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SIMPLE STATE-SYNCHRONIZATION FOR A CLASS OF POSSIBLY NONSMOOTH CHAOTIC SYSTEMS

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Abstract
Synchronization of chaotic systems is a widely studied problem, it has been cast using several techniques relying on different control schemes in order to force an accesible variable of some system to behave exactly in the same way that some other variable pertaining to another system or systems. In this paper, we develop a general framework to deal with state-synchronization of a class of chaotic systems with full relative degree. Conditions for synchronization are proposed and simulation results are shown for identical with parameter mismatch and for nonidentical chaotic systems.

Key words
Synchronization, chaos, non-smooth systems.

1 Introduction

Great advances have been achieved since Pecora and Carroll proposed a synchronization scheme in their seminal paper [Pecora and Carrol, 1990]; in this paper, they cast the synchronization problem proposing a master-slave approach in which some available signal from the master system is input to a slave subsystem having the same structure of the master. This approach has been used in several works later on. In [Alvarez, 1996] synchronizability conditions are formally stated and applied to the Lorenz system [Lorenz, 1963]. It has been applied also to dual inverted pendula [Dongfang and Di, 2008]. Non-forced synchronization has been analized including bifurcation structures for non-identical Duffing oscillators [Vincent and Kenfack, 2008]. Since then, multiple synchronization strategies have been proposed to deal with the problem in many aspects. An important still open problem is to synchronize chaotic systems through a driving signal in a non-master-slave framework. An example of this kind of strategies can be found in [Hong, Qin, and Chen, 2001] in which authors develop an adaptive synchronization approach or in [Sarasola *et al.*, 2003] where a synchronization technique was proposed using a linear feedback coupling. A robust synchronization strategy was proposed in [Alvarez *et al.*, 2010] using sliding modes control. Lately, synchronization problem has been naturally extended to complex networks [Duan, Chen, and Huang, 2007], consensus and pinning are significant examples [Olfati-Saber, Alex Fax, and Murray, 2007]. In this paper we deal with synchronization problem for a class of chaotic systems known as full relative degree systems, we propose a nonlinear feedback synchronization law for state synchronization of systems with parameter mismatch and non-identical possibly nonsmooth systems.

The paper is organized as follows. Section 2 states the synchronization problem and defines the class of chaotic systems under consideration. Section 3 gives conditions and formally states the proposed synchronization strategy. In section 4 some results are shown on synchronization strategy applied to well-known chaotic systems. Finally we conclude with a discussion and some future work.

2 Systems under Consideration

Consider systems of the form

$$\begin{aligned} \dot{x}^j &= F^j(x) + G^j(x)u \\ y^j &= H^j(x) \end{aligned} \quad (1)$$

for $j = 1, 2$ and vector fields $F^j(x)$ and $G^j(x)$, states $x^j \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $F^j(x)$ possibly describing chaotic behavior.

Definition 1. (*State synchronization*) *Let a couple of systems of the form (1) to have chaotic behavior and $F^j(x)$ possibly being a nonsmooth vector field for some i. Both systems state-synchronized if*

$$\lim_{t\to\infty} |x^1 - x^2| = 0 \tag{2}$$

Thus, the state-synchronization error can be expressed as

$$e(t) = x^1(t) - x^2(t) \tag{3}$$

Systems under consideration are in the so-called normal form, and for a system like (1) to be written in normal form, it is needed that it has full relative degree, we give the following definitions.

Definition 2. (*Full relative degree*) *System (1) is said to have* full relative degree *if*
i) $L_G L_F^k H(x) = 0$, $k = 0, 1, ..., n-2$
ii) $L_G L_F^{n-1} H(x) \neq 0$
where $L_F H(x) = \frac{\partial H(x)}{\partial x} F(x)$ *denotes the Lie derivative of H(x) along F(x),* $L_F^0 H(x) = H(x)$.

Let us now consider systems that are written in normal form

Theorem 1. *The following system has full relative degree*

$$\begin{aligned} \dot{x}_i &= x_{i+1}, (i = 1, ..., n-1) \\ \dot{x}_n &= f(x) + g(x)u \\ y &= x_1 \end{aligned} \tag{4}$$

for $f(x)$ and $g(x) \neq 0$ scalar functions and $x = \left[x_1, x_2, ..., x_n\right]^T$ the state vector.

Proof. Let us define

$$F(x) = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ f(x) \end{bmatrix}; G(x) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ g(x) \end{bmatrix} \tag{5}$$

and $H(x) = h(x) = x_1$, in order to prove condition $i)$ from definition 2, we find

$$\begin{aligned} L_F H(x) &= x_2 \\ L_F^2 H(x) &= x_3 \\ &\vdots \\ L_F^{n-2} H(x) &= x_{n-1} \\ L_G L_F^{n-2} H(x) &= 0 \end{aligned}$$

Condition $ii)$ is proven as follows

$$\begin{aligned} L_F^{n-1} H(x) &= x_n \\ L_G L_F^{n-1} H(x) &= g(x) \neq 0 \end{aligned}$$

Corollary 1. *Let a system like (1) to have full relative degree. Then it can be rewritten in its normal form through the following coordinates change $x^j = \varphi(x)$ for system j.*

$$x_i^j = \varphi_i(x) = L_F^{i-1} F H(x) \tag{6}$$

for $i = 1, 2, ..., n$.

For proof of Corollary 1 please refer to [Isidori, 2000] and [Nijmeijer and Van Der Schaft, 1996]. We cope with nonlinear chaotic full relative degree systems possibly containing nonsmooth dynamics.

3 Synchronization Strategy

Consider a couple of systems of the form:

$$\begin{aligned} \dot{x}_i^j &= x_{i+1}^j, (i = 1, ..., n-1) \\ \dot{x}_n^j &= f^j(x^j) + g^j(x^j)u \\ y^j &= x_1^j \end{aligned} \tag{7}$$

for $j = 1, 2$, $x_i \in \mathbb{R}^n$, $i = 1, 2..., n$, and f^j and g^j scalar functions. Let the state synhcronization error be defined by (3), then dynamics for the synchronization error can be expressed by

$$\begin{aligned} \dot{e}_i &= e_{i+1}, (i = 1, ..., n-1) \\ \dot{e}_n &= f(x^1, x^2) + g(x^1, x^2)u \\ y &= e_1 \end{aligned} \tag{8}$$

where $f(x^1, x^2) = f^1(x^1) - f^2(x^2)$ and $g(x^1, x^2) = g^1(x^1) - g^2(x^2)$. The synchronization error dynamics is again in normal form.

3.1 Controllabillity for Synchronization: Synchronizability

A system like (8) with functions involving states from different systems should be analized for assuring controllability. For error synchronization dynamics, let us now define:

$$F(e) = \begin{bmatrix} e_2 \\ e_3 \\ \vdots \\ f(x^1, x^2) \end{bmatrix}; G(e) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ g(x^1, x^2) \end{bmatrix} \tag{9}$$

and extended state $x = \left[(x^1)^T \ (x^2)^T\right]^T$ Let us give some preliminary definitions [Nijmeijer and Van Der Schaft, 1996].

Definition 3. *(Lie Brackets) Consider vector fields $F(e)$ and $G(e)$ in $\mathbb{R}^n$ according to (5). Then the* Lie Bracket *operation generates a new vector field:*

$$[F,G] = \frac{\partial G}{\partial x}F - \frac{\partial F}{\partial x}G. \qquad (10)$$

Higher order Lie Brackets can be obtained as

$$\begin{aligned}(ad_F^1, G) &= [F, G]\\ (ad_F^2, G) &= [F, [F, G]]\\ &\vdots\\ (ad_F^k, G) &= [F, (ad_F^{k-1}, G)].\end{aligned}$$

Thus, we introduce the following proposition

Proposition 1. *(Synchronizability) The system defined by (8), is locally accesible about $e = 0$ if the accesibility distribution $\mathcal{C}$ spans n space. $\mathcal{C}$ is defined by:*

$$\mathcal{C} = [g, [ad_F^k g]] \qquad (11)$$

for $k = 1, 2, ...$, i.e. distribution $\mathcal{C}$ is involutive.

This will be considered as the synchronizability condition further on.

3.2 The Synchronization Law

Let us define the error vector $e = \begin{bmatrix} e_1 \; e_2 \; ... \; e_n \end{bmatrix}^T$, and some constant vector $a = \begin{bmatrix} -a_n \; -a_{n-1} \; ... \; -a_1 \end{bmatrix}^T$, such that the polynomial $\lambda^n + a_1\lambda^{n-1} + ... + a_n$ is strictly Hurwitz. Then, provided *synchronizability* conditions a synchronization control law can be stated for (8) in the following form

$$u = \frac{a^T e - f(x^1, x^2)}{g(x^1, x^2)} \qquad (12)$$

Application of (12) over (8) leads to the following linear, asymptotically stable dynamics:

$$\begin{aligned}\dot{e}_i &= e_{i+1}, (i = 1, ..., n-1) \qquad (13)\\ \dot{e}_n &= a^T e\\ y &= e_1\end{aligned}$$

which can be expressed as $\dot{e} = Ae$ whose equilibrium point $e = 0$ is asymptotically stable, meaning that synchronization error vanishes when $t \to \infty$.

4 Simulation Results

In the following subsections we will describe some simulation results for synchronization law proposed applied to several well-known chaotic systems.

4.1 Continuous Case

For simulation purposes we have chosen a slightly different system from one of the circuits proposed in [Sprott, 2000] which may exhibit chaotic behavior for certain set of parameters. Circuit chosen is of the form:

$$\dddot{x} = -\mu\ddot{x} + \dot{x}^2 - x + \beta u \qquad (14)$$

which exhibits chaotic behavior for $\mu = -2.017$ and $\beta = 0$, Lyapunov exponents are $(0.055, 0, -2.072)$. It is easy to show that (14) has full relative degree by selecting output $y = x$, thus a couple of Sprott-like circuits can be written as follows

$$\begin{aligned}\dot{x}_1^j &= x_2^j \qquad (15)\\ \dot{x}_2^j &= x_3^j\\ \dot{x}_3^j &= -x_1^j + (x_2^j)^2 + \mu_j x_3^j + \beta_j u\\ y_j &= x_1^j\end{aligned}$$

for $j = 1, 2$. Let us define the error $e_1 = x_1^1 - x_1^2 = y_1 - y_2$, then the error dynamics are depicted as

$$\begin{aligned}\dot{e}_1 &= e_2 \qquad (16)\\ \dot{e}_2 &= e_3\\ \dot{e}_3 &= -e_1 + e_2(e_2 + 2x_2^2) + \mu_1 x_3^1 - \mu_2 x_3^2 + \tilde{\beta}u\end{aligned}$$

for $\tilde{\beta} = \beta_1 - \beta_2$ and $\beta_1 \neq \beta_2$. Systems (15) show chaotic behavior for $\mu_j = -2.017$, $\beta_j = 0$, $(j = 1, 2)$ and initial conditions $x^j(0) = \begin{bmatrix} 0 \; 0 \; 1 \end{bmatrix}^T$.

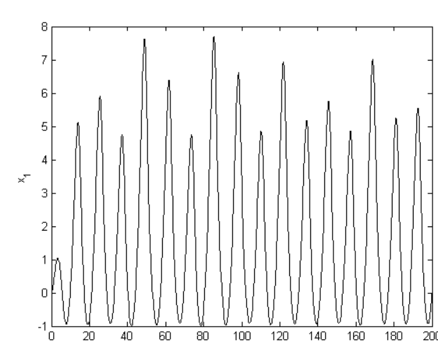

Figure 1. Time evolution for x_1 variable of system (15) for $\mu_1 = -2.017$

Clearly, if $\mu_1 = \mu_2$ and $x^1(0) = x^2(0)$, circuits are perfectly synchronized. In order to prove our proposal, we use $\mu_1 = -2.017$ and $\mu_2 = -2.02$. Fig. 1 shows solution for x_1 for system 14. Although this parameter mismatch appears to be nonsignificant, time responses for both systems diverge considerably as shown in Fig. 2.

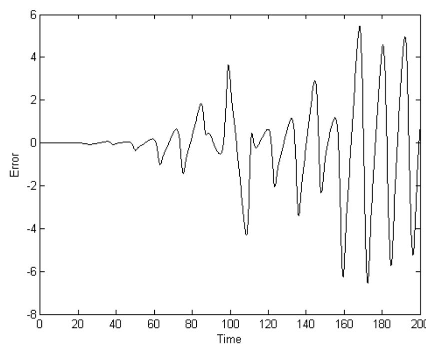

Figure 2. Error for x_1 of systems (15) with respect to time considering parameter mismatch

In order to synchronize the states (and outputs) of both systems $e \to 0$, we propose the synchronization law:

$$u = \frac{1}{\tilde{\beta}}[a^T e + e_1 - e_2(e_2 + 2x_2^2) - \mu_1 x_3^1 + \mu_2 x_3^2] \quad (17)$$

For demonstration purposes, let us now choose $a = \begin{bmatrix}-6 & -11 & -6\end{bmatrix}^T$ for behavioral modes located at $\lambda = 1, 2, 3$. For this synchronization law, an asymptotically stable equilibrium point is expected for the closed loop system.

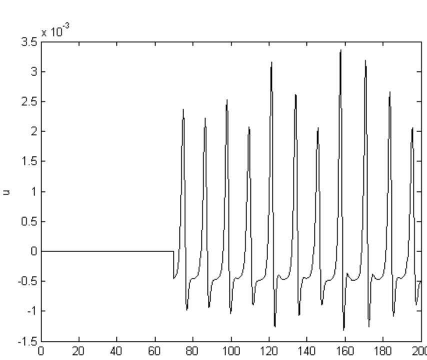

Figure 3. Synchronization law for (15)

Synchronization law was applied at $t = 70s$ and its time evolution is depicted in Fig. 3. Synchronization error is shown in Fig. 4.

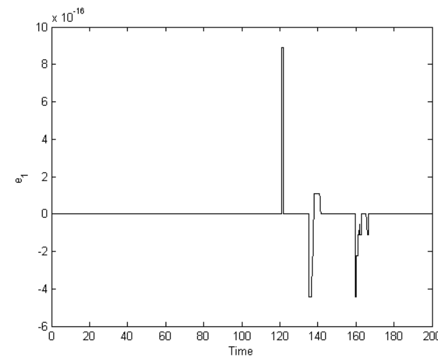

Figure 4. Synchronization error for systems (15)

4.2 Discontinuous Case

Second example under consideration is a discontinuous chaotic system based on [Sprott, 2000] of the form

$$\dddot{x} = -\mu\ddot{x} - \dot{x} - x + sgn(x) + \beta u \quad (18)$$

Such system exhibits chaotic behavior for $\mu = 0.5$ and $\beta = 0$ with Lyapunov exponents $(0.152, 0, -0.652)$. A projection of the chaotic attractor onto the $x_1 x_2$-plane is shown in Fig. 5.

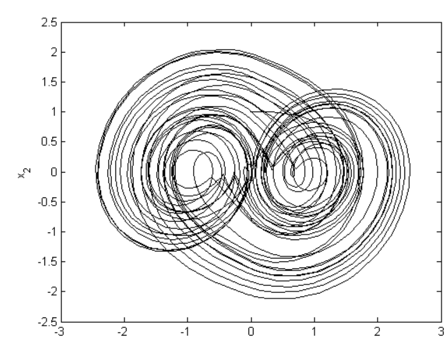

Figure 5. Projection of chaotic atractor for (19) onto the $x_1 - x_2$ plane

Let us consider a couple of non-identical non-smooth systems in the following form:

$$\begin{aligned} \dot{x}_1^j &= x_2^j \\ \dot{x}_2^j &= x_3^j \\ \dot{x}_3^j &= -x_1^j - x_2^j - \mu_j x_3^j + sgn(x_1^j) + \beta_j u \\ y^j &= x_1^j \end{aligned} \quad (19)$$

for $j = 1, 2$, and

$$sgn(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x > 0 \end{cases}$$

Let us consider a parameter mismatch, $\mu_1 = 0.5$ and $\mu_2 = 0.49$. Error in x_1 between both systems is shown in Fig. 6.

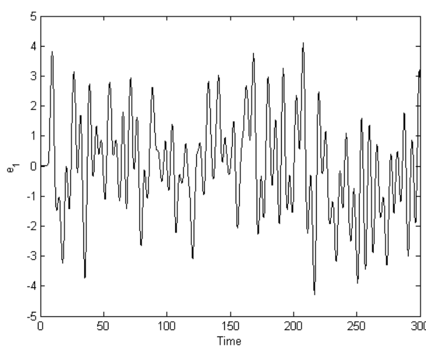

Figure 6. Error for variable x_1 of systems (19) with parameter mismatch

Synchronization error can be defined as $e_1 = y^1 - y^2 = x_1^1 - x_1^2$ in such a way that synchronization error dynamics can be cast in the following form

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= e_3 \\ \dot{e}_3 &= -e_1 - e_2 - \mu_1 x_3^1 + \mu_2 x_3^2 + \tilde{\beta} u \end{aligned} \tag{20}$$

provided that $\tilde{\beta} = \beta_1 - \beta_2 \neq 0$. For simplicity we have chosen $\beta_1 = 2$ and $\beta_2 = 1$. Then, a suitable synchronization law can be stated as

$$u = \frac{1}{\tilde{\beta}}[a^T e + e_1 + e_2 + \mu_1 x_3^1 - \mu_2 x_3^2] \tag{21}$$

Again we may choose a^T as in previous example in order to force the closed loop system to be asymptotically stable. Synchronization law was applied at $t = 5s$ and it is shown in Fig. 7.

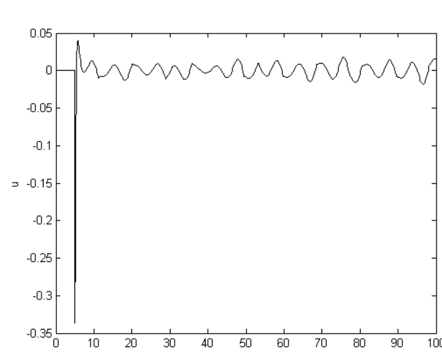

Figure 7. Synchronization law for example (2)

Synchronization error for variables x_1 of both systems is shown in Fig. 8.

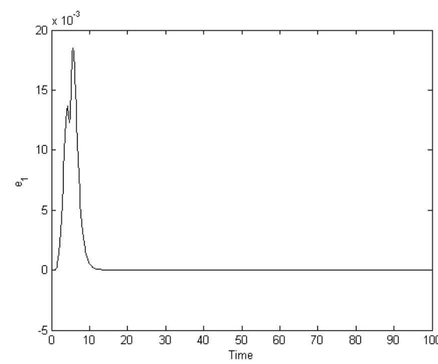

Figure 8. Synchronization error for example (2)

4.3 Non-Identical Systems

In this section we deal with the synchronization of a couple of different systems applying the proposed strategy. We choose a normal-form Lorenz system and a Sprott system. Lorenz dynamics can be described as

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1) \\ \dot{x}_2 &= \rho x_1 - x_2 - x_1 x_3 \\ \dot{x}_3 &= x_1 x_2 - \beta x_3 + u \\ y &= x_1 \end{aligned} \tag{22}$$

It is well known that for parameters $\sigma = 10$, $\rho = 28$ and $\beta = 8/3$ it shows chaotic behavior. Clearly, system (22) is not in the form (7), but it can be expressed in normal form by using coordinates transformation shown in (6), i.e. $x^1 = \varphi(x)$. Thus, for the Lorenz system, new coordinates are:

$$\begin{aligned} \varphi_1(x) &= = x_1 \\ \varphi_2(x) &= \sigma(x_2 - x_1) \\ \varphi_3(x) &= -\sigma^2(x_2 - x_1) + \sigma(\rho x_1 - x_2 - 20 x_1 x_3) \end{aligned} \tag{23}$$

leading to the following dynamics

$$\begin{aligned} \dot{x}_1^1 &= x_2^1 \\ \dot{x}_2^1 &= x_3^1 \\ \dot{x}_3^1 &= f^1(x^1) + g^1(x^1)u \\ y &= x_1^1 \end{aligned} \tag{24}$$

for

$$f^1(x^1) = (\rho - 1)\sigma x_2^1 - (\sigma + 1)x_3^1 - (x_1^1)^2(x_2^1 + \sigma x_1^1) - (\beta x_1^1 - x_2^1)\left[\frac{\sigma(1-\rho)x_1^1 + (\sigma+1)x_2^1 + x_3^1}{x_1^1}\right] \quad (25)$$

and

$$g^1(x^1) = -\sigma x_1^1 \quad (26)$$

System (24) is now in normal form. Let us now consider another system to synchronize with:

$$\dddot{x} = -\mu\ddot{x} - \dot{x} + x - x^3 + \beta u \quad (27)$$

Equation (27) represent an electronic circuit described in [Sprott, 2000] and exhibits chaotic behavior for $\mu = 0.39$ and $\beta = 0$. Normal form for the system is shown below

$$\begin{aligned} \dot{x}_1^2 &= x_2^2 \\ \dot{x}_2^2 &= x_3^2 \\ \dot{x}_3^2 &= -\mu x_3^2 - x_2^2 + x_1^2 - (x_1^2)^3 + \beta u \\ y^2 &= x_1^2 \end{aligned} \quad (28)$$

Synchronization error is defined as $e_1 = y^1 - y^2 = x_1^1 - x_1^2$, leading to the following synhronization error dynamics

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= e_3 \\ \dot{e}_3 &= f(x^1, x^2) + g(x^1, x^2)u \end{aligned} \quad (29)$$

with $f(x^1, x^2) = f^1(x^1) - f^2(x^2)$ and $g(x^1, x^2) = \beta - \sigma x_1^1$. Output error between both systems is shown in Fig. 9.

Thus, a synchronization law can be stated in the form

$$u = \frac{a^T e - f(x^1, x^2)}{g(x^1, x^2)} \quad (30)$$

For demonstration purposes, we apply synchronization law at $t = 100s$, Fig. 10 shows time response for e_1.

5 Concluding Remarks and Future Work

The proposed strategy has shown successful synchronicity for a class of chaotic systems and it can be implemented for systems depicting some nonsmoothness features. It has been implemented for systems not in normal form but fully linearizable with coordinates transformation. Future work is focused in synchronization for more than two systems, and on robust synchronization using Internal model controller.

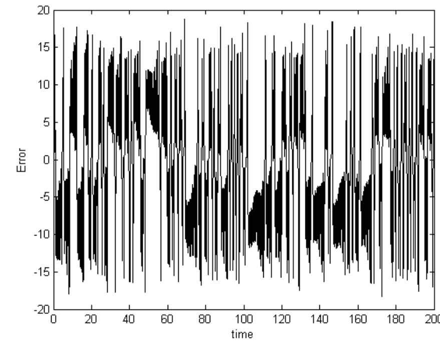

Figure 9. Time evolution for error between Lorenz and Sprott systems in open loop

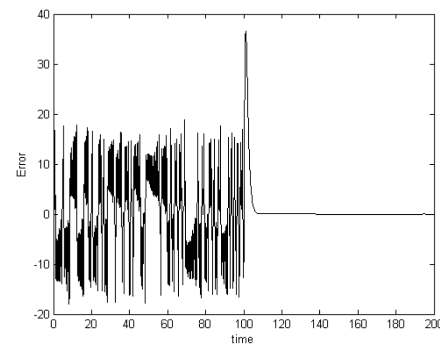

Figure 10. Error e_1 between (24) and (28) with synchronization law applied at $t = 100s$.

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