A General Criterion for Control of Chaos with Feedback

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It is a well-known fact that many processes occurring in nature exhibit chaotic behavior. In many practical applications, such behavior is highly undesirable. It is, therefore, worthwhile to determine chaos suppression criteria, applicable to as wide a variety of systems as possible.

In this paper, we consider a two-dimensional dynamical system described by:

$$\dot{z} = f(z) + \varepsilon[a(z)\sin\omega t + u(z)]\,. \tag{1}$$

with $z=[x,y]^T$. We assume that when $\varepsilon = 0$ the system has two saddles (x_1,y_1) and (x_2,y_2) connected by a separatrix Γ described, for the sake of certainty, by the equation $y = \phi(x)$. The results apply as well with some minor modifications to the case of looping separatrix.

The question to be answered is formulated as follows: Find the conditions on the feedback $u(z)$ that guarantee the absence of chaos in the system regardless of the frequency ω. For the two-dimensional systems, the investigation of chaos can be reduced to investigation of homoclinic bifurcation, which, in turn, can be accomplished by using the Melnikov's method. Furthermore, it turns to be possible to express the result in terms of line integrals along the separatrix, which does not require solving the system explicitly, usually, an impossible task.

The result is stated as a Theorem.

THEOREM. Suppose that the system

$$\dot{z} = f(z)$$

has a separatrix $\Gamma : y = \phi(x)$ connecting the two saddles (x_1,y_1) and (x_2,y_2) .Suppose further that the following inequality holds:

$$\sqrt{\left|x_2 - x_1\right|\int_{x_1}^{x_2} g(x)dx} < \int_\Gamma dz \wedge u(z)\,, \tag{2}$$

where

$$g(x) = a_2^2(x,\phi(x)) + a_1^2(x,\phi(x))[\phi'(x)]^2 - 2a_1(x,\phi(x))a_2(x,\phi(x))\phi'(x)$$

with indices denoting the components of the vector $a(z)$.

Then for sufficiently small ε the system (1) has no homoclinic bifurcations for any value of ω.

The proof of this theorem involves the use of the Melnikov's method. It is then shown that the integrals can be reduced to line integrals along the separatrix and then to quadratures. Further, the use is made of the integral identity applicable to any two functions $b(x)$ and $c(x)$:

$$\left[\int_A^B b(x)c(x)dx\right]^2 = \int_A^B b^2(x)dx\int_A^B c^2(x)dx - \frac{1}{2}\int_A^B\int_A^B [b(x)c(y) - b(y)c(x)]^2\,dxdy\,.$$

This further reduces the integrals to the forms described in the conditions of the Theorem. The desired inequality is obtained by bearing in mind that the absolute value of sine and cosine does not exceed one.

It is worth noting that the inequality in the conditions of the Theorem does not include explicitly the solution of the unperturbed system. All that is needed is the equation for the separatrix, which can often be obtained analytically, especially if the system is Hamiltonian. The latter, however is not required for the applicability of the Theorem. The result has the same form regardless of whether the system is Hamiltonian or not.

For the special case of the single second-order differential equation

$$\ddot{x} = f(x, \dot{x}) + \varepsilon[a(x, \dot{x}) \sin \omega t + u(x, \dot{x})],$$

it is possible to further simplify the result. Let the separatrix Γ be described by the equation $\dot{x} = \phi(x)$. Then the inequality (2) takes the form

$$\sqrt{|x_2 - x_1| \int_{x_1}^{x_2} a^2(x, \phi(x)) dx} < \int_{x_1}^{x_2} u(x, \phi(x)) dx \qquad \textbf{(3)}$$

If, in addition, the function *a(x)* does not depend on the derivative of the unknown function, the feedback *u* can be selected in a similar way, and the Theorem can be applied without knowing the equation for the separatrix – only the coordinates of the saddle points are needed. Indeed, the inequality (3) reduces to:

$$\sqrt{|x_2 - x_1| \int_{x_1}^{x_2} a^2(x) dx} < \int_{x_1}^{x_2} u(x) dx \qquad \textbf{(4)}$$

The conclusion, therefore, is that the problem of controlling chaos can be solved for a certain class of systems without having to solve the differential equations explicitly.