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PERIOD DOUBLING BIFURCATION IN DISCRETE PHASE-LOCKED LOOPS

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Abstract

Bifurcation theory is very important in digital phase-locked loops (DPLLs) which are frequently encountered in radio engineering and communication and have been used during 60 years. Calculation of bifurcation values of the parameters are very important problem for analysis of working regimes of DPLLs.

Mathematical model of discrete digital phase-locked loop with sinusoidal characteristic of phase discriminator is considered. The Feigenbaum's effect for nonunimodal maps which describe such DPLL is investigated by theoretical approach and numerical calculations. Bifurcations parameters of period doubling bifurcation are calculated.

Key words

Phase-locked loops, period-doubling bifurcations, Feigenbaum's numbers.

1 Introduction

The study of limit cycles for discrete dynamical systems has a long history and important applications in various areas of research. The following nonlinear difference equation of the first order:

$$x(t+1) = rx(t)(1-x(t)), \quad t \in N, \quad r > 0, \quad (1)$$

was introduced by Pierre Verhulst in 1845 as a mathematical model of population dynamics within a closed environment that takes into account internal competition [Schuster, 1984]. Logistic equation (1), which can be generalized to the form

$$x(t+1) = f(x(t)), \quad x \in R, \quad t \in N, \quad (2)$$

has an extremely complex limiting structure of solutions and was intensively studied in the second part of the 20^{th} century [Sharkovsky, 1995; Li & Yorke, 1975; Sharkovsky *et al.,* 1997; May, 1976; Metropolis, M. Stein & P. Stein, 1973; Feigenbaum, 1978; Weisstein, 1999]. In particular, in equation (1), period-doubling bifurcations were discovered.

Surprisingly, while solutions to a linear multidimensional discrete equation

$$x(t+1) = Ax(t), \quad x \in R^n, \quad t \in N,$$

and its continuous analogue

$$\dot{x} = Ax(t), \quad x \in R^n, \quad t \in R$$

(where A is a constant $n \times n$ matrix), to a large extent, possess similar behavior, solutions to equation (2) and its continuous one-dimensional analogue

$$\dot{x} = f(x(t)), \quad x, t \in R,$$

bear qualitatively different structure (see for example [Leonov & Seledzhi, 2002; Neittaanmäki & Ruotsalainen, 1985; Keller, 1977; Marsden & McCracken, 1976]).

One of examples of a nonlinear difference equation which are important in applications is the following equation of the first order:

$$x(t+1) = x(t) - \alpha \sin x(t) + \gamma, \quad t \in N, \quad (3)$$

where α and γ are nonnegative parameters. Over the last 40 years, many authors conducted rigorous studies of equation (3) both as a pure mathematical object [Arnold, 1983; Jakobson, 1971] and as a mathematical model of a phase-locked loop [Osborne, 1980; Gupta, 1975; Lindsey, 1972; Lindsey & Chie, 1981].

Equation (3) with $\gamma = 0$ describes a wide class of digital phase-locked loops (DPLLs) with the sinusoidal characteristic of phase detector [Banerjee & Sarkar, 2005-2008; Leonov *et al.,* 1992; Leonov *et al.,* 1996; Leonov, 2001; Leonov, 2002; Leonov & Seledzhi, 2005]. Osborne [Osborne,1980] pioneered in the use of exact methods, such as the Contraction Mapping Theorem, applicable to the direct study of nonlinear effects in this system. However, even the exact methods used by Osborne, while revealing what then appeared as multiple cycle slipping, followed by a divergent behavior of iterations, did not allow for a precise interpretation of nonlinear effects discovered at transition from global asymptotic stability to chaos through period doubling bifurcations [Leonov & Seledzhi, 2005].

The research and development of mathematical theory of DPLLs for array processors are commonly used in radio engineering, communication, and computer architecture [Banerjee & Sarkar, 2005–2008; Zoltowski, 2001; Mannino *et al.,* 2006; Hussain & Boashash, 2002; Kudrewicz & Wasowicz, 2007; Gardner, 1966; Lindsey, 1972; Lindsey & Chie, 1981; Leonov, Reitmann & Smirnova, 1992; Kuznetsov, Leonov & Seledzi, 2006; Leonov, Ponomarenko & Smirnova, 1996; Lapsley *et al.,* 1997; Kroupa, 2003; Best, 2003; Abramovitch, 2002]. For example, such digital control systems exhibit high efficiency in eliminating clock skew - an undesirable phenomenon arising in parallel computing [Leonov & Seledzhi, 2002; Leonov & Seledzhi, 2005]. DPLLs have gained widespread recognition and preference over their analog counterparts because of their ability to deal with this phenomenon effectively. From a mathematical perspective, this gives rise to a problem associated with the analysis of global stability of nonlinear difference equations that serve as mathematical models of discrete phase-locked loops [Leonov, 2001]; that is, the analysis can be formulated in terms of parameters for such systems.

The present paper is devoted to study of bifurcations of discrete system (3) with $\gamma = 0$ and to computation of bifurcation parameters. Using the qualitative theory of dynamical systems, special analytical methods, and advanced mathematical packages designed to work with long numbers allowed us to succeed in computation of bifurcation values of parameter of the investigated system. The first 14 bifurcation values are calculated with good accuracy. Also it is shown that for the obtained bifurcation values of investigated system, which is not a unimodal map, an effect of convergence similar to famous Feigenbaum's effect is observed.

2 Digital phase-locked loops

The nonlinear dynamics of different nonlinear electronic systems is studied by researchers for at least three decades. Occurrence of such complex behaviors as bifurcation, chaos, intermittency etc. in electronic systems has been revealed from these studies [Banerjee & Sarkar, 2008; Kilias *et al.,* 1995; Chen, Chau & Chan, 1999; Giannakopoulos & Deliyannis, 2005]. In addition, control chaos and bifurcation in electronic circuits and systems is an active area of research [Chen, Hill & Yu, 2003; Collado & Suarez, 2005].

By controlling chaos and bifurcation, one can suppress chaotic behavior where it is unwanted (e.g. in power electronics and mechanical systems). On the other hand, in electronic systems one can harness the richness of chaotic behavior in chaos based electronic communication system. Possibility of exploiting the chaotic signal in chaos based secure communication system has boosted up the research on the chaotic dynamics of electronic circuits and systems [Kennedy, 2000].

Owing to potential application in synchronous communication system and rich nonlinear dynamical behavior, PLL is probably the most widely studied system among all electrical systems [Gardner, 1966; Kudrewicz & Wasowicz, 2007]. At the advent of digital communication systems, DPLLs have rapidly replaced the conventional analog PLLs because they overcome the problems of sensitivity to DC drift, periodic adjustment, and the building of higher order loops [Lindsey & Chie, 1981].

DPLLs are widely used in frequency demodulators, frequency synthesizers, data and clock synchronizers, modems, digital signal processors, and hard disk drives to name a few [Banerjee & Sarkar, 2008; Zoltowski, 2001; Mannino *et al.,* 2006]. A DPLL is a discrete time nonlinear feedback controlled system whose nonlinear behavior is complicated, and it poses exact solutions only in particular cases. To understand the complete behavior of a DPLL, it is necessary to resort to modern nonlinear dynamical tools of bifurcation and chaos theories. Also in this regard bifurcation control of DPLL has not been explored yet. The study of nonlinear dynamics of DPLL has two fold applications. First, using the insight of nonlinear behaviors of DPLLs, an optimum DPLL system can be designed. Second, by characterizing the chaos from DPLLs, one can explore the possibility of using DPLLs in chaos based secure electronic communication systems. Thus, research of nonlinear dynamics of DPLLs is an important problem.

There are different types of DPLLs: positive zero crossing DPLLs (ZC1-DPLL) [Bernstein, Liberman & Lichtenberg, 1989; Banerjee & Sarkar, 2005; Banerjee & Sarkar, 2005[1]; Banerjee & Sarkar, 2008; Leonov & Seledzhi, 2005], uniform sampling DPLL [Zoltowski, 2001], bang-bang DPLLs [Dalt, 2005], and tanlock DPLLs [Hussain & Boashash, 2002].

In the present paper, we will consider a discrete dynamical system which describes the nonlinear dynamics of dual sampler based zero crossing DPLLs (ZC2-DPLL). Unlike ZC1-DPLL, in a ZC2-DPLL sampling is done at the positive and negative zero crossings of the input signal [Banerjee & Sarkar, 2008[1]]. For this particular sampling technique, it has a wide frequency acquisition range in comparison with a ZC1-DPLL, and

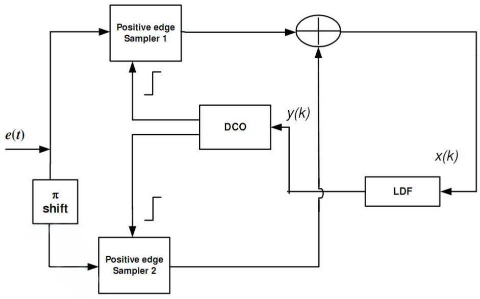

Figure 1. Functional block diagram of a $ZC_2 - DPLL$, [Banerjee & Sarkar, 2008].

that is why ZC2-DPLLs have drawn the attention of researchers for a long time [Majumdar, 1979; Frias & Rocha, 1980; Banerjee & Sarkar, 2006].

Following [Banerjee & Sarkar, 2008[1]; Majumdar, 1979; Leonov & Seledzhi, 2002], we formulate the equation of DPLL. Fig. 1 shows the block diagram of a ZC2-DPLL. It contains two positive edge triggered samplers. Input signal is fed directly into sampler-1, and a π shifted version of input signal is fed into sampler-2. Let $e(t)$ be the noise-free analog input signal to the system with a phase angle $\theta_i(t)$ relative to the loop DCO phase. Then the system equation can be written as

$$e(t) = A_0 sin\left[\omega_0 t + \theta_i(t)\right],$$

where $\theta_i(t) = (\omega_i - \omega_0)t + \theta_0$. Here A_0 is the amplitude, and ω_i and θ_0 are the angular frequency and phase of the input signal, respectively. ω_0 is the nominal angular frequency of the DCO having time period T. Writing the sampled version of $e(t)$ at the kth sampling instant (SI) $t(k)$ as $x(k)$, one can write the output signals of sampler-1 and sampler-2, respectively, as follows [Majumdar, 1979]:

$$x_1(k') = A_0 sin[\omega_0 t(k') + \theta_i(k')],$$
$$x_1(k'') = A_0 sin[\omega_0 t(k'') + \theta_i(k'') - \pi],$$

where $k' = 2k$, $k'' = (2k+1)$, $k = 0, 1, 2, 3 \ldots$.

Here sampling instants (SIs) are occurring at the end of each half period of the DCO.

The sequence $x(k)$, $k = 0, 1, 2, \ldots$, is filtered digitally by a loop digital filter (LDF). The transfer function of LDF in a first order loop is written as a constant gain G1 (s/volt). The LDF output sequence is given by $y_k = G_1 x(k)$. The sequences $y(k)$ are used to control the next half period of the DCO. The kth half period $T'(k)$ of DCO can be written as

$$T'(k) = T(k)/2 = t(k+1) - t(k),$$

in terms of the kth and $(k+1)$st SIs, respectively. DCO period $T(k+1)$ at that instant is governed by the relation

$$T(k+1) = \frac{T}{2} - y(k).$$

In case $t(0) = 0$, we get

$$t(k) = \frac{kT}{2} - \sum_{i=0}^{k-1} y(i).$$

Thus, the sampler output at the kth instant is

$$x(k) = A_0 sin[\phi(k)],$$

where

$$\phi(k) = \theta_i(k) - \omega_0 \sum_{i=0}^{k-1} y(i)$$

is the phase error between the input signal and the DCO output at $t(k)$.

Then, the equation for the phase of the ZC2-DPLL can be written as

$$\phi(k+1) = \phi(k) + \pi(z-1) - \frac{1}{2} z K_1 sin\phi(k), \quad (4)$$

where z has been substituted in place of (ω_i/ω_0) and $K_1 = A_0\omega_0 G_1$ is the closed loop gain of ZC2-DPLL.

3 Analytical investigation

In engineering DPLL's practice, the case of initial frequency of master and local generators coincidence is very important [Banerjee & Sarkar, 2008[1]; Leonov & Seledzhi, 2002]. For this instance, in (4),

$$z = (\omega_i/\omega_0) = 1,$$

and the equation for DPLL can be given by

$$\sigma(t+1) = \sigma(t) - r \sin\sigma(t), \quad t \in N, \quad (5)$$

where $r = \dfrac{1}{2} A_0 \omega_0 G_1$ is a positive number [Leonov & Seledzhi, 2002].

One of the first works dedicated to analysis of system (5) belongs to Osborne. In [Osborne, 1980], was considered the algorithm of investigation of periodic solutions, and it was shown that even in a simple discrete model of PLL, the bifurcation phenomenon involved to arising of new stable periodical solutions and to changing of their period are observed. Later, in the papers [Belykh & Maksakov, 1979; Belykh & Lebedeva, 1983] for such systems, a model of transition to chaos through a cascade of period-doubling bifurcations was considered. Association and development of these ideas in the works [Leonov & Seledzhi, 2002; Leonov & Seledzhi, 2005] has allowed to construct bifurcation tree of transition to chaos through a cascade of period doubling (Fig. 2).

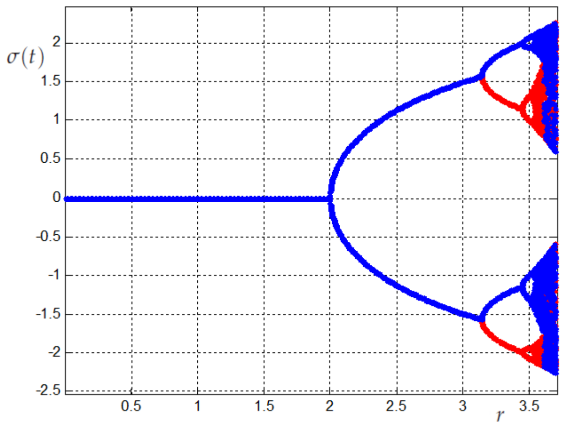

Figure 2. Seledzhi's bifurcation tree.

In [Leonov & Seledzhi, 2002], it was proved that system (5) is globally asymptotically stable for $r \in (0, 2)$.

Following the works [Osborne, 1980; Leonov & Seledzhi, 2002], let us consider behavior of periodic solutions of system (5) for $r \geq 2$.

Let $r \in (2, r_1)$, where r_1 is the root of the equation

$$\sqrt{r^2 - 1} = \pi + \arccos \frac{1}{r}.$$

Then the following theorem takes place.

Theorem. *If* $r \in (2, r_1)$ *and* $\sigma(0) \in [-\pi, \pi]$, *then* $\sigma(t) \in [-\pi, \pi]$ *for all* $t = 1, 2, \ldots$.

This theorem determines system (5) as a map of the interval $[-\pi, \pi]$ into itself for $r \in (2, r_1)$.

For $r < 2$, equation (5) is global asymptotically stable: $\sigma(t)$ with any initial conditions $\sigma(0)$ aspire to the states of equilibrium $\sigma = 2\pi j$, $j \in Z$, $t \mapsto +\infty$.

The value $r = 2$ is the first point of bifurcation. For $r > 2$ all the stationary points become Lyapunov unstable. There is an asymptotically stable, symmetric with respect to $\sigma = 0$, solution with period 2 for $r \in (2, \pi)$.

Let $\sigma(t + 1) = -\sigma(t)$, then $2\sigma(t) = r \sin(\sigma(t))$, and the symmetric solutions of equation (5) with period 2 have the properties:

$$\sigma(2j) = \sigma(0), \quad j \in Z,$$
$$\sigma(2j + 1) = -\sigma(0), \quad j \in Z,$$

where the initial conditions of $\sigma(0)$ satisfy equality

$$2\sigma(0) = r \sin(\sigma(0)). \tag{6}$$

Equation (6) in the interval $[-\pi, \pi]$ has two roots for all $r > 2$: $\sigma(0)$ and $-\sigma(0)$.

For $r \in (\pi, \beta)$, where $\beta = \sqrt{\pi^2 + 2} \approx 3.445229$, there are two asymptotically stable solutions with period 2 which satisfy the relation

$$\sigma(t + 1) = \sigma(t) \pm \pi.$$

From here it follows that

$$\pi = r \sin(\sigma(t)).$$

Then, the first periodic solution of period 2 for $r \in (\pi, \beta)$ has the properties

$$\sigma(2j) = \sigma(0), \quad \forall j \in Z,$$
$$\sigma(2j + 1) = \sigma(0) - \pi, \quad \forall j \in Z,$$

where the initial conditions of $\sigma(0)$ satisfy the equality

$$\sin(\sigma(0)) = \frac{\pi}{r}. \tag{7}$$

The second periodic solution of period 2 for $r \in (\pi, \beta)$ has the properties

$$\sigma(2j) = \sigma(0), \quad \forall j \in Z,$$
$$\sigma(2j + 1) = \sigma(0) + \pi, \quad \forall j \in Z,$$

where the initial conditions of $\sigma(0)$ satisfy the equality

$$\sin(\sigma(0)) = -\frac{\pi}{r}. \tag{8}$$

Equations (7), (8) have on $[-\pi, \pi]$ two roots for all $r > 2$.

According to analytical investigations described above, the following is known:

For $r = r_1 = 2$, the first bifurcation occurs. The global asymptotic stability of the stationary set vanishes, and a globally asymptotically stable cycle of period 2 appears.

The second bifurcation value $r = r_2 = \pi$ value corresponds to bifurcation of splitting: the globally stable cycle of period 2 loses its stability, and two locally stable cycles of period 2 appear. Note that for the first time, this phenomena was described in [Leonov & Seledzhi, 2002]: a cycle of some period T loses its stability, and two cycles of the same period T appear.

For $r = r_3 = \sqrt{\pi^2 + 2} \approx 3.4452$, the third bifurcation occurs: two cycles of period 2 lose stability, and two 4-periodical cycles appear.

In Fig. 3, we show an enlarged domain of the bifurcation tree where the second and third bifurcations are clearly seen.

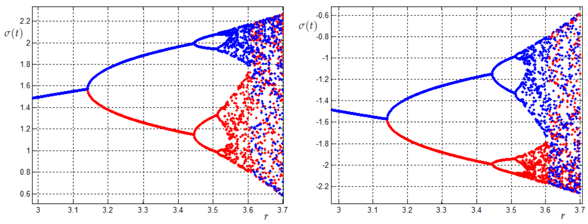

Figure 3. Seledzhi's bifurcation tree. Enlarged domains.

Further transition to chaos through a cascade of period-doubling bifurcations takes place.

Note that the phenomenon of transition to chaos through a cascade of period- doubling bifurcations is well studied for the whole class of maps of an interval into itself. In 1975 M. Feigenbaum noticed that for the equation

$$x_{n+1} = \lambda x(1 - x),$$

the following is observed: if

$$\frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n} = \delta_n,$$

then

$$\lim_{n \to \infty} \delta_n = 4.6692...,$$

where $\lambda_{n-1}, \lambda_n, \lambda_{n+1}$ are consecutive bifurcation values.

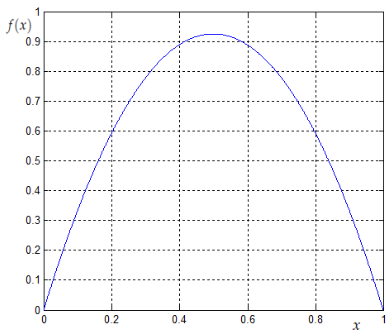

Figure 4. Plot of the function $f(x) = \lambda x(1-x), \quad \lambda = 3.7.$

He performed similar calculations with another logistical map, and found a geometric progression with the same denominator. After that, the hypothesis that δ does not depend on the type of a specific map was born.

It was found that the convergence is universal for one-dimensional one-parameter families of maps of an interval into itself [Campanin & Epstain, 1981; Ostlund *et al.*, 1983; Lanford, 1982; Hu & Rudnick, 1982]. The value $\delta = 4.6692...$ is the famous Feigenbaum's constant. The Renorm-group Theory explains this phenomenon for the class of unimodal maps (continuous map an interval into itself which has a unique critical point in the interval and is strictly monotonous on either side of extremum (Fig. 4)) and for some special cases [Feigenbaum, 1980; Vul, Sinai & Khanin, 1984; Cvitanovich, 1989; Bensimon, Jensen & Kadanoff, 1986; Kuznetsov, 2001; Shirkov, Kazakov & Vladimirov, 1988].

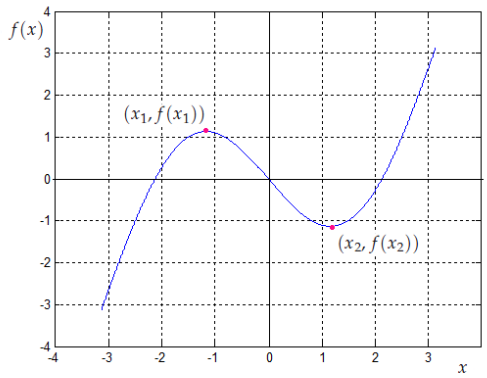

Figure 5. Plot of the function $f(x) = x - r sin(x), r = 2.5.$

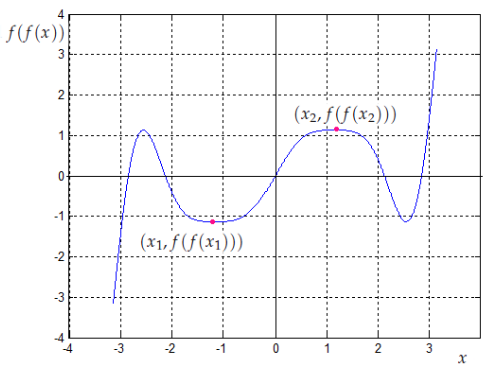

Figure 6. Plot of the function $f(f(x)) = f(x) - r sin(f(x)), r = 2.5.$

The calculations obtained in the present work allow us to show that, for system (5), the effect of convergence similar to Feigenbaum's effect is observed.

Note that for the function $f(x) = x - r sin(x)$, which has two critical points $(x_1, f(x_1))$ and $(x_2, f(x_2))$

in the interval $[-\pi, \pi]$ (Fig. 5), and for the function $f(f(x)) = f(x) - rsin(f(x))$ (Fig. 6) we have for $r > 2$:

$$f(x_1) \neq x_1, \quad f(x_1) \neq x_2,$$
$$f(f(x_1)) \neq x_1, \quad f(f(x_1)) \neq x_2.$$

4 Computer modeling

First numerical calculations of bifurcation values of parameter r for system (5) are presented in [Osborne, 1980; Banerjee & Sarkar, 2006; Leonov & Seledzhi, 2002].

In [Leonov & Seledzhi, 2002; Leonov & Seledzhi, 2005], methods for analysis of behavior of periodic trajectories of the system were developed. They allowed to justify the using of computational procedures for calculating the following bifurcation values. In present paper with the help of these analytical methods and specialized mathematical packages, the first 14 bifurcation values of parameter r were obtained.

The algorithm for computation is based on application of the method of multipliers [Vul, Sinai & Khanin, 1984; Kuznetsov, 2001].

The multiplier of a periodic trajectory of period T for a discrete dynamical system $x_{n+1} = f(x_n, r)$ can be written as

$$M_T(r) = \prod_{i=1}^{T} f'(x_i(r), r),$$

where $x_i(r)$, $i = 1, \ldots, T$, are the points (limit values) which form a stable periodic trajectory of period T. The multiplier is responsible for the stability of the cycle: for $r = r_T$ for which $M_T(r_T) = -1$, there occurs a period-doubling bifurcation at which the periodic trajectory of period T loses stability and there appears a periodic trajectory of period $2T$.

With a good accuracy, exact values of the first 14 bifurcation values of parameter r for initial data $\sigma(0) = 1$ for system (5) were obtained. For computation, multipliers of all periods 4, 16, 32, ..., 8192, for each value of parameter r greater than the analytically obtained r_3, with small step were calculated. This allowed to avoid admission of bifurcation values.

The values are obtained under the condition of convergence of limiting values up to 15 signs after comma:

$$|\sigma(t) - \sigma(t + T)| < 10^{-15}.$$

The specified condition demands $t = 2 \times 10^8$ iterations.

Table (1) shows the first 14 calculated bifurcation values of parameter r for system (5).

As was said earlier, the bifurcation parameter $r = r_2 = \pi$ does not correspond to period-doubling bifurcation. There is a bifurcation of splitting of the cycle: the cycle of period 2 loses its stability, and two locally stable cycles of period 2 appear.

j	Bifurcation parameter, r_j	Feigenbaum's number, δ_j
1	2	
2	π	3.7597337326
3	3.445229223301312	4.4874675842
4	3.512892457411257	4.6240452067
5	3.527525366711579	4.6601478320
6	3.530665376391086	4.6671765089
7	3.531338162105000	4.6687679883
8	3.531482265584890	4.6690746582
9	3.531513128976555	4.6691116965
10	3.531519739097210	4.6690257365
11	3.531521154835959	4.6686408913
12	3.531521458080261	4.6678177276
13	3.531521523045159	4.6657974003
14	3.531521536968802	

Table 1. Values of bifurcation parameters and Feigenbaum's numbers for a discrete dynamical system.

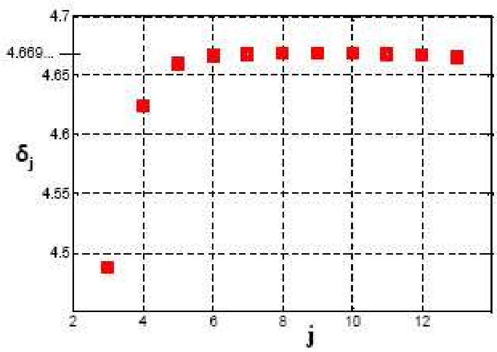

Figure 7. Feigenbaum's numbers

For the calculated bifurcation values of parameters r_j (Table 1), with the help of the relation

$$\delta_j = \frac{r_j - r_{j-1}}{r_{j+1} - r_j},$$

the values of Feigenbaum's numbers δ_j are calculated. They are presented in the last column of Table 1.

In the Fig. (7) is shown that the obtained Feigenbaum's numbers δ_j have a good convergence to Feigenbaum's constant $\delta = 4.6692016\ldots$. Thus, for system (5) an effect of convergence of bifurcation values of parameter r similar to Feigenbaum's effect is observed.

5 Conclusion

Application of qualitative theory of dynamical systems, special analytical methods and modern mathematical packages has helped to promote considerably in calculation of bifurcation values of parameter for a one-dimensional discrete system describing operation of digital phase-locked loop.

Numerically calculated fourteen bifurcation values of parameter of the investigated system are presented. It is shown that for the obtained bifurcation values of nonunimodal map, an effect of convergence similar to Feigenbaum's effect is observed.

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