Control for a Weakly Perturbed Stochastic Oscillator with a Guaranteed Lifetime

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Abstract: A problem of controlling a noisy oscillatory system so as to prevent it from leaving a prescribed domain covers a wide range of applications. In this paper, in contrast to the great majority of control approaches, we suggest a control strategy aimed at building a system in which the escape rate and/or escape probability are independent of noise (in the small noise limit). An explicit formula for feedback control is derived. Our results exploit the properties of the Euler-Lagrange equations of motion. We demonstrate that for Lagrangian systems, in contrast to the great majority of nonlinear problems, one can construct a closed-form asymptotic solution to the first exit time problem. An explicit formula allows choosing the parameters of a regulator guaranteeing weak dependence of the escape rate on noise strength. An application of this result to the problem of trapping a particle in the betatron accelerator illustrates the theory.

Keywords: Control of oscillations; stochastic systems; nonlinear control; nonlinear dynamics

1. INTRODUCTION

The problem of controlling a stochastic system so as to prevent it from leaving a prescribed domain G covers a large number of applications. There are two criteria commonly associated with control against escape: the probability of escape over a specified time interval [0; T] and the rate of escape. Problems for which escape time criteria are appropriate fit into one of two categories. In the first category, exit from the region of the desired operation is in certain sense catastrophic, and avoiding such an event is a high priority. An example in this category is the failure of a nuclear power engine. In the second category, exit from the admissible domain is not fatal but it is still an event to be avoided because of a drastic decline in performance when a system is outside the good region. Examples in this category are control of plasma and particles, control of nano- and microstructures, etc. The mean time the system spends in the safe region is interpreted as the lifetime of the system.

In the context of optimal control, the lifetime must be maximized. We recall that the solution of the optimal control problem is sensitive to the properties of the perturbation. This paper discusses an alternative approach to escape control. In practice, limitations imposed on the system are defined by the task to be performed under a broad range of excitations. This implies a control strategy aimed at building a system in which performance costs are insensitive or, at least, weakly sensitive to noise.

We suggest a convenient control strategy for a class of systems described by the Lagrangian equations with small noise. The term "small noise" essentially means that escape from an admissible domain is a relatively rare event. Note that a large number of physical and engineering problems fall into the "small noise" category. Given that the technical requirements are quite strict, a system in which escape is common might not be worth considering. A number of relevant examples are discussed, e.g., in (Meerkov and Runolfsson, 1988; Dupuis and McEneaney, 1997).

Throughout the paper, strength of noise is characterized by a small parameter ε. We denote by τ^ε the first moment at which the weakly perturbed system leaves G. We recall that the probability P_T^ε of rare escape from G over a fixed time interval [0; T] is approximated by the Poisson law (Gardiner, 2004)

$$P_T^\varepsilon \approx 1 - \exp(-\lambda^\varepsilon T). \tag{1}$$

This implies that the criterion of interest in the small noise model is the mean escape time $\mathbf{E}\tau^\varepsilon$ or rate $\lambda^\varepsilon = 1/\mathbf{E}\tau^\varepsilon$.

For the sake of simplicity, we consider in details a system with white noise perturbation. Note that the diffusion model can be interpreted as an approximation of more complicated phenomena described by systems with wide-band ergodic or fast noise, see e.g., (Kushner, 1984; Gulinskii and Liptser, 2000; Liptser, Spokoiny and Veretennikov, 2002; Kovaleva, 2006; Kovaleva and Akulenko, 2007).

The direct calculation of $\mathbf{E}\tau^\varepsilon$ for a degenerate small noise diffusion requires considering the Dirichlet problem for a singular Fokker-Plank equation. Both analytic and numerical solutions to this equation are prohibitively difficult but asymptotic approaches might be of help. In this paper, we employ the large deviation theory as an appropriate vehicle for estimating statistical quantities in weakly perturbed systems.

We use the main results of the theory in the form presented by Kushner (1984) and Freidlin and Wentzell (1998). Recent advances in theory and applications are discussed, among others, by Olivieri and Vares (2005) and Feng and Kurtz (2006). Despite the well-developed theory, most of the existing solutions are related to one-dimensional systems; there are only few explicit solutions for multidimensional systems. Comprehensive results have been obtained for multidimensional linear systems (Meerkov and Runolfsson, 1988; Freidlin and Wentzell, 1998). The large deviations principle for Hamiltonian-type systems was derived by Wu (2001) but no closed-form solutions have been constructed.

Kovaleva (2005, 2006) and Kovaleva and Akulenko (2007) have derived a closed-form asymptotic estimate of $\ln(\mathbf{E}\tau^{\varepsilon})$ for Lagrangian systems with linear dissipation and additive noise. In this paper we obtain an explicit asymptotics of $\ln(\mathbf{E}\tau^{\varepsilon})$ for a class of Lagrangian systems with nonlinear dissipation and state-dependent noise.

The idea of applying the large deviations approach to minimize escape probability has been advocated, for the first time in a control framework, by Dupuis and Kushner (1989). A powerful development of this idea has been achieved in the theory of risk-sensitive escape control (Dupuis and McEneaney, 1997; Boue and Dupuis, 2001). However, optimal values of the risk-sensitive criteria directly depend on the noise strength.

This paper is organized as follows. Section 2 is devoted to the asymptotic analysis of the problem and includes main results. Section 3 illustrates the theory.

2. BASIC METHODOLOGY

In this Section we recall the main issues of the large deviations theory requisite for the further analysis. A few brief comments will be made on the derivation of the large deviation principle. For details, see (Kushner, 1984; Freidlin and Wentzell, 1998).

2.1 The Lagrangian model

For the sake of simplicity, consider a system with mass matrix $M = I_n$, where I_n is the n-dimensional identity matrix. An extension to the case of $M(q)$ is given at the end of this section.

The kinetic energy of the system with mass matrix $M = I_n$ is written as $T(\dot q) = (\dot q, \dot q)/2$; the potential energy is denoted by $U(q)$; the total energy $H(q, \dot q) = T(\dot q) + U(q)$; the Lagrangian of the system $L(q, \dot q) = T(\dot q) - U(q)$; $q \in \mathbf{R}^n$ is the vector of generalized configuration coordinates. All vectors defined in the paper are column vectors.

The Euler-Lagrange equation of the controlled system has the form

$$\ddot q + \frac{\partial U}{\partial q} = \varepsilon\sigma(q,\dot q)\dot w(t) + u(q,\dot q)\ ,\ q, \dot q \in G, \qquad (2)$$

where $u \in \mathbf{R}^n$ is the vector of control forces acting on the system; $w(t)$ is standard Wiener process in $\mathbf{R}^m$; $\sigma(q, \dot q)$ is a non-degenerate $n\times m$-matrix. The control u is chosen in the form

$$u(q, \dot q) = -kA(q, \dot q)\,\dot q\ , \qquad (3)$$

where $A = \sigma\sigma'$, the gain $k > 0$ ensures the desired escape rate. The prime denotes the transpose matrix or vector.

Throughout this paper, we assume that

A.1. The reference domain G is a connected open bounded set in $\mathbf{R}^{2n}$ with smooth boundary ∂G and compact $\bar G$ (closure of G); the origin O:$\{q = 0, \dot q = 0\} \in \mathrm{int}G$.

A.2. $U(q)$ has a minimum at $q = 0$; $U(0) = 0$, $U(q) > 0$ in $\bar G$ if $q \neq 0$.

A.3. $A(q, \dot q)$ is a positive definite symmetric matrix in $\bar G$.

A.4. $A(q, \dot q)$ and $H(q, \dot q)$ are analytic functions of $q, \dot q$ in $\bar G$.

Assumptions *A*.3 implies that the functions $A(q,p)$ and $H(q,p)$ are sufficiently smooth to ensure the requisite transformations and uniqueness of the solution.

In view of *A*.1 - *A*.4, the LaSalle Invariance Principle with the Lyapunov function $V(q, \dot q) = T(\dot q) + U(q)$ can be invoked to prove that point O is an asymptotically stable state of the system

$$\ddot q + \frac{\partial U}{\partial q} = -kA(q,\dot q)\dot q\ . \qquad (4)$$

In addition, we assume that

A.5. System (4) has a unique asymptotically stable point O in G, and all trajectories of (4) originating in $\bar G$ tend to O not leaving G.

Assumption *A*.5 implies that no escapes can occur from G in the absence of noise. However, noise, however small it might be, induces escape from any bounded domain with a non-zero escape rate λ^{ε}. Our goal is to show that, in the small noise limit, the logarithmic escape rate $\ln\lambda^{\varepsilon}$ in system (2) is independent of noise provided the control u is chosen in the form (3). The definition of the noise-independent limit will be given below.

B. The large deviations principle

We recall that the large deviations principle provides a cost (action) functional that must be minimized by the "most likely" exit path. The solution of the minimization problem determines the limiting values of $\ln(\mathbf{E}\tau^{\varepsilon})$ and related quantities as $\varepsilon \to 0$.

Omitting a general concept, we construct a variational problem associated with the calculation of the mean exit time in system (2). We note that, by assumption *A*.4, the functions

$H(q, \dot{q})$ and $\sigma(q, \dot{q})$ are sufficiently smooth to ensure the requisite transformations and uniqueness of the solution.

Introducing the momentum $p = \partial L(q, \dot{q})/\partial \dot{q} = \dot{q}$, we rewrite (2) as

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial q} - kA(q,p)\frac{\partial H}{\partial p} + \varepsilon\sigma(q, p)\,\dot{w}(t)\,, \qquad (5)$$

where $H(q, p) = T(p) + U(q)$. It is obvious that $H(0,0) = 0$; otherwise $H(q, p) > 0$.

Following (Kushner, 1984), the action functional for system (5) has the form

$$S_\tau(Q,P) = \tfrac{1}{2}\int_0^\tau (F, A^{-1}F)dt\,,\quad \dot{Q} = \frac{\partial H}{\partial P} \qquad (6)$$

if $Q(t)$, $P(t)$ are absolutely continuous, and $S_\tau(Q, P) = \infty$ if $Q(t)$, $P(t)$ are not absolutely continuous. Here we denote

$$F(Q,P) = \dot{P} + \frac{\partial H}{\partial Q} + kA(Q,P)\frac{\partial H}{\partial P}$$

Let $S(q, p)$ be a lower bound of (6) calculated along an extremal forwarded from the initial point O to the terminal point $Q(\tau) = q$, $P(\tau) = p$:

$$S(q, p) = \inf\{S_\tau(Q, P):$$
$$Q(0) = 0, P(0) = 0;\ Q(\tau) = q,\ P(\tau) = p\}. \qquad (7)$$

Note that the terminal moment τ is not fixed but must be identified as a solution of the variational problem given below.

The key relation derived in (Kushner, 1984) is

$$\lim_{\varepsilon\to 0} \varepsilon^2(\ln\mathbf{E}\,\tau^\varepsilon) = \inf_{\partial G} S(q, p) = S_0, \qquad (8)$$

where τ^ε is the first moment the orbit $Q(t)$, $P(t)$ reaches the boundary ∂G.

Hence, (7) is the variational problem to be solved. Using the well-known relations between variational problems and Hamilton-Jacobi equations, (e.g., Gelfand and Fomin, 2000), we calculate $S(q, p)$ as a solution of the Hamilton-Jacobi equation

$$\left(\frac{\partial S}{\partial q}, \frac{\partial H}{\partial p}\right) - \left(\frac{\partial S}{\partial p}, \frac{\partial H}{\partial q}\right) - \left(\frac{\partial S}{\partial p}, kA(q, p)\frac{\partial H}{\partial p}\right)$$
$$+\frac{1}{2}\left(\frac{\partial S}{\partial p}, A(q, p)\frac{\partial S}{\partial p}\right) = 0,\ S \in G, \qquad (9)$$

with initial condition $S(0,0) = 0$.

Arguing as in (Kovaleva, 2005, 2006; Kovaleva and Akulenko, 2007), we find the solution of (9) in the form

$$S(q, p) = 2kH(q, p). \qquad (10)$$

The uniqueness of the smooth solution is discussed in Kovaleva and Akulenko (2007).

We now introduce the definition of *the noise-independent limit*. It follows from (8) that $\mathbf{E}\,\tau^\varepsilon \sim \exp(S_0/\varepsilon^2)$ if ε is sufficiently small. Therefore, for any σ we have $\mathbf{E}\,\tau^\varepsilon \to \infty$ and $\lambda^\varepsilon \to 0$ as $\varepsilon \to 0$. In order to obtain a well-defined limit, it is worth considering the logarithmic asymptotics (8). Since $|\ln\lambda^\varepsilon| = \ln\mathbf{E}\,\tau^\varepsilon$, the logarithmic escape rate is independent of noise (as $\varepsilon \to 0$) if the right-hand side of (8) is independent of σ. It is obvious that function (10) and, therefore, the limit (8) are independent of σ under any conditions on ∂G. It follows from (9) – (11) that the noise-independent asymptotics is due to a proper choice of velocity feedback $u_1 = -kA(q, \dot{q})\,\dot{q}$.

Finally, we note that a sharp asymptotic analysis yields the estimate (Kamin, 1978)

$$\mathbf{E}\,\tau^\varepsilon = C(\varepsilon)\exp(S_0/\varepsilon^2)(1 + o(1)), \qquad (11)$$

where $\varepsilon^2C(\varepsilon) \to 0$ as $\varepsilon \to 0$ and, in general, $C(\varepsilon)$ depends on σ. A closed-form expression for $C(\varepsilon)$ is available only in some low-dimensional cases. In Section 3 we give an example of numerical simulation demonstrating weak dependence of $\mathbf{E}\,\tau^\varepsilon$ on σ.

Remark. If the mass matrix $M = M(q)$, the equation of motion is written as

$$\frac{d}{dt}\frac{\partial L(q,\dot{q})}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \varepsilon\sigma(q,\dot{q})\dot{w}(t) + u(q,\dot{q})\,,$$

where the Lagrangian $L(q, \dot{q}) = T(q, \dot{q}) - U(q)$, the kinetic energy $T(q, \dot{q}) = (M(q)\,\dot{q}, \dot{q})/2$. Introducing the momentum $p = \partial L(q, \dot{q})/\partial \dot{q} = M(q)\,\dot{q}$ and using the function $H(q, p) = T(q, \dot{q}(p)) + U(q, \dot{q}(p))$, we can reproduce the above transformations and obtain the solution in the form (10).

3. EXAMPLE

We consider the problem of controlling a particle in the betatron accelerator. The betatron is essentially a transformer with a torus-shaped vacuum tube of elliptic cross section (Fig. 1). Alternating current accelerates electrons in the vacuum around a circular axis of the torus but small imperfections result in deviations from this axis and generate oscillations in the cross section. The safe operation is associated with circular motion within the tube; as soon as a particle reaches the internal surface of the tube, the system becomes unstable. The control tusk is thus to secure the particle within the tube.

The equations of controlled oscillations in the presence of noise take the form (Blaquiere, 1966)

$$\ddot{x}_1 + \Omega_1^2 x_1 = -\tfrac{\alpha}{2}(x_1^2 - x_2^2) + \Delta_1\,\dot{w}_1(t) + f_1$$

$$\ddot{x}_2 + \Omega_2^2 x_2 = -\alpha\, x_1x_2 + \Delta_2\,\dot{w}_2(t) + f_2, \qquad (12)$$

where t is the azimuthal coordinate calculated along the circular axis; x_1, x_2 are, respectively, the radial and vertical deviations from the circular axis in the cross section of the torus (Fig. 1).

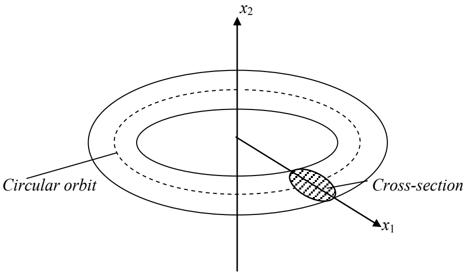

Fig. 1. Model of a vacuum tube

Since the velocity of circular rotation is a constant equal to the electric field frequency, the phase t can be interpreted as the dimensionless time variable of the system. By Ω_1 and Ω_2 we denote the frequencies of linear oscillations near the equilibrium state $x_1 = x_2 = 0$; the factor α is due to the interplay of the radial and vertical oscillations; $\Delta_i \dot{w}_i(t)$ are the projections of planar excitation onto the axes x_i; f_i are the projections of the counteracting control force; $i = 1, 2$. For brevity, we take $\Omega_1 = \Omega_2$, $\Delta_1 = \Delta_2 = \Delta$.

In order to reduce system (12) to the dimensionless form, we introduce the new variables $q_i = \alpha x_i/\Omega^2$ and denote $u_i = \alpha f_i/\Omega^2$, $\varepsilon\sigma = \alpha\Delta/\Omega^2$. Now, by (3), we find

$$u_i = -\, b\,\dot{q}_i,\ b = k\sigma^2.$$

Using the new notations, we rewrite (12) in the form

$$\ddot{q}_i + \frac{\partial U}{\partial q_i} = \varepsilon\sigma_i\,\dot{w}_i(t) - b\,\dot{q}_i,\ i = 1, 2, \quad (13)$$

where

$$U(q) = \tfrac{1}{2}\,(q_1^{\,2} + q_2^{\,2} + 2q_1^{\,2}q_2 - \tfrac{2}{3}\,q_2^{\,3}) \quad (14)$$

is the standard Hénon-Heiles potential (Blaquiere, 1966; Tabor, 1989); q is the vector with entries q_1, q_2. The small parameter ε is identified below. The total energy of the system is

$$H(q, p) = \tfrac{1}{2}\,\|p\|^2 + U(q), \quad (15)$$

where p is the vector with entries $p_i = \dot{q}_i$ and norm $\|p\| = (p_1^{\,2} + p_1^{\,2})^{1/2}$.

The direct calculation shows that (14) is a two-dimensional potential with the minimum $U(0) = 0$; the equality $U(q) = U^* = 1/6$ determines the potential barrier (Blaquiere, 1966, Tabor, 1989, and references therein). Once $U(q) = U^* = 1/6$, the particle reaches the internal surface of the tube with the resulting loss of stability. Hence, the admissible domain of variation for the variable q is

$$G_q\!: \{U(q) < U^* = 1/6\},\ \partial G_q\!: \{U(q) = U^* = 1/6\} \quad (16)$$

Formally, no constraints are imposed on the variable p. However, as shown in (Blaquiere, 1966, Tabor, 1989), the conservative system

$$\ddot{q}_i + \frac{\partial U}{\partial q_i} = 0$$

has (non-asymptotically) stable periodic solutions in the domain (16); additional dissipation makes the system

$$\ddot{q}_i + \frac{\partial U}{\partial q_i} = -\, b\,\dot{q}_i,\ i = 1, 2,$$

asymptotically stable, with an attracting point O: ($q = 0$, $p = 0$). This implies that assumption A.5 holds and the momentum p is bounded if $q \in G_q$. This can be formalized as $p \in G_p$, where G_p is an open bounded set in $\mathbf{R}^2$ with boundary ∂G_p. The total reference domain can thus be described as G: $G_q \times G_p$; ∂G: $\partial G_q \times \partial G_p$.

Although the domain G and boundary ∂G cannot be explicitly defined, $\inf H(q,p)$ on ∂G can be found. Since the first term in (15) is a positive definite quadratic form, and, by A.1, A.5, the point O:$\{q = 0,\ p = 0\} \in G$, the lower bound of $H(q, p)$ is achieved at $p = 0$ for any $U(q)$. Hence,

$$\inf_{\partial G} H(q,p) = \inf_{\partial G_q} U(q) = 1/6 \quad (17)$$

Now, using (8), (10), (17), we obtain the main term of the logarithmic asymptotics

$$3\ln(\mathbf{E}\,\tau^{\varepsilon}) = k/\varepsilon^2. \quad (18)$$

If the mean escape time $\mathbf{E}\,\tau^{\varepsilon}$ is known, then, using (18), we can calculate a relevant gain k.

For example, let the control task be to ensures the safe operation over the time interval [0; T] with probability $P_s > P^{\varepsilon} = 1 - \varepsilon^2$, $\varepsilon \ll 1$. We first analyze the case $P_s = P^{\varepsilon}$. Invoking the escape probability $P_T^{\,\varepsilon} = 1 - P^{\varepsilon} = \varepsilon^2$, using the Poisson law (1), and skipping negligible terms, we obtain the mean exit time

$$\mathbf{E}\,\tau_P^{\,\varepsilon} = T/\varepsilon^2. \quad (19)$$

It now follows from (18), (19) that $k = 3\varepsilon^2\ln(T/\varepsilon^2)$. Recalling that $b = (k/\varepsilon^2)(\varepsilon\sigma)^2$ and $\varepsilon\sigma = \alpha\Delta/\Omega^2$, we obtain $b = 3\ln(T/\varepsilon^2)(\Omega^2\Delta/\alpha)^2$. It is easy to see that the gain b depends on the pregiven time T and escape probability $P_T^{\,\varepsilon} = \varepsilon^2$. If we take $\varepsilon_1 < \varepsilon$, then the required gain $b_1 > b$. In the converse case, if we implement the regulator with the gain $b_1 > b$, we obtain $\varepsilon_1 < \varepsilon$ and, therefore, $P_s = 1 - \varepsilon_1^{\,2} > P^{\varepsilon}$.

Numerical experiments were carried out to compare the logarithmic escape rate $|\ln\lambda^{\varepsilon}| = \ln\mathbf{E}\,\tau^{\varepsilon}$ in system (13) with the predicted value $\ln\mathbf{E}\,\tau_P^{\,\varepsilon}$. We chose $T = 10^4$, $P^{\varepsilon} = 0.99$, $\varepsilon = 0.1$. This yielded $\mathbf{E}\,\tau_P^{\,\varepsilon} = 10^6$, $k/\varepsilon^2 = 41.4$, $b = 0.414\sigma^2$. The averaged results of simulations obtained in series of 30 experiments are depicted in Fig. 2. As seen in Fig. 2, the discrepancy is about ± 6%.

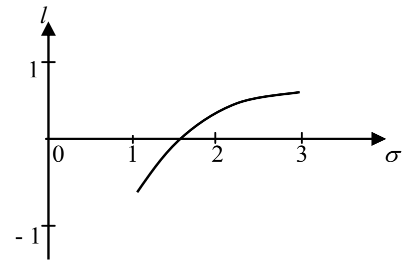

Fig. ***2***. Simulation with $b = 0.414\sigma^2$; $l = 10\ln(\mathbf{E}\tau^\varepsilon/\mathbf{E}\tau_P^\varepsilon)$

Formula (11) can be used to explain the discrepancy between the predicted and experimental values. It follows from (11), (17) that $\mathbf{E}\tau^\varepsilon/\mathbf{E}\tau_P^\varepsilon = C(\varepsilon)(1 + o(1))$ if ε is small enough. Now, if we recall that a rough approximation for the rate of escape from a one-dimensional potential well yields the estimate $C(\varepsilon) \sim b$ (Gardiner, 2004) and presume a similar dependence for system (13), we conclude that $C(\varepsilon) < 1$ and, hence, $\ln(\mathbf{E}\tau^\varepsilon/\mathbf{E}\tau_P^\varepsilon) < 0$ if b is sufficiently small, and $C(\varepsilon) > 1$, $\ln(\mathbf{E}\tau^\varepsilon/\mathbf{E}\tau_P^\varepsilon) > 0$ if b is not very small. This explains the shape of the curve in Fig. 2.

4. CONCLUSIONS

In this paper we have suggested a regulator for a weakly perturbed Lagrangian system. The resulting control law ensures that the lifetime of the controlled system is independent of noise in small noise limit. The simplicity of this controller, which results from the physical structure of the system, may constitute an interesting alternative to optimal regulators.

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