(\xi)d\xi\right)$$

$$+(s(t) \times r)^\top B^{-1}\int_{t-\tau}^{t} \widetilde{D}(\xi+\tau)(s(\xi) - r)d\xi$$

$$+\int_{t-\tau}^{t}(\lambda + \beta(\xi - t + \tau)\|s(\xi) - r\|^2 d\xi,$$

$$V_3(t, \omega(t), s_t) = V_2(\omega(t), s_t)$$

$$-(s(t) \times r)^\top B^{-1}\int_0^{t+\tau} e^{\varepsilon(u-t-\tau)}\widetilde{D}(u)du(s(t) - r)$$

$$-(s(t) \times r)^\top B^{-1}\int_0^{t} e^{\varepsilon(u-t)}\widetilde{C}(u)du(s(t) - r),$$

where $\varepsilon, \lambda, \beta$ are positive parameters.

With the aid of the same arguments as in the proof of Theorem 1, it can be verified that, under appropriate choice of tuning parameters $\varepsilon, \lambda, \beta$ and for sufficiently large value of h, there exist positive numbers δ, c_1, c_2, c_3 such that

$$c_1\left(\|\omega(t)\|^2+\|s(t)-r\|^2+\int_{t-\tau}^{t}\|s(\xi)-r\|^2 d\xi\right)$$

$$\leq V_3(t,\omega(t),s_t)\leq c_2\Bigg(\|\omega(t)\|^2$$

$$+\|s(t)-r\|^2+\int_{t-\tau}^{t}\|s(\xi)-r\|^2 d\xi\Bigg),$$

$$\dot V_3\leq -c_3\Bigg(\|\omega(t)\|^2$$

$$+\|s(t)-r\|^2+\int_{t-\tau}^{t}\|s(\xi)-r\|^2 d\xi\Bigg)$$

for $\|\omega(t)\|+\|s(t)-r\|<\delta$. This completes the proof. □

5 Example

In [Sosnitskii, 2010], with the aid of the averaging method, asymptotic stability conditions were obtained for the scalar equation

$$\ddot x(t)+h\dot x(t)+(1+b\cos\omega t)x(t)=0, \qquad (7)$$

where $x(t)\in\mathbb{R}$ and h,b,ω are positive parameters. It is worth mentioning that (7) can be interpreted as linear approximation for the equation modeling oscillations of a pendulum in a periodically varying gravitational field and a resisting medium.

In this section, we consider the corresponding time-delay equation

$$\ddot x(t)+h\dot x(t)+(1+b\cos\omega t)x(t-\tau)=0, \qquad (8)$$

where $\tau=\mathrm{const}>0$. In this case the appearance of delay may be caused by network communication of the control signal, whereas the term $b\cos\omega t$ may characterize control deviations from a prescribed values due to nonstationary perturbations.

It is easy to verify that Assumptions 1 and 2 are fulfilled for the equation (8). Hence, for sufficiently large h, this equation is asymptotically stable.

To derive lower bounds for admissible values of the parameter h, one can apply the approach proposed in the proof of Theorem 1. It should be noted that, in this case, when constructing the functional V_3, we can take $\varepsilon=0$.

As a result, we arrive at the following conditions:

$$h>1+\frac{2b}{\omega}+\frac{\tau(1+b)^2}{2\lambda},$$

$$h>1+\frac{b^2}{\omega^2(1-\lambda-\beta\tau)}+\frac{1+b}{4\lambda}+\frac{\tau}{\beta},$$

where tuning positive parameters λ and β satisfy the inequality $\lambda+\beta\tau<1$.

Eliminating the parameter β, we obtain

$$h>1+\min_{\lambda\in(0,1)}\max\{\psi_1(\lambda),\ \psi_2(\lambda)\}.$$

Here

$$\psi_1(\lambda)=\frac{2b}{\omega}+\frac{\tau(1+b)^2}{2\lambda},$$

$$\psi_2(\lambda)=\frac{1+b}{4\lambda}+\frac{(b+\tau\omega)^2}{\omega^2(1-\lambda)}.$$

It should be noted that $\psi_2'(\lambda)=0$ as

$$\lambda=\lambda_*=\frac{\omega\sqrt{1+b}}{\omega\sqrt{1+b}+2(b+\tau\omega)}.$$

Hence, if $\psi_1(\lambda_*)\leq\psi_2(\lambda_*)$, then $h>\psi_2(\lambda_*)$, and if $\psi_1(\lambda_*)>\psi_2(\lambda_*)$, then $h>\psi_2(\hat\lambda)$, where

$$\hat\lambda=\frac{-p_2+\sqrt{p_2^2-4p_1p_3}}{2p_1},\quad p_1=\frac{2b}{\omega},$$

$$p_2=\left(\frac{b}{\omega}+\tau\right)^2-\frac{2b}{\omega}+\frac{\tau(1+b)^2}{2}-\frac{1+b}{4},$$

$$p_3=\frac{1+b}{4}-\frac{\tau(1+b)^2}{2}.$$

6 Conclusion

In the present contribution, new asymptotic stability conditions are derived for considered nonstationary mechanical systems. Using original constructions of Lyapunov–Krasovskii functionals, it was shown that, for any given constant delay, one can choose sufficiently large multiplier at the vector of the dissipative forces for which the asymptotic stability can be guaranteed. The proofs of the theorems provide us a constructive approach for finding lower bounds for admissible values of the above multiplier.

An interesting direction for further research is an extension of the obtained results to systems with distributed delays.

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