# DYNAMICS OF A BUSH-SHAFT SYSTEM WITH IMPACT AND FRICTION

**Jan Awrejcewicz**
Department of Automatics and Biomechanics
Technical University of Łódź
Poland
awrejcew@p.lodz.pl

**Yuriy Pyryev**
Department of Automatics and Biomechanics
Technical University of Łódź
Poland
jupyrjew@p.lodz.pl

**Abstract**
In this work dynamics of a rigid body (bush) with viscous-elastic constraints and subject to impact and friction actions is studied. The body is in dynamical contact with another moving body (rotated with constant velocity thermo-elastic shaft), and friction generated heat and wear (on the contact surface) are taken into account. The problem is solved analytically and the obtained prediction is verified numerically showing surprisingly good agreement.

**Key words**
Impact, friction, heat, wear, bush-shaft system.

## 1 Introduction

Simple model of a contact bush-shaft system with heat and wear generated by friction and/or impacts is proposed and studied. The bush motion is bounded by rigid barriers and it takes place within the introduced clearance. The bush is considered as a rigid body being fixed to a foundation via mass-less springs and dampers, and it is mounted on the rotating with constant angular velocity thermo-elastic shaft. The so far stated problem is reduced to analysis of the equations governing bush dynamics taking into account impacts and nonlinear friction. The latter one is a product of time depended contact pressure and relative velocity of the shaft and the bush. The contact pressure value is estimated by a second order Volterra-type equation. In the case of a small slope of the kinematical friction coefficient, the restitution coefficients required to realize the system periodic impact motion either with one or two impacts are estimated analytically.

Analytically predicted vibro-impact stick-slip and slip-slip dynamics has been also verified numerically.

## 2 The system under analysis

We are focused on modeling of non-linear dynamics of two bodies consisting of a stiff bush 2 with clearance $2\Delta_\varphi$ (see Figure 1) and solid isotropic circular shaft 1 of radius $R_1$. The bush external radius $R_2$, whereas internal bush radius is $R_1 - U_0$ ($U_0/R_1 << 1$), and $U_0$ corresponds to initial shaft compression. The bush is linked with housing by springs with stiffness $k_2$ and the damper with viscous coefficient $c$, and is mounted on the rotating thermo-elastic shaft 1.

The following assumptions are taken [Awrejcewicz, Pyryev, 2002]:

(i) The external excitation of the system allows neglecting of the inertial term in the Lamé equation;

(ii) The vector components related to displacement as well as the shaft temperature $T(R,t)$ depend only on a radius coordinate $R$ and time $t$;

(iii) The following friction model is introduced between shaft and bush:

$$F_{\mathrm{fr}}(V_r,\varphi_2) = \begin{cases} f(V_r)2\pi R_1 P(t), & V_r \neq 0, \text{ slip} \\ \min\left(R_1^{-1}R_2^2\left|c\Omega + k_2\varphi_2\right|,\right. & \\ \left.F_0\, 2\pi R_1\, P(t)\right)\mathrm{sgn}(c\Omega + k_2\varphi_2), & V_r = 0, \text{ stick} \end{cases} \tag{1}$$

where: $F_0 2\pi R_1 P(t)$ is static friction force, $P(t)$ is the contact pressure, $F_0 = f(+0)$, $f(V_r)$ is the kinetic friction coefficient depending on relative velocity of the contact bodies $V_r = R_1\Omega - R_1\varphi_2'(t)$, $\varphi_2(t)$ is the angle of bush deviation;

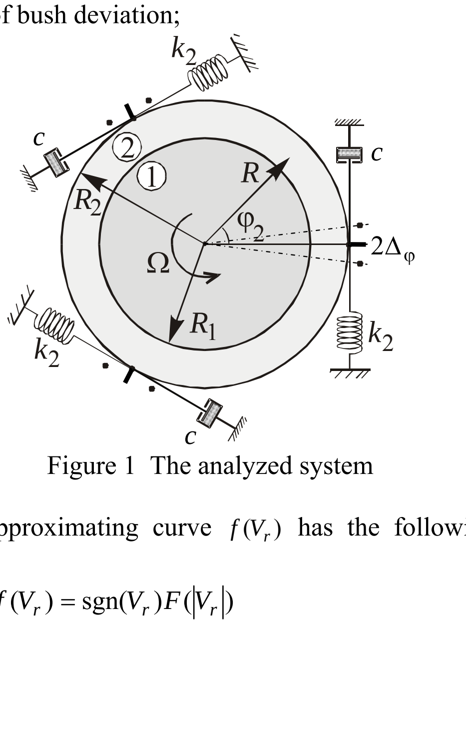


Figure 1 The analyzed system

(iv) Approximating curve $f(V_r)$ has the following form

$$f(V_r) = \mathrm{sgn}(V_r)F(|V_r|) \tag{2}$$

$$F(V_r)=\begin{cases}F_0-\kappa V_r, & 0<V_r\le V_{\min}\\ F_0-\kappa V_{\min}, & V_{\min}<V_r\end{cases} \tag{3}$$

where $F_0$, $\kappa$, $V_{\min}$ are constant coefficients;

(v) Heat flows $q_1$ and $q_2$ are generated on the contact surface $R=R_1$ and governed by the equation $q_1+q_2=(1-\eta)f(V_r)P(t)V_r$, where $\eta\in[0,1]$ denotes the heat energy part, which is dissipated by wear. Both flows $q_1$ and $q_2$ go into shaft and bush, respectively:

$$q_1=\lambda_1\frac{\partial T(R_1,t)}{\partial R},\ q_2=-\alpha_T\left(T_0-T(R_1,t)\right), \tag{4}$$

where $\lambda_1$ is the thermal conductivity, $\alpha_T$ is the heat transfer coefficient between the shaft and the bush, $T_0$ is the temperature of the bush. Furthermore, we assume that bush transforms heat ideally and $T_0=0$;

(vi) Velocity of the bush wear is proportional to a certain power of friction force. We assume Archard's law of wear [Awrejcewicz, Pyryev, 2002] of the form

$$dU^w(t)/dt=K^w\left|V_r(t)\right|P(t), \tag{5}$$

where the coefficient $K^w$ is usually identified experimentally.

## 3 Mathematical formulation of the problem

The dimensionless equations governing dynamics of the analyzed system have the form

$$\ddot\varphi(\tau)+2h\dot\varphi(\tau)+\omega_0^2\varphi(\tau)=\operatorname{sgn}(\omega_1-\dot\varphi)\Psi(\dot\varphi)p(\tau),$$

$$\left|\varphi(\tau)\right|<1,\ \dot\varphi(\tau)\neq\omega_1, \tag{6}$$

$$\ddot\varphi(\tau)=0,\ \left|\varphi(\tau)\right|<1,\ \dot\varphi(\tau)=\omega_1, \tag{7}$$

$$\dot\varphi^+=-k\dot\varphi^-,\ \left|\varphi\right|=1,\ \dot\varphi^-\varphi>0, \tag{8}$$

$$\varphi(0)=x,\ \dot\varphi(0)=y, \tag{9}$$

where $k$ is the coefficient of restitution, and $\dot\varphi^-$ ($\dot\varphi^+$) is the bush velocity just before (after) impact,

$$\Psi(\dot\varphi)=\begin{cases}1+\varepsilon\omega_1\eta_0, \dot\varphi<\omega_1\eta_0,\ \omega_1(2-\eta_0)<\dot\varphi\\ 1+\varepsilon\dot\varphi, & \omega_1\eta_0<\dot\varphi<\omega_1\\ 1+2\varepsilon\omega_1-\varepsilon\dot\varphi,\ \omega_1<\dot\varphi<\omega_1(2-\eta_0)\end{cases}.$$

In order to solve the motion equations (5) one needs to know contact pressure $p(\tau)$ and wear $u^w(\tau)$ [1]:

$$p(\tau)=H(\tau)-u^w(\tau)+ \\ +2\gamma_1\tilde\omega^{-1}\int_0^\tau\dot G_p(\tau-\xi)\Psi\left(\dot\varphi(\xi)\right)\left|\omega_1-\dot\varphi(\xi)\right|p(\xi)d\xi \tag{10}$$

$$u^w(\tau)=k^w\int_0^\tau\left|\omega_1-\dot\varphi(\tau)\right|p(\tau)d\tau. \tag{11}$$

Then the problem is reduced to consideration of equations (5) and (10), which yield both dimensionless pressure $p(\tau)$ and velocity $\dot\varphi(\tau)$. The temperature is defined by the following formula

$$\theta(r,\tau)=\gamma_1\tilde\omega^{-1}\int_0^\tau\dot G_\theta(r,\tau-\xi)\Psi\left(\dot\varphi(\xi)\right)\left|\omega_1-\dot\varphi(\xi)\right|p(\xi)d\xi,$$

where

$$\left\{G_p(\tau),G_\theta(1,\tau)\right\}=\frac{\{0.5,1\}}{Bi}-\sum_{m=1}^{\infty}\frac{\left\{2Bi,2\mu_m^2\right\}}{\mu_m^2(Bi^2+\mu_m^2)}e^{-\mu_m^2\tilde\omega\tau}$$

and $\mu_m$ are the roots of the characteristic equation $BiJ_0(\mu)-\mu J_1(\mu)=0$.

In the above the following dimensionless parameters are introduced:

$$\tau=\frac{t}{t_*},\ r=\frac{R}{R_1},\ \varphi(\tau)=\frac{\varphi_2}{\Delta_\varphi},\ p=\frac{P}{P_*},\ \theta=\frac{T_1}{T_*},$$

$$u^w=\frac{U^w}{U_0},\ \alpha_0=\frac{f_0}{F_0},\ \omega_1=\frac{\Omega t_*}{\Delta_\varphi},\ \mu_0=\frac{1-\alpha_0}{\alpha_0},$$

$$\eta_0=\frac{V_0-V_{\min}}{V_0},\ \varepsilon=\frac{\mu_0}{\omega_1},\ \gamma=\sqrt{\frac{2\pi R_1^2P_*F_0\Delta_\varphi}{B_2\Omega^2}},$$

$$\omega_0^2=\frac{k_2R_2^2t_*^2}{B_2},\ 2h=\frac{cR_2^2t_*}{B_2},\ Bi=\frac{\alpha_TR_1}{\lambda_1},\Psi=\frac{F}{f_0},$$

$$f(V_0)=f_0,\ \tilde\omega=\frac{t_*}{t_T},\ k^w=\frac{K^w\Delta_\varphi E_1}{(1-2\nu_1)(1+\nu_1)},$$

$$\gamma_1=\frac{(1-\eta)E_1\alpha_1R_1^2f_0\Delta_\varphi}{\lambda_1(1-2\nu_1)t_T},\ x=\frac{\varphi_2^\circ}{\Delta_\varphi},\ y=\frac{\omega_2^\circ t_*}{\Delta_\varphi},$$

where:

$$t_*=\sqrt{\frac{B_2\Delta_\varphi}{f_0 2\pi R_1^2P_*}},\ V_*=\frac{R_1\Delta_\varphi}{t_*},\ T_*=\frac{U_0}{\alpha_1(1+\nu_1)R_1},$$

$$P_*=\frac{E_1U_0}{(1-2\nu_1)(1+\nu_1)R_1},\ t_T=\frac{R_1^2}{a_1},\ V_0=\Omega R_1.$$

$F_d=cR_2\varphi'(t)$ is the damping force related to the bush, $F_s=k_2\varphi_2R_2$ is springs force related to the bush, $B_2$ is the moment of inertia of the bush, $F_{\text{fr}}$ is friction force between the bush and the shaft, and $B_2$, $k_2$, $c$, $F_{\text{fr}}$ are the quantities measured per length unit bush.

## 4 Analysis in the case of lack of tribological processes

First the case of bush vibrations without tribological processes is studied ($\gamma_1=0$, $k^w=0$). For this case we have $p(\tau)=H(\tau)$ ($H(\tau)=1$, $\tau>0$, $H(\tau)=0$, $\tau\le 0$). Our system governed by equations (6) may exhibit four different periodic motions. Namely: (i) periodic orbit with one impact, where a stick does not appear (Figure 2, curve 1); (ii) periodic orbit with two impacts, where a slip of the contacting bodies occurs (Figure 2, curve 2); (iii) periodic orbit with one impact, where a stick-slip occurs (Figure 2, curve 3); (iv) periodic orbit with two impacts, where a stick-slip appears (Figure 2, curve 4).

In what follows we assume that $\varepsilon << 1$, $\omega_0^2 << 1$, $2h << 1$ and $\eta_0 \le -1$. It means that the system dynamics is exhibited in the interval ($0 < V_0 < V_{\min}$), where a decreasing slope of the kinetic friction coefficient is observed.

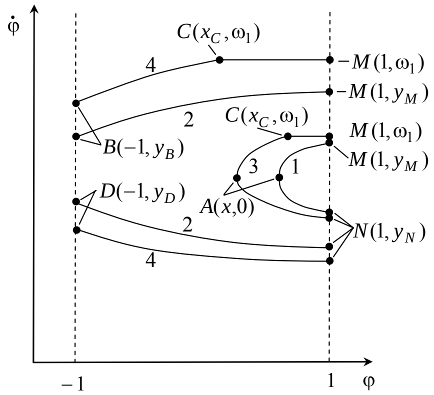


Figure 2 Periodic bush phase orbit

The reported below results are yielded by a standard perturbation approach. It allows us to give formulas for the coefficient of restitution $k(x,\omega_1)$

$$
k(x,\omega_1)=\begin{cases}
k_{ACMNA}, & 0<\omega_1<2-(4/3)(1-\chi)\varepsilon, \quad x_1<x<x_0\\
k_{AMNA}, & 0<\omega_1<2-(4/3)(1-\chi)\varepsilon, \quad x_0<x<1\\
k_{BCMNDB}, & 2-(4/3)(1-\chi)\varepsilon<\omega_1<2, \quad x_1<x<-1\\
k_{ACMNA}, & 2-(4/3)(1-\chi)\varepsilon<\omega_1<2, \quad -1<x<x_0\\
k_{AMNA}, & 2-(4/3)(1-\chi)\varepsilon<\omega_1<2, \quad x_0<x<1\\
k_{BCMNDB}, & 2<\omega_1<2+(4/3)(1-\chi)\varepsilon, \quad x_2<x<-1\\
k_{ACMNA}, & 2<\omega_1<2+(4/3)(1-\chi)\varepsilon, \quad -1<x<x_0\\
k_{AMNA}, & 2<\omega_1<2+(4/3)(1-\chi)\varepsilon, \quad x_0<x<1\\
k_{BCMNDB}, & 2+(4/3)(1-\chi)\varepsilon<\omega_1<\infty, \quad x_2<x<x_0\\
k_{BMNDB}, & 2+(4/3)(1-\chi)\varepsilon<\omega_1<\infty, \quad x_0<x<-1\\
k_{AMNA}, & 2+(4/3)(1-\chi)\varepsilon<\omega_1<\infty, \quad -1<x<1
\end{cases}
$$

and dimensionless period $\tau(x,\omega_1)$ for various cases of the system periodic dynamics

$$
\tau(x,\omega_1)=\begin{cases}
\tau_{ACMNA}, & 0<\omega_1<2-(4/3)(1-\chi)\varepsilon, \quad x_1<x<x_0\\
\tau_{AMNA}, & 0<\omega_1<2-(4/3)(1-\chi)\varepsilon, \quad x_0<x<1\\
\tau_{BCMNDB}, & 2-(4/3)(1-\chi)\varepsilon<\omega_1<2, \quad x_1<x<-1\\
\tau_{ACMNA}, & 2-(4/3)(1-\chi)\varepsilon<\omega_1<2, \quad -1<x<x_0\\
\tau_{AMNA}, & 2-(4/3)(1-\chi)\varepsilon<\omega_1<2, \quad x_0<x<1\\
\tau_{BCMNDB}, & 2<\omega_1<2+(4/3)(1-\chi)\varepsilon, \quad x_2<x<-1\\
\tau_{ACMNA}, & 2<\omega_1<2+(4/3)(1-\chi)\varepsilon, \quad -1<x<x_0\\
\tau_{AMNA}, & 2<\omega_1<2+(4/3)(1-\chi)\varepsilon, \quad x_0<x<1\\
\tau_{BCMNDB}, & 2+(4/3)(1-\chi)\varepsilon<\omega_1<\infty, \quad x_2<x<x_0\\
\tau_{BMNDB}, & 2+(4/3)(1-\chi)\varepsilon<\omega_1<\infty, \quad x_0<x<-1\\
\tau_{AMNA}, & 2+(4/3)(1-\chi)\varepsilon<\omega_1<\infty, \quad -1<x<1
\end{cases}
$$

where

$k_{AMNA}=1-(2/3)\tau_1\varepsilon+o(\varepsilon^2)$, $\tau_1=\sqrt{2(1-x)}$,

$k_{BMNDB}=1+\dfrac{\tau_2^3-\tau_1^3}{3(2+\tau_2^2)}\varepsilon+o(\varepsilon^2)$, $\tau_2=\sqrt{-2(1+x)}$,

$$k_{ACMNA}=\tau_1/\omega_1-(1/3)(\tau_1^2/\omega_1)\varepsilon-(1/8)(\tau_1/\omega_1)(4-\tau_1^2)\delta\varepsilon+o(\varepsilon^2),$$

$$k_{BCMNDB}=\frac{\tau_0}{\omega_1}+\frac{-16\tau_0^2+\tau_2^3\omega_1^2(\tau_0+\omega_1)-\tau_2^2\omega_1^2(4+\tau_0^2)}{6\omega_1(\tau_2^2\omega_1^2+2\tau_0^2)}\varepsilon+\frac{\tau_2^2\tau_0\omega_1(\tau_2^2+4)}{16(\tau_2^2\omega_1^2+2\tau_0^2)}\delta\varepsilon+o(\varepsilon^2),$$

$$x_0(\omega_1)=1-(1/2)\omega_1^2+(1/3)\omega_1^3(1-\chi)\varepsilon+(1/8)\omega_1^2(\omega_1^2-4)\delta\varepsilon+o(\varepsilon^2)$$

$$x_1(\omega_1)=1-0.5\omega_1^2-(1/3)\omega_1^3(1-\chi)\varepsilon+(1/8)\omega_1^2(\omega_1^2-4)\delta\varepsilon+o(\varepsilon^2)$$

$$x_2(\omega_1)=x_1(\omega_1)+(2/3)(\omega_1^2-4)^{3/2}(1-\chi)\varepsilon.$$

A detailed analysis shows that the function $k(x,\omega_1)$ possesses the following values $k(x_1,\omega_1)=1,\ 0<\omega_1<2$ $k(x_2,\omega_1)=1,\ 2<\omega_1<\infty$, $k(1,\omega_1)=1$ at the boundaries, whereas inside the considered interval it has the minima, which can be presented in the form

$$
k_{\min}=\begin{cases}
1-(2/3)\omega_1(1-\chi)\varepsilon, & 0<\omega_1<2+(4/3)(1-\chi)\varepsilon\\
1-(4/3)(1-\chi)\varepsilon, & 2+(4/3)(1-\chi)\varepsilon<\omega_1<\infty
\end{cases}
$$

Note that for an arbitrary $k^*\in(k_{\min},1)$ there are two values of $x_1^*$, $x_2^*$. Let us introduce the following intervals

$x_1<x_1^*<x_0, x_0<x_2^*<1$ for $0<\omega_1<2$,

$x_2<x_1^*<x_0, x_0<x_2^*<1$ for $2<\omega_1<2+(4/3)(1-\chi)\varepsilon$,

$x_2<x_1^*<-1, -1<x_2^*<1$ for $2-(4/3)(1-\chi)\varepsilon<\omega_1<\infty$.

It is not difficult to check that the periodic orbit associated with $x_1^*$ (decreasing part of the coefficient $k(x)$) is stable, whereas the periodic orbit associated with $x_2^*$ (increasing part of the coefficient $k(x)$) is unstable.

## 5 Numerical analysis

Periodic orbits of bush motion are presented in Figure 3 for various values of the coefficient $\gamma_1$ responsible for the heat transfer processes occurred on the contact surface due to friction ($\varepsilon=0.1$, $k=0.967$, $\omega_0^2=0.4$, $\eta_0=-2$, $\chi=0.5$, $Bi=10$, $\tilde{\omega}=0.1$, $k^w=0$).

Curve 1 corresponds to the case when $\gamma_1=0$ (lack of heat extension) and it crosses point $x=-1$. Increase of the parameter $\gamma_1$ yields decrease of the vibration period.

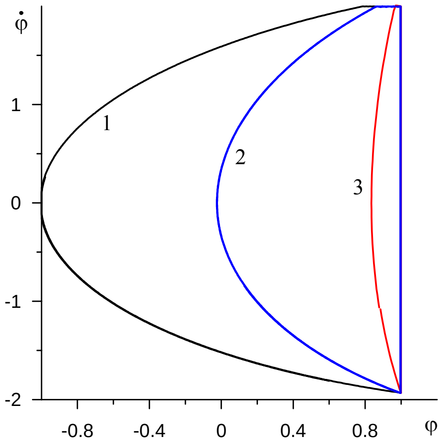

Figure 3 Phase trajectory of the bush movement for different values of $\gamma_1$: curve 1 – $\gamma_1 = 0$; curve 2 – $\gamma_1 = 0.3$; curve 3 – $\gamma_1 = 0.5$

Time histories of contact pressure, temperature on surface contact are reported in Figure 4.

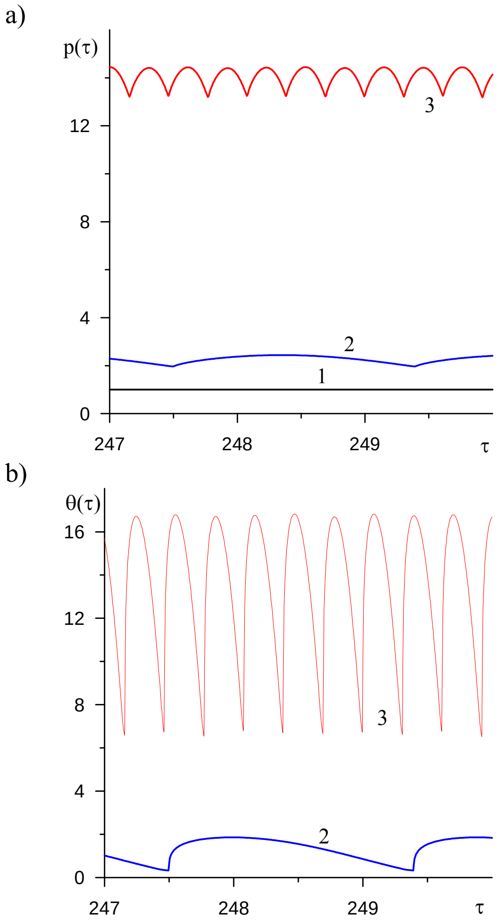


Figure 4 Time histories of dimensionless contact pressure $p(\tau)$ (a) and dimensionless contact temperature $\theta(\tau)$ (b) versus dimensionless time $\tau$ for different values of $\gamma_1$ : curves 1 – $\gamma_1 = 0$; curves 2 – $\gamma_1 = 0.3$; curves 3 – $\gamma_1 = 0.5$.

Curves 2 (3) correspond to the case, where the shaft heat expansion is taken into account $\gamma_1 = 0.3$ ($\gamma_1 = 0.5$), but the bush wear is neglected ($k^w = 0$). Observe that $\gamma_1$ increase introduces changes of amplitude of the contact pressure and amplitude of the contact temperature.

## 6 Conclusion

Mathematical model of periodic bush impact-type vibrations of the bush-shaft system taking into account tribological processes is derived. We have also address an analytical approach to the stated problem by introduction a perturbation parameter associated with the friction cinematic slope. The latter technique allowed us to derive the appropriate restitution coefficients responsible for realization of impact periodic vibration of stick-slip and/or slip types and either with one or with two impacts during the motion period. The period is analytically estimated, and the predicted analytically results have been compared with numerical ones showing very good agreement. Observe that this approach extends our earlier studies. Namely, dynamics of the mentioned bush-shaft system without impacts is studied in references [Awrejcewicz, Pyryev, 2002, 2004, 2005], whereas influence of the introduced gap between both bodies on their non-linear dynamics is reported in [Balandin, 1993].

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