Viscous Flows in a Half Space caused by Tangential Vibrations on its Boundary.[1]

V. A. Vladimirov

Department of Mathematics, University of York, YO10 5DD, UK[2]

Abstract

The paper is devoted to the studies of viscous flows caused by a vibrating boundary. The fluid domain is a half space, its boundary is a non-deformable plane that exhibits purely tangential vibrations. Such a simple geometrical setting allows us to study general boundary velocity fields and to obtain general results. From a practical viewpoint such boundary conditions may be seen as the tangential vibrations of the material points of a plane stretchable membrane. In contrast to the classical boundary layer theory we aim to build a global solution. In order to achieve this goal we employ the Vishik-Lyusternik approach combined with two-timing and averaging methods. Our main result is: we obtain a uniformly valid in the whole fluid domain approximation to the global solutions. This solution corresponds to general boundary conditions and to three different settings of the main small parameter. Our solution always include the 'inner' part and 'outer' part that contain both oscillating and non-oscillating components. It is shown that the non-oscillating 'outer' part of the solution is governed by the either full Navier-Stokes equations or the Stokes equations (both with the unit viscosity) and can be interpreted as a steady or unsteady streaming. In contrast to the existing theories of a steady streaming our solutions do not contain any secular (infinitely growing with the inner normal coordinate) terms. The examples of the spatially periodic vibrations of the boundary and the angular torsional vibrations of an infinite rigid disc are considered. These examples are still brief and illustrative, while the core of the paper is devoted to the adapting of the Vishik-Lyusternik method to the development of the general theory of vibrational boundary layers.

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1 Introduction

The classical boundary layer theory is aimed to describe the viscous flows in the vicinity of boundaries. At the same time it is well known that flows caused by an oscillating rigid

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[2]E-mail: vv500@york.ac.uk

boundary are not concentrated near the boundary: in addition to the vibrational boundary layers the celebrated phenomenon of a steady streaming takes place [1, 2]. The steady streaming in a viscous incompressible fluid is the subject of the large number of papers. There are different geometrical settings, including the steady streaming due to an oscillating solid in the fluid unbounded at infinity, the steady streaming in channels caused by oscillating walls, *etc.* Excellent reviews on the subject can be found in [3, 4, 5]. The adaptation of the classical boundary layer theory for the building of a steady streaming solution contains a number of theoretical difficulties that include the unbounded linear growth of the normal velocity component (a secular term) in a normal to the vibrating surface inner coordinate (see *e.g.* [3, 2]). Therefore it would be a valuable improvement to construct the *global* asymptotic solution valid in *the whole flow region* and does not contain any secular terms. It is especially interesting to build such solutions corresponding to the most general oscillating boundary conditions and to describe them analytically in order to create a unified viewpoint for the consideration of various special examples. There are other challenging problems, such us: to study the different ways of introduction of a small parameter, to understand how flexible the related scaling is, and to build analytic solutions for different asymptotic limits in the space of dimensionless parameters. In this paper we have partially met the above requirements by the employment of the Vishik-Lyusternik (V-L) asymptotic method that was originally used by von Karman [6] for the considering of the Blasius problem in fluid dynamics and then introduced and actively developed in the mathematical asymptotic theory and in the elasticity theory [7, 8, 9, 10, 11]. In fluid mechanics its usage is still rather restricted[3]: one can find its first applications to the steady boundary layers [6, 12, 13, 14] and more recently to the hydrodynamic impact theory [15][4]. However the V-L method has at least two clear advantages: (i) it is logically simpler than the classical boundary layer theory and its extension to the steady streaming theory, and (ii) it is mathematically rigorous and allows to build a solution that is *uniformly valid* in the whole flow region[5].

We apply the V-L method to the three-dimensional viscous incompressible fluid flows in a half space $z > 0$ with the plane undeformed boundary $z = 0$. The flows are caused by the given tangential oscillations (vibrations) of the material points of this plane. The motions of the boundary has the representative scales of time T_*, velocity U_*, length L_*, and the frequency ω; together with the constant viscosity coefficient ν they form three independent

[3]The observed delay in the use of this method in fluid dynamics may be partially explained by the fact that there are no streaming flows in the already available applications [6, 12, 13, 14, 15], hence the absence of the global solution can be tolerated and the use of the classical boundary layer theory is quite sufficient for practical purposes.

[4]It is in current use by K.I.Ilin and M.A.Sadiq for the description of the steady streaming caused by a vibrating cylinder or sphere (private communications).

[5]The simplicity of the method (in comparison with the Prandtl boundary layer theory) was emphasised by von Karman, who wrote about it: ‘we carried through this calculation in the manner of the calculus of perturbations’ [6]. However, the further developments of the method have been mainly on the mathematical side, *e.g.* there are theorems within this method, that assess the degree of proximity of asymptotic and exact solutions for important classes of ODEs and PDEs [7, 8, 9].

dimensionless scaling parameters. We express them in terms of the only small parameter $\varepsilon^2 = \nu/(\omega L_*^2)$ in such a way that the viscous term in the Navier-Stokes equations is always proportional to ε^2, while the dimensionless coefficient at the nonlinear term is taken as ε^α, with three different options $\alpha = 1, 2, 3$. To treat the arising asymptotic problems we use the general two-timing framework with two mutually dependent time variables t and $\tau = \omega t$, which we justify *a posteriori* by proving that it leads to the asymptotic solutions of the original governing equations. In order to split up the solution into its oscillating and non-oscillating component we also use the operation of τ-averaging that is purely mathematical and does not require any justification. All our asymptotic solutions are uniformly (in ε) valid in the whole domain $z > 0$ and do not contain any secular terms. In every case ($\alpha = 1, 2, 3$) the calculations of successive approximations are continued until the leading non-oscillating terms (we also call them 'slow motion' or 'streaming') appear. It is shown that the slow motion is described by the full Navies-Stokes equations for $\alpha = 1$ and by the unsteady Stokes equations for $\alpha = 2, 3$. In both cases the effective viscosity is equal to the unity and the effective 'slow' boundary conditions have been precisely derived. It is remarkable that the general mathematical expression for these effective boundary conditions related to the slow part of velocity is the same for all α's. Its structure reveals a sharp qualitative difference between the divergence-free boundary conditions (when the two-dimensional tangent divergence of velocity at the boundary plane is vanishing) and the other boundary conditions with the non-vanishing tangential divergence. Physically, the appearance of such a distinction is quite natural: the presence of a nonzero boundary divergence causes collisions between the different parts of the boundary layer, while the divergence-free boundary conditions correspond to the layers without collisions. The main difference between the solutions for these two cases is the presence (or the absence) of an oscillating peaks of pressure and correspondent potential terms in the velocity fields that are decaying inside a fluid domain with a characteristic length scale L^* related to the inhomogeneity of the boundary conditions (not with the viscous length scale).

As instructive brief examples we consider the wave-like motions of the material points of the boundary in the form of either a standing wave or a travelling wave[6], and the torsional vibrations of an infinite rigid disc, considered by different methods earlier in [19, 20, 21]. For these examples we discuss both cases of steady and unsteady streaming. For the oscillated disc we show that the streaming flow is the same as one caused by a stretching plate [22, 23, 24, 25]. The exploring of interesting examples can be easily continued, however we restrict ourselves only with these few representative flows, emphasising the ways of their obtaining from the general global solution.

[6]The case of a travelling wave can be considered as a high Reynolds model of a 'flying carpet' [16, 17] or a self-swimming sheet [18]

2 Formulation of the Problem

A viscous incompressible fluid fills the upper half of the three-dimensional space ($z > 0$) with Cartesian coordinates $\boldsymbol{x} = (x, y, z)$. The Navier-Stokes equations and the boundary conditions are

$$\boldsymbol{v}^*_t + (\boldsymbol{v}^*\nabla)\boldsymbol{v}^* = -\nabla p^* + \nu\Delta\boldsymbol{v}^*, \quad \operatorname{div}\boldsymbol{v}^* = 0 \quad \text{for} \quad z > 0, \tag{2.1}$$

$$\boldsymbol{v}^* = (\widetilde{a}, \widetilde{b}, 0) = (\widetilde{\boldsymbol{a}}, 0) \quad \text{at} \quad z = 0 \tag{2.2}$$

$$\boldsymbol{v}^* \to 0, \ p^* \to 0 \quad \text{as} \quad z \to \infty \tag{2.3}$$

$$\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right), \quad \Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

where t is time; the subscripts of independent variables denote partial derivatives; $\boldsymbol{v}^*(\boldsymbol{x}, t)$ and $p^*(\boldsymbol{x}, t)$ are the fields of velocity and pressure; ν – constant viscosity; the density of fluid is equal to unity and suppressed that changes the dimension of pressure; two-dimensional 'vector'-field $\widetilde{\boldsymbol{a}} = (\widetilde{a}, \widetilde{b})$ has only (x, y)- components

$$\widetilde{a} = \widetilde{a}(x, y, t, \tau), \quad \widetilde{b} = \widetilde{b}(x, y, t, \tau) \tag{2.4}$$

that are given functions of x, y, t and $\tau \equiv \omega t$; ω is a constant frequency parameter. The dependence of $\widetilde{\boldsymbol{a}}$ on τ is 2π-periodic. The notations with 'tilde' $\widetilde{f} = \widetilde{f}(\boldsymbol{x}, t, \tau)$ have been adapted throughout this paper for the 2π-periodic in τ functions with zero τ-average $\langle\widetilde{f}\rangle \equiv 0$, while the τ-average of any function $f = f(\boldsymbol{x}, t, \tau)$ is defined as the 'partial' integral[7]

$$\langle f\rangle \equiv \frac{1}{2\pi}\int_0^{2\pi} f(\boldsymbol{x}, t, \tau)d\tau \tag{2.5}$$

As one can see τ is a dimensionless variable with the characteristic scale of order unity. Physically, the boundary condition (2.2)[8] gives the no-slip on the plane boundary where material particles are moving with velocity $\widetilde{\boldsymbol{a}} = \widetilde{\boldsymbol{a}}(x, y, t, \omega t)$. The setting of the initial data at $t = 0$ for (2.1)-(2.3) depends on the particular problem; it will be addressed later.

The time variable t is involved into the functional form of the boundary functions (2.4) via two mutually dependent variables t and $\tau \equiv \omega t$. It leads us to the classical 'two-timing' setting: in order to exploit it systematically *we accept that solutions to (2.1)-(2.4) depend on $\boldsymbol{x}$, t, and τ, where the dependence on τ is 2π-periodic*:

$$\boldsymbol{v}^*(\boldsymbol{x}, t) \equiv \widehat{\boldsymbol{v}}(\boldsymbol{x}, t, \tau), \quad p^*(\boldsymbol{x}, t) \equiv \omega\widehat{p}(\boldsymbol{x}, t, \tau) \tag{2.6}$$

[7]The integral (2.5) performs the average over τ while $t = \text{const}$. This operation may look controversial from the physical viewpoint, however it is completely rigorous: for any function $f(t, \tau)$ that is defined in a vicinity of the straight line $\tau = \omega t$ in the plane (t, τ) one can consider partial derivatives and partial integrals without reference to the mutual dependence or independence of the variables t and τ. The existence of such a vicinity is a standard hidden assumption in any multi-scale method.

[8]There is an incompleteness of the boundary conditions at infinity: condition (2.3) at $z \to 0$ is not sufficient and must be complemented by the appropriate restrictions at $x \to \infty$, $y \to \infty$ that appear differently for the different problem settings (such as decreasing, increasing or periodic in variables x, y function $\widetilde{\boldsymbol{a}}$). We imply that such conditions are implicitly formulated and we shall address this issue later.

In our notations all 'hat' functions are 2π-periodic in τ: $\widehat{f}(\boldsymbol{x},t,\tau)=\widehat{f}(\boldsymbol{x},t,\tau+2\pi)$; they generally have nonzero τ-averages $\langle\widehat{f}\rangle\neq 0$ and can be decomposed into their 'tilde' and 'bar' parts,

$$\widehat{f}=\widetilde{f}+\overline{f},\quad \text{with}\quad \overline{f}=\langle\widehat{f}\rangle \tag{2.7}$$

where all 'bar' functions $\overline{f}$ do not depend on τ. For the solutions (2.6) we replace the partial t-derivative in (2.1) by $\partial/\partial t\to\partial/\partial t+\omega\partial/\partial\tau$, following to the chain rule.

Let the dimensional constants of time T_*, velocity U_*, and length L_* be the representative (characteristic) scales involved in $\widetilde{\boldsymbol{a}}$ (2.4) [9]. Then the governing equations (2.1)-(2.3) in their dimensionless form can be rewritten as

$$\widehat{\boldsymbol{v}}_\tau+\nabla\widehat{p}=-S^{(1)}(\widehat{\boldsymbol{v}}\nabla)\widehat{\boldsymbol{v}}+R\Delta\widehat{\boldsymbol{v}}-S^{(2)}\widehat{\boldsymbol{v}}_t,\quad \operatorname{div}\widehat{\boldsymbol{v}}=0\qquad \text{for}\quad z>0 \tag{2.8}$$

$$\widehat{\boldsymbol{v}}=(\widetilde{\boldsymbol{a}},0)\qquad \text{at}\quad z=0 \tag{2.9}$$

$$\widehat{\boldsymbol{v}}\to 0,\quad \widehat{p}\to 0\qquad \text{as}\quad z\to\infty \tag{2.10}$$

$$R=\frac{\nu}{\omega L_*^2},\quad S^{(1)}=\frac{U_*}{\omega L_*},\quad S^{(2)}=\frac{1}{\omega T_*} \tag{2.11}$$

where $\tau\equiv\omega t$ and for brevity we use the same notations for the dimensionless functions and independent variables as for their dimensional counterparts:

$$\widehat{\boldsymbol{v}}/U^*\to\widehat{\boldsymbol{v}},\ \widehat{p}/(L^*U^*)\to\widehat{p},\ \widetilde{\boldsymbol{a}}/U^*\to\widetilde{\boldsymbol{a}},\ \boldsymbol{x}/L^*\to\boldsymbol{x},\ t/T^*\to t \tag{2.12}$$

We are faced with the dimensionless problem (2.8)-(2.10) in the space of three independent dimensionless scaling parameters (two Strouhal numbers $S^{(1)},S^{(2)}$ and the inverse Reynolds number R)[10]. In order to use a rigorous asymptotic procedure we choose in this space one-parametric pathes

$$S^{(1)}(\varepsilon)=\varepsilon^\alpha,\quad S^{(2)}(\varepsilon)=\lambda\varepsilon^\beta,\quad R(\varepsilon)=\kappa\varepsilon^2\quad \text{with}\quad \alpha=1,2,3;\quad \beta=2 \tag{2.13}$$

with constants $\kappa\sim 1$, $\lambda\sim 1$ and a single parameter ε. Below we take $\lambda=1$, since it can be always adsorbed into the definition of the slow time t. Similarly, we can take $\kappa=1$, however we retain κ in our formulas for the tracing of viscous terms. The choice $\beta=2$ (2.13) is aimed to balance the terms with $\partial/\partial t$ and Δ in the final 'slow equations'[11]. The use of (2.13) will allow us to build the asymptotic solutions for tree different limits $\varepsilon\to 0$ for $\alpha=1,2,3$ that correspond to three particular pathes $\varepsilon\to 0$ in the space of independent parameters (2.11),(2.13). In order to reflect the ratio between nonlinear and viscous terms in (2.8) we

[9]These scales can be either mutually dependent or independent.

[10]Our dimensionless equations use $\widehat{\boldsymbol{v}}_\tau$ as the 'reference term' of order unity. Therefore this terminology is not conventional. For example, using the analogy with rotating fluids, R and $S^{(1)}$ can be called the Eckman number and the Rossby number [26].

[11]It is not crucial and can be relaxed by allowing any $\beta\geqslant 2$ (see Section 6). The case of the steady streaming corresponds to the pure oscillatory boundary condition where T_* is absent ($\beta\to\infty$ or $S^{(2)}=0$).

call the cases of $\alpha = 1, 2, 3$ the *strong, moderate, and weak nonlinearity* correspondingly[12]. Hence the system (2.8)-(2.10) takes the form

$$\widehat{\boldsymbol{v}}_\tau + \nabla \widehat{p} = -\varepsilon^\alpha (\widehat{\boldsymbol{v}}\nabla)\widehat{\boldsymbol{v}} + \varepsilon^2(\kappa\Delta\widehat{\boldsymbol{v}} - \widehat{\boldsymbol{v}}_t), \quad \operatorname{div}\widehat{\boldsymbol{v}} = 0 \quad \text{for} \quad z > 0 \tag{2.14}$$

$$\widehat{\boldsymbol{v}} = (\widetilde{\boldsymbol{a}}, 0) \quad \text{at} \quad z = 0 \tag{2.15}$$

$$\widehat{\boldsymbol{v}} \to 0, \quad \widehat{p} \to 0 \quad \text{as} \quad z \to \infty \tag{2.16}$$

In (2.14)-(2.16) we assume that all involved functions and their derivatives are of order unity, and the only available small parameter ε is given explicitly. As the next step in our asymptotic procedure we temporary consider the originally dependent variables t and τ in (2.14)-(2.16) as being independent. This is a standard auxiliary step for the two-timing method, it will be *a posteriori* mathematically justified.

Following the general procedure of the V-L method we look for a solution to (2.14)-(2.16) in the form[13]

$$\widehat{\boldsymbol{v}} = (U + u, V + v, W + \varepsilon w), \quad \widehat{p} = P + p \quad \text{for } z > 0 \text{ and } s > 0 \tag{2.17}$$

where $\boldsymbol{V} \equiv (U, V, W)$ and P are functions of x, y, z, t, τ, while $\boldsymbol{v} \equiv (u, v, w)$ and p are functions of x, y, s, t, τ with a 'stretched' vertical variable $s \equiv z/\varepsilon$. All functions involved are 2π-periodic in τ, however to simplify the notations we suppress the 'hats' in the right hand sides of (2.17) and everywhere below.

According to its traditional meaning $\boldsymbol{V}, P$ give the 'external solution' outside the boundary layer, while $\boldsymbol{v}, p$ represent the 'inner part' of the solution[14] in the boundary layer itself and must rapidly decrease $\boldsymbol{v} \to 0, p \to 0$ (along with their derivatives) when $s \to \infty$. For convenience, the small parameter ε is introduced as the multiplier to w in (2.17); if we do not do it, then the term of zero order ($\varepsilon^0 = 1$) in w always vanishes in all asymptotic procedures below.

The next key assumption is: the 'external' solution $\boldsymbol{V}, P$ satisfies the exact governing equations (2.14):

$$\boldsymbol{V}_\tau - \nabla P = -\varepsilon^\alpha (\boldsymbol{V}\nabla)\boldsymbol{V} + \varepsilon^2(\kappa\Delta\boldsymbol{V} - \boldsymbol{V}_t), \quad \operatorname{div}\boldsymbol{V} = 0 \quad \text{for} \quad z > 0 \tag{2.18}$$

[12] One can also recall that in the general asymptotic procedure all parameters $L_*, U_*, T_*, \omega, \nu$ in (2.13) can be considered as arbitrary functions of ε (as $\varepsilon \to 0$), provided that we stay on the chosen path (2.13). A simple example $L_* \sim$ const, $T_* \sim$ const, $\nu \sim$ const, $\omega \sim \varepsilon^{-2}$ corresponds to $U_* \sim \varepsilon^{\alpha-2}$, hence the case $\alpha = 2$ looks the most 'natural' physically; however other cases can have their own areas of applicability. As a well known example of an 'unusual' dependence of physical parameters on ε one can mention that the rigorous asymptotic procedure for the 'inverted pendulum' is based on the vibrations of a pivot with the spatial amplitude $\sim \varepsilon^{-1}$; another example is the Stokes drift where the amplitude of the oscillating velocity $\sim \sqrt{\varepsilon}$ [27, 28].

[13] Notice, that we use the notations u, v, w for the components of the partial (artificially defined) field $\boldsymbol{v}$, NOT for the physical velocities $\boldsymbol{v}^*$ or $\widehat{\boldsymbol{v}}$.

[14] In the mathematical papers on asymptotic methods the standard terminology is the opposite, *e.g.* the 'inner solution' corresponds to the exterior of the boundary layer. We avoid the use of the purely mathematical terminology; the reader can find this terminology along with the rigorous exposition of the method in [7, 8, 9].

Then the substitution of (2.17) into (2.14) with taking (2.18) into account and replacing z with $s \equiv z/\varepsilon$ produces the equations for the 'inner' part of the solution:

$$\begin{aligned}
\boldsymbol{u}_\tau - \kappa \boldsymbol{u}_{ss} + \nabla_{\boldsymbol{y}} p &= -\varepsilon^2(\boldsymbol{u}_t - \kappa\Delta_{\boldsymbol{y}}\boldsymbol{u}) - \varepsilon^{\alpha-1} W \boldsymbol{u}_s - \quad \text{for} \quad s > 0 \\
&\quad -\varepsilon^\alpha[(\boldsymbol{U}\nabla_{\boldsymbol{y}})\boldsymbol{u} + (\boldsymbol{u}\nabla_{\boldsymbol{y}})\boldsymbol{U} + (\boldsymbol{u}\nabla_{\boldsymbol{y}} + w\nabla_s)\boldsymbol{u}] - \varepsilon^{\alpha+1}\boldsymbol{U}_z w, \\
p_s &= -\varepsilon^2(w_\tau - \kappa w_{ss}) - \varepsilon^{\alpha+1}[(\boldsymbol{u}\nabla_{\boldsymbol{y}})W + W w_s] - \\
&\quad -\varepsilon^{\alpha+2}[(\boldsymbol{U}+\boldsymbol{u})\nabla_{\boldsymbol{y}} w + w(W_z + w_s)] - \varepsilon^4(w_t - \kappa\Delta_{\boldsymbol{y}} w), \\
w_s &= -\nabla_{\boldsymbol{y}}\boldsymbol{u}
\end{aligned} \tag{2.19}$$

where we have introduced several auxiliary two-dimensional 'vectors' with (x, y)-components only: $\boldsymbol{y} \equiv (x, y)$, $\boldsymbol{u} \equiv (u, v)$, $\boldsymbol{U} \equiv (U, V)$, $\nabla_{\boldsymbol{y}} \equiv (\partial/\partial x, \partial/\partial y)$, as well as $\Delta_{\boldsymbol{y}} \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$. We also use $\nabla_z \equiv \partial/\partial z$ and $\nabla_s \equiv \partial/\partial s$. Equations (2.19) are expressed in the spatial variables $\boldsymbol{y}, s$; hence in the functions $\boldsymbol{V} = (U, V, W)$ one must replace z with εs and use the Taylor series

$$\boldsymbol{V}(\varepsilon s) = \boldsymbol{V}(0) + \sum_1^\infty \frac{\varepsilon^n}{n!}(\nabla_z^n \boldsymbol{V})_{z=0} s^n \quad \text{for} \quad s \geqslant 0 \tag{2.20}$$

where the variables $\boldsymbol{y}, t, \tau$ have been suppressed. The substitution of (2.20) into (2.19) reveals the complete dependence of the 'inner' equations on ε. For brevity we will use in all 'inner' equations below the notations $\boldsymbol{V}$, $\nabla_z^n \boldsymbol{V}$, *etc.*, omitting the reference that they all are taken at $z = 0$.

Equations (2.18)-(2.20) represent the system of equations we have to solve. It has to be complemented by the boundary conditions (2.15), (2.16) that take form:

$$\boldsymbol{U} + \boldsymbol{u} = \widetilde{\boldsymbol{a}}, \quad W + \varepsilon w = 0, \qquad \text{at} \quad z = s = 0 \tag{2.21}$$

$$\boldsymbol{V} \to 0, \ P \to 0 \qquad \text{as } z \to \infty \tag{2.22}$$

$$\boldsymbol{v} \to 0, \ p \to 0 \qquad \text{as } s \to \infty \tag{2.23}$$

We look for the solution U, V, W, P, and u, v, w, p (all together eight unknown functions that satisfy (2.18)-(2.23)) in the form of regular series in ε such as

$$f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots \quad \text{where} \quad f = U, V, W, P, u, v, w, p \tag{2.24}$$

where all terms in the series are 2π-periodic in τ.

For the further use we introduce the following shorthands:

- Operator $\mathfrak{D}[\phi, q]$ ('Diffusion operator') for the solution of 1+1 diffusion problem

  $$f = \mathfrak{D}[\phi, q]: \quad f_\tau - \kappa f_{ss} = q \text{ for } s > 0, \quad f = \phi \text{ at } s = 0, \quad f \to 0 \text{ as } s \to \infty \tag{2.25}$$

  where an unknown function is $f = f(\boldsymbol{y}, t, s, \tau)$, the boundary datum and the source terms are $\phi = \phi(\boldsymbol{y}, t, \tau)$ and $q = q(\boldsymbol{y}, t, s, \tau)$; the variables $\boldsymbol{y}, t$ play part of parameters[15]. In the special case of the homogeneous equation $q \equiv 0$ we adapt a reduced notation

  $$\mathfrak{D}[\phi, 0] \equiv \mathfrak{D}[\phi] \quad \text{for} \quad s \geqslant 0 \tag{2.26}$$

[15]The initial data at $\tau = 0$ are not relevant, since we work within the class of 2π-periodic in τ functions.

- Operator $\mathfrak{A}[g]$ ('s-averaging operator') for the averaging over $s < \xi < \infty$:

$$\mathfrak{A}[g] \equiv \int_s^\infty g(\boldsymbol{y}, \xi, t, \tau) d\xi \quad \text{for} \quad s \geqslant 0 \tag{2.27}$$

  We will also use

$$\mathfrak{A}^*[g] \equiv \int_0^s g(\boldsymbol{y}, \xi, t, \tau) d\xi; \quad \mathfrak{A}^2[g] = \int_s^\infty \left(\int_\xi^\infty g(\boldsymbol{y}, \eta, t, \tau) d\eta \right) d\xi, \quad etc. \tag{2.28}$$

  One can see that $\mathfrak{A}^n[g] \to 0$ when $s \to \infty$ for any integer n.

- Operator $\mathfrak{P}[\psi]$ ('Potential operator') for the solution of Neumann's problem for Laplace's equation[16]

$$\Phi = \mathfrak{P}[\psi]: \qquad \Delta\Phi = 0 \text{ for } z > 0; \quad \Phi_z = \psi \text{ at } z = 0, \quad \nabla\Phi \to 0 \text{ as } z \to \infty \tag{2.29}$$

  where $\Phi = \Phi(\boldsymbol{y}, z, t, \tau)$ and $\psi = \psi(\boldsymbol{y}, t, \tau)$.

All three introduced operators (or functionals $\mathfrak{D}, \mathfrak{A}, \mathfrak{P}$) are linear, they provide the uniqueness of the solutions, and are equal to zero on the zero functions $\mathfrak{D}[0, 0] = \mathfrak{D}[0] = \mathfrak{P}[0] = \mathfrak{A}[0] = 0$. Their commuting rules (between themselves as well as with $\nabla_{\boldsymbol{y}}$) are apparent. Below the functions ϕ and ψ (that play a part of the initial data at $s = 0$ or $z = 0$ in operators $\mathfrak{D}$ or $\mathfrak{P}$) can also depend on s or z. In this case in the arguments of ϕ or ψ one must put $s = 0$ or $z = 0$. For example, the sequence of operators $\mathfrak{P}\mathfrak{A}[g]$ means that the previous to $\mathfrak{P}$ operation $\mathfrak{A}$ (averaging over s (2.27)) is automatically taken from 0 to ∞. The explicit analytical forms of operators $\mathfrak{D}$ (2.25), (2.26) and $\mathfrak{P}$ (2.29) (*e.g.* in terms of Green's functions) are standard in mathematical physics [29].

3 The Strong Nonlinearity $\alpha = 1$

3.1 The zeroth approximation for $\alpha = 1$

The terms of order $\varepsilon^0 = 1$ in (2.18)-(2.24) are:

$$\boldsymbol{V}_{0\tau} = -\nabla P_0, \quad \operatorname{div} \boldsymbol{V}_0 = 0 \qquad \text{for} \quad z > 0 \tag{3.1}$$

$$\boldsymbol{u}_{0\tau} - \kappa \boldsymbol{u}_{0ss} + \nabla_{\boldsymbol{y}} p_0 = -W_0 \boldsymbol{u}_{0s}, \quad p_{0s} = 0, \quad w_{0s} = -\nabla_{\boldsymbol{y}} \boldsymbol{u}_0 \qquad \text{for } s > 0 \tag{3.2}$$

$$\boldsymbol{U}_0 + \boldsymbol{u}_0 = \widetilde{\boldsymbol{a}}, \quad W_0 = 0, \qquad \text{at} \quad z = s = 0 \tag{3.3}$$

$$\boldsymbol{V}_0 \to 0, \ P_0 \to 0 \qquad \text{as } z \to \infty \tag{3.4}$$

$$\boldsymbol{v}_0 \to 0, \ p_0 \to 0 \qquad \text{as } s \to \infty \tag{3.5}$$

It is convenient to describe its' solution as a sequence of steps.

[16] Again, we imply that the boundary conditions at infinity in (2.29) are complemented by the appropriate restrictions in the variables $\boldsymbol{y} = (x, y)$; it will be discussed later.

Step 1: The integration of the equation $p_{0s} = 0$ (3.2) over s gives $p_0 = p_0(\boldsymbol{y}, t, \tau)$ that after the use of a zero limit for p_0 in (3.5) yields $p_0 \equiv 0$ (which implies both $\overline{p}_0 \equiv 0$ and $\widetilde{p}_0 \equiv 0$). Also in the equation for $\boldsymbol{u}_0$ (3.2) the function $W_0 = W_0(\boldsymbol{y}, 0, t, \tau) \equiv 0$ as it is stated as the boundary condition for W_0 (3.3).

Step 2: We use the fact that all considered functions are 2π-periodic in τ (2.6): hence they can be split into the 'bar' (independent of τ) and 'tilde' parts (2.7). The 'bar' part[17] of (3.1)-(3.5) is:

$$
\begin{aligned}
0 = -\nabla \overline{P}_0, \quad \operatorname{div} \overline{\boldsymbol{V}}_0 = 0 \quad & \text{for } z > 0 & (3.6)\\
\kappa \overline{\boldsymbol{u}}_{0ss} = 0, \quad \overline{w}_{0s} = -\nabla_{\boldsymbol{y}} \overline{\boldsymbol{u}}_0 \quad & \text{for } s > 0 & (3.7)\\
\overline{\boldsymbol{U}}_0 + \overline{\boldsymbol{u}}_0 = 0, \quad \overline{W}_0 = 0, \quad & \text{at} \quad z = s = 0 & (3.8)\\
\overline{\boldsymbol{V}}_0 \to 0, \ \overline{P}_0 \to 0 \quad & \text{as } z \to \infty & (3.9)\\
\overline{\boldsymbol{v}}_0 \to 0 \quad & \text{as } s \to \infty & (3.10)
\end{aligned}
$$

The integration of $\nabla \overline{P}_0 = 0$ (3.6) over x, y, z and the use of (3.9) yield $\overline{P}_0 \equiv 0$. Similarly, the integration of $\overline{\boldsymbol{u}}_{0ss} = 0$ (3.7) gives $\overline{\boldsymbol{u}}_0$ as a linear function of s, hence the decay of $\overline{\boldsymbol{u}}_0$ at $s \to \infty$ (3.10) means that $\overline{\boldsymbol{u}}_0 \equiv 0$. Then the latter equation (3.7) is $\overline{w}_{0s} = 0$ that together with the condition at infinity for $\overline{w}_0$ (3.10) produces $\overline{w}_0 \equiv 0$. The result is $\overline{\boldsymbol{v}}_0 = (\overline{u}_0, \overline{v}_0, \overline{w}_0) \equiv 0$ for $s \geqslant 0$. The use of $\overline{\boldsymbol{u}}_0 = 0$ in the boundary conditions (3.8) gives $\overline{\boldsymbol{V}}_0 = 0$ at $z = 0$. Summarizing all 'bar' results we write:

$$
\begin{aligned}
\overline{P}_0 \equiv 0, \ \operatorname{div} \overline{\boldsymbol{V}}_0 = 0 \quad & \text{for } z > 0\\
\overline{\boldsymbol{V}}_0 = 0 \quad & \text{at } z = 0\\
\overline{\boldsymbol{v}}_0 \equiv 0 \quad & \text{for } s \geqslant 0
\end{aligned}
$$

Step 3: The 'tilde' part of (3.1)-(3.5) is:

$$
\begin{aligned}
\widetilde{\boldsymbol{V}}_{0\tau} = -\nabla \widetilde{P}_0, \quad \operatorname{div} \widetilde{\boldsymbol{V}}_0 = 0 \quad & \text{for} \quad z > 0 & (3.11)\\
\widetilde{\boldsymbol{u}}_{0\tau} = \kappa \widetilde{\boldsymbol{u}}_{0ss}, \quad \widetilde{w}_{0s} = -\nabla_{\boldsymbol{y}} \widetilde{\boldsymbol{u}}_0 \quad & \text{for} \quad s > 0 & (3.12)\\
\widetilde{\boldsymbol{U}}_0 + \widetilde{\boldsymbol{u}}_0 = \widetilde{\boldsymbol{a}}, \quad \widetilde{W}_0 = 0, \quad & \text{at} \quad z = s = 0 & (3.13)\\
\widetilde{\boldsymbol{V}}_0 \to 0, \ \widetilde{P}_0 \to 0 \quad & \text{as } z \to \infty & (3.14)\\
\widetilde{\boldsymbol{v}}_0 \to 0, \ \widetilde{p}_0 \to 0 \quad & \text{as } s \to \infty & (3.15)
\end{aligned}
$$

Equations (3.11), the latter one in (3.13), and (3.14) form a self-contained system

$$
\begin{aligned}
\widetilde{\boldsymbol{V}}_{0\tau} = -\nabla \widetilde{P}_0, \quad \operatorname{div} \widetilde{\boldsymbol{V}}_0 = 0 \quad & \text{for} \quad z > 0\\
\widetilde{W}_0 = 0 \quad & \text{at } z = 0,\\
\widetilde{\boldsymbol{V}}_0 \to 0, \ \widetilde{P}_0 \to 0 \quad & \text{as } z \to \infty
\end{aligned}
$$

[17] It mathematically appears as the result of the τ-averaging (2.5) of (3.1)-(3.5).

which describes a potential motion[18] $\widetilde{\boldsymbol{V}}_0 = \nabla\widetilde{\Phi}_0$, $\widetilde{\Phi}_{0\tau} = -\widetilde{P}_0$ with zero boundary conditions for the normal velocity and with decay at infinity:

$$\begin{aligned} \Delta\widetilde{\Phi}_0 = 0 \quad & \text{for} \quad z > 0 \\ \widetilde{\Phi}_{0z} = 0 \quad & \text{at} \quad z = 0 \\ \nabla\widetilde{\Phi}_0 \to 0 \quad & \text{as } z \to \infty \end{aligned}$$

By the uniqueness of Neumann's problem ($\mathfrak{P}[0] = 0$ (2.29)) we get $\widetilde{\boldsymbol{V}}_0 \equiv 0$. Then the remaining equations (3.11)-(3.15) give us a sequence of two problems. The first one

$$\begin{aligned} \widetilde{\boldsymbol{u}}_{0\tau} = \kappa\widetilde{\boldsymbol{u}}_{0ss}, \quad & \text{for } s > 0 \\ \widetilde{\boldsymbol{u}}_0 = \widetilde{\boldsymbol{a}} \quad & \text{at } s = 0 \\ \widetilde{\boldsymbol{u}}_0 \to 0 \quad & \text{as } s \to \infty \end{aligned}$$

can be solved according to (2.25), (2.26) as:

$$\widetilde{\boldsymbol{u}}_0 = \mathfrak{D}[\widetilde{\boldsymbol{a}}] \quad \text{for} \quad s \geqslant 0 \tag{3.16}$$

The second problem is

$$\begin{aligned} \widetilde{w}_{0s} = -\nabla_{\boldsymbol{y}} \cdot \widetilde{\boldsymbol{u}}_0 \quad & \text{for } s > 0 \\ \widetilde{w}_0 \to 0 \quad & \text{as } s \to \infty \end{aligned}$$

with the solution presented as

$$\widetilde{w}_0 = \nabla_{\boldsymbol{y}} \int_s^\infty \widetilde{\boldsymbol{u}}_0(\boldsymbol{y}, \eta, t, \tau) d\eta \equiv \nabla_{\boldsymbol{y}}\mathfrak{A}[\widetilde{\boldsymbol{u}}_0] = \nabla_{\boldsymbol{y}}\mathfrak{A}\mathfrak{D}[\widetilde{\boldsymbol{a}}] = \mathfrak{A}\mathfrak{D}[\nabla_{\boldsymbol{y}}\widetilde{\boldsymbol{a}}] \text{ for } s \geqslant 0 \tag{3.17}$$

where the operator $\mathfrak{A}$ is given in (2.27).

Step 4: Putting all results of the zero approximation together we write that

$$\overline{P}_0 \equiv 0,\ \widetilde{P}_0 \equiv 0,\ \widetilde{\boldsymbol{V}}_0 \equiv 0 \quad \text{for} \quad z \geqslant 0 \tag{3.18}$$

$$\overline{p}_0 \equiv 0,\ \widetilde{p}_0 \equiv 0,\ \overline{\boldsymbol{v}}_0 \equiv 0,\ \widetilde{\boldsymbol{u}}_0 = \mathfrak{D}[\widetilde{\boldsymbol{a}}],\ \widetilde{w}_0 = \mathfrak{A}\mathfrak{D}[\nabla_{\boldsymbol{y}}\widetilde{\boldsymbol{a}}] \quad \text{for} \quad s \geqslant 0 \tag{3.19}$$

while for the function $\overline{\boldsymbol{V}}_0$ we obtained only the constraints

$$\begin{gathered} \operatorname{div} \overline{\boldsymbol{V}}_0 = 0 \qquad \text{for } z > 0 \\ \overline{\boldsymbol{V}}_0 = 0 \text{ at } z = 0,\ \overline{\boldsymbol{V}}_0 \to 0 \text{ as } z \to \infty \end{gathered} \tag{3.20}$$

[18]The potential $\widetilde{\Phi}_0$ is completely determined via pressure $\widetilde{P}_0$ by the equation $\widetilde{\Phi}_{0\tau} = -\widetilde{P}_0$, since an arbitrary function of $\boldsymbol{x}, t$ that appears after the integration over τ vanishes due to the constraints of τ-periodicity and zero τ-average of both functions.

3.2 The first approximation for $\alpha = 1$

The terms of order $\varepsilon^1 = \varepsilon$ in (2.18)-(2.24) are:

$$\boldsymbol{V}_{1\tau} = -\nabla P_1 - (\boldsymbol{V}_0\nabla)\boldsymbol{V}_0, \quad \mathrm{div}\,\boldsymbol{V}_1 = 0 \qquad \text{for} \quad z > 0 \tag{3.21}$$

$$\begin{aligned} \boldsymbol{u}_{1\tau} - \kappa\boldsymbol{u}_{1ss} + \nabla_{\boldsymbol{y}} p_1 = -W_0\boldsymbol{u}_{1s} - (W_1 + sW_{0z})\boldsymbol{u}_{0s} - \\ -(\boldsymbol{U}_0\nabla_{\boldsymbol{y}})\boldsymbol{u}_0 - (\boldsymbol{u}_0\nabla_{\boldsymbol{y}})\boldsymbol{U}_0 - (\boldsymbol{u}_0\cdot\nabla_{\boldsymbol{y}} + w_0\nabla_s)\boldsymbol{u}_0 \end{aligned} \qquad \text{for} \quad s > 0 \tag{3.22}$$

$$p_{1s} = 0, \quad w_{1s} = -\nabla_{\boldsymbol{y}}\cdot\boldsymbol{u}_1 \qquad \text{for} \quad s > 0 \tag{3.23}$$

$$\boldsymbol{U}_1 + \boldsymbol{u}_1 = 0, \quad W_1 + w_0 = 0, \qquad \text{at} \quad z = s = 0 \tag{3.24}$$

$$\boldsymbol{V}_1 \to 0, \; P_1 \to 0 \qquad \text{as } z \to \infty \tag{3.25}$$

$$\boldsymbol{v}_1 \to 0, \; p_1 \to 0 \qquad \text{as } s \to \infty \tag{3.26}$$

Again, we solve this problem as a sequence of steps.

Step 1: The 'bar' part of the former equation in (3.21) with taking into account $\widetilde{\boldsymbol{V}}_0 \equiv 0$ (3.18) gives:

$$(\overline{\boldsymbol{V}}_0\cdot\nabla)\overline{\boldsymbol{V}}_0 = -\nabla\overline{P}_1, \quad \text{for} \quad z > 0$$

The combining of this equation with (3.20) produces the full problem for steady Euler's equation with zero boundary conditions for all components of $\overline{\boldsymbol{V}}_0$: the only solution available is $\overline{\boldsymbol{V}}_0 \equiv 0$. Combining it with $\widetilde{\boldsymbol{V}}_0 \equiv 0$ (3.18) we get

$$\boldsymbol{V}_0 = \overline{\boldsymbol{V}}_0 = \widetilde{\boldsymbol{V}}_0 \equiv 0 \quad \text{for} \quad z \geqslant 0 \tag{3.27}$$

and the finding of all functions (3.18)-(3.20) involved into the zero approximation is completed.

Step 2: The integration of the equation $p_{1s} = 0$ (3.23) gives $p_1 = p_1(\boldsymbol{y}, t, \tau)$, which together with the condition for pressure in (3.26) yields $p_1 \equiv 0$.

Step 3: Using $\boldsymbol{V}_0 \equiv 0$ and $p_1 \equiv 0$ we write the remaining unsolved part of (3.21)-(3.26) as

$$\boldsymbol{V}_{1\tau} = -\nabla P_1, \quad \mathrm{div}\,\boldsymbol{V}_1 = 0 \qquad \text{for} \quad z > 0 \tag{3.28}$$

$$\boldsymbol{u}_{1\tau} - \kappa\boldsymbol{u}_{1ss} = -\boldsymbol{q}_1 \qquad \text{for} \quad s > 0 \tag{3.29}$$

$$w_{1s} = -\nabla_{\boldsymbol{y}}\boldsymbol{u}_1 \qquad \text{for} \quad s > 0 \tag{3.30}$$

$$\boldsymbol{U}_1 + \boldsymbol{u}_1 = 0, \quad W_1 = -w_0 \qquad \text{at} \quad z = s = 0 \tag{3.31}$$

$$\boldsymbol{V}_1 \to 0, \; P_1 \to 0 \qquad \text{as } z \to \infty \tag{3.32}$$

$$\boldsymbol{v}_1 \to 0 \qquad \text{as } s \to \infty \tag{3.33}$$

Combining $\overline{\boldsymbol{v}}_0 \equiv 0$ from (3.19) with $\overline{W}_1 = 0$, $\widetilde{W}_1 = -\widetilde{w}_0$ at $z = 0$ (that both follow from the latter equality in (3.31)) one can get

$$\boldsymbol{q}_1 \equiv \{(\boldsymbol{u}_0\nabla_{\boldsymbol{y}}) + (W_1 + w_0)\nabla_s\}\boldsymbol{u}_0 = \boldsymbol{Q}, \tag{3.34}$$

$$\boldsymbol{Q} \equiv \{\widetilde{\boldsymbol{u}}_0\nabla_{\boldsymbol{y}} + (\widetilde{W}_1 + \widetilde{w}_0)\nabla_s\}\widetilde{\boldsymbol{u}}_0 = \{\widetilde{\boldsymbol{u}}_0\nabla_{\boldsymbol{y}} - (\mathfrak{A}^*\nabla_{\boldsymbol{y}}\cdot\widetilde{\boldsymbol{u}}_0)\nabla_s\}\widetilde{\boldsymbol{u}}_0 \tag{3.35}$$

where $\mathfrak{A}^*$ is defined in (2.28). The introduced value $\boldsymbol{Q}$ plays a key part in all our results below; by virtue of (3.16),(3.19) it can be explicitly expressed as:

$$\boldsymbol{Q} = \{\mathfrak{D}[\widetilde{\boldsymbol{a}}]\nabla_{\boldsymbol{y}} - (\mathfrak{A}^*\mathfrak{D}[\nabla_{\boldsymbol{y}}\widetilde{\boldsymbol{a}}])\nabla_s\}\mathfrak{D}[\widetilde{\boldsymbol{a}}] \quad \text{for} \quad s \geqslant 0 \tag{3.36}$$

Step 4: The 'bar' part of (3.28)-(3.33) can be written as:

$$\overline{P}_1 \equiv 0, \quad \operatorname{div} \overline{\boldsymbol{V}}_1 = 0 \quad \text{for} \quad z > 0 \tag{3.37}$$
$$\kappa \overline{\boldsymbol{u}}_{1ss} = \overline{\boldsymbol{q}}_1 \quad \text{for} \quad s > 0 \tag{3.38}$$
$$\overline{w}_{1s} = -\nabla_{\boldsymbol{y}}\overline{\boldsymbol{u}}_1 \quad \text{for} \quad s > 0 \tag{3.39}$$
$$\overline{\boldsymbol{U}}_1 + \overline{\boldsymbol{u}}_1 = 0, \quad \overline{W}_1 = 0 \quad \text{at} \quad z = s = 0 \tag{3.40}$$
$$\overline{\boldsymbol{V}}_1 \to 0, \ \overline{P}_1 \to 0 \quad \text{as } z \to \infty \tag{3.41}$$
$$\overline{\boldsymbol{v}}_1 \to 0 \quad \text{as } s \to \infty \tag{3.42}$$

Now $\overline{\boldsymbol{q}}_1 = \overline{\boldsymbol{Q}}$ (3.34) is known, hence the integrations of (3.38),(3.39) with the use of (3.42) gives $\overline{\boldsymbol{v}}_1$ as

$$\overline{\boldsymbol{u}}_1 = \frac{1}{\kappa}\int_s^\infty \left(\int_\xi^\infty \overline{\boldsymbol{Q}}(\boldsymbol{y},\eta,t)d\eta\right)d\xi = \frac{1}{\kappa}\mathfrak{A}^2\overline{\boldsymbol{Q}} \quad \text{for} \quad s \geqslant 0 \tag{3.43}$$
$$\overline{w}_1 = \nabla_{\boldsymbol{y}}\int_s^\infty \overline{\boldsymbol{u}}_1(\boldsymbol{y},\xi,t)d\xi = \nabla_{\boldsymbol{y}}\mathfrak{A}\overline{\boldsymbol{u}}_1 = \frac{1}{\kappa}\nabla_{\boldsymbol{y}}\mathfrak{A}^3\overline{\boldsymbol{Q}}$$

with the operator $\mathfrak{A}^n$ defined in (2.28).

Step 5: The 'tilde' part of (3.28)-(3.33) is:

$$\widetilde{\boldsymbol{V}}_{1\tau} = -\nabla\widetilde{P}_1, \quad \operatorname{div}\widetilde{\boldsymbol{V}}_1 = 0 \quad \text{for} \quad z > 0 \tag{3.44}$$
$$\widetilde{\boldsymbol{u}}_{1\tau} - \kappa\widetilde{\boldsymbol{u}}_{1ss} = -\widetilde{\boldsymbol{Q}} \quad \text{for} \quad s > 0 \tag{3.45}$$
$$\widetilde{w}_{1s} = -\nabla_{\boldsymbol{y}}\widetilde{\boldsymbol{u}}_1 \quad \text{for} \quad s > 0 \tag{3.46}$$
$$\widetilde{\boldsymbol{U}}_1 + \widetilde{\boldsymbol{u}}_1 = 0, \quad \widetilde{W}_1 = -\widetilde{w}_0 \quad \text{at} \quad z = s = 0 \tag{3.47}$$
$$\widetilde{\boldsymbol{V}}_1 \to 0, \ \widetilde{P}_1 \to 0 \quad \text{as } z \to \infty \tag{3.48}$$
$$\widetilde{\boldsymbol{v}}_1 \to 0 \quad \text{as } s \to \infty \tag{3.49}$$

First we consider its subsystem

$$\widetilde{\boldsymbol{V}}_{1\tau} = -\nabla\widetilde{P}_1, \quad \operatorname{div}\widetilde{\boldsymbol{V}}_1 = 0 \quad \text{for} \quad z > 0$$
$$\widetilde{W}_1 = -\mathfrak{A}\mathfrak{D}[\nabla_{\boldsymbol{y}}\widetilde{\boldsymbol{a}}] \quad \text{at} \quad z = s = 0$$
$$\widetilde{\boldsymbol{V}}_1 \to 0, \ \widetilde{P}_1 \to 0 \quad \text{as } z \to \infty$$

where the expression for $\widetilde{w}_0$ at $s = 0$ is taken from (3.19) and the s-averaging $\mathfrak{A}$ is taken over $(0,\infty)$. It can be rewritten as Neumann's problem for the potential[19] $\widetilde{\Phi}_1(\boldsymbol{x},t,\tau)$

$$\Delta\widetilde{\Phi}_1 = 0 \quad \text{for} \quad z > 0$$
$$\widetilde{\Phi}_{1z} = -\mathfrak{A}\mathfrak{D}[\nabla_{\boldsymbol{y}}\widetilde{\boldsymbol{a}}] \quad \text{at } z = 0$$
$$\nabla\widetilde{\Phi}_1 \to 0 \quad \text{as } z \to \infty$$

[19] Again, the potential $\widetilde{\Phi}_1$ is uniquely determined via pressure $\widetilde{P}_1$ due to the constraints of τ-periodicity and zero τ-average.

such that:

$$\widetilde{\boldsymbol{V}}_1 = \nabla\widetilde{\Phi}_1, \quad \widetilde{\Phi}_{1\tau} = \widetilde{P}_1 \tag{3.50}$$

According to (2.29) its solution is

$$\widetilde{\Phi}_1(\boldsymbol{x}, t, \tau) = -\mathfrak{P}\mathfrak{A}\mathfrak{D}[\nabla_{\boldsymbol{y}}\widetilde{\boldsymbol{a}}] \quad \text{for } z \geqslant 0 \tag{3.51}$$

Next we can consider the remaining part of the system (3.44)-(3.49):

$$\begin{aligned} \widetilde{\boldsymbol{u}}_{1\tau} - \kappa\widetilde{\boldsymbol{u}}_{1ss} &= -\widetilde{\boldsymbol{q}}_1 && \text{for } s > 0 \\ \widetilde{w}_{1s} &= -\nabla_{\boldsymbol{y}}\widetilde{\boldsymbol{u}}_1 && \text{for } s > 0 \\ \widetilde{\boldsymbol{u}}_1 &= \nabla_{\boldsymbol{y}}\mathfrak{P}\mathfrak{A}\mathfrak{D}[\nabla_{\boldsymbol{y}}\widetilde{\boldsymbol{a}}] && \text{at} \quad s = 0 \\ \widetilde{\boldsymbol{v}}_1 &\to 0 && \text{as } s \to \infty \end{aligned}$$

with $\widetilde{\boldsymbol{U}}_1 = \nabla_{\boldsymbol{y}}\widetilde{\Phi}_1$ taken from (3.51). It gives us an inhomogeneous 1+1 diffusion problem for $\widetilde{\boldsymbol{u}}_1$. According to (2.25),(3.34) its solution is:

$$\widetilde{\boldsymbol{u}}_1 = \mathfrak{D}[\nabla_{\boldsymbol{y}}\mathfrak{P}\mathfrak{A}\mathfrak{D}[\nabla_{\boldsymbol{y}}\widetilde{\boldsymbol{a}}], -\widetilde{\boldsymbol{Q}}] \quad \text{for } s \geqslant 0 \tag{3.52}$$

Step 6: At this stage we have found all unknown functions of the first approximation except $\overline{\boldsymbol{V}}_1$. In order to find $\overline{\boldsymbol{V}}_1$ we have to consider the original $\boldsymbol{V}$-equation (2.18) for the next two approximations:

$$\begin{aligned} \boldsymbol{V}_{2\tau} &= -\nabla P_2 + \kappa\Delta\boldsymbol{V}_0 - \boldsymbol{V}_{0t} - (\boldsymbol{V}_0\nabla)\boldsymbol{V}_1 - (\boldsymbol{V}_1\nabla)\boldsymbol{V}_0 \\ \boldsymbol{V}_{3\tau} &= -\nabla P_3 + \kappa\Delta\boldsymbol{V}_1 - \boldsymbol{V}_{1t} - (\boldsymbol{V}_0\nabla)\boldsymbol{V}_2 - (\boldsymbol{V}_2\nabla)\boldsymbol{V}_0 - (\boldsymbol{V}_1\nabla)\boldsymbol{V}_1, \end{aligned}$$

With $\boldsymbol{V}_0 \equiv 0$ (3.27) these equations are reduced to

$$\boldsymbol{V}_{2\tau} = -\nabla P_2, \quad \boldsymbol{V}_{3\tau} = -\nabla P_3 + \kappa\Delta\boldsymbol{V}_1 - \boldsymbol{V}_{1t} - (\boldsymbol{V}_1\nabla)\boldsymbol{V}_1$$

The former equation allows us to find $\boldsymbol{V}_2$ that is out of the scope of our calculations. Taking the 'bar' part of the latter equation yields

$$\overline{\boldsymbol{V}}_{1t} + (\overline{\boldsymbol{V}}_1\nabla)\overline{\boldsymbol{V}}_1 + \langle(\widetilde{\boldsymbol{V}}_1\nabla)\widetilde{\boldsymbol{V}}_1\rangle = -\nabla\overline{P}_3 + \kappa\Delta\overline{\boldsymbol{V}}_1$$

that with the use of (3.37),(3.40),(3.41),(3.50) can be transformed into the exact Navier-Stokes system in the slow time variable t:

$$\begin{aligned} \overline{\boldsymbol{V}}_{1t} + (\overline{\boldsymbol{V}}_1\nabla)\overline{\boldsymbol{V}}_1 &= -\nabla\overline{\Pi}_1 + \kappa\Delta\overline{\boldsymbol{V}}_1, \quad \operatorname{div}\overline{\boldsymbol{V}}_1 = 0 && \text{for} \quad z > 0 \\ \overline{\boldsymbol{U}}_1 &= -\overline{\boldsymbol{u}}_1, \quad \overline{W}_1 = 0 && \text{at} \quad z = 0 \\ \overline{\boldsymbol{V}}_1 &\to 0, && \text{as } z \to \infty \end{aligned} \tag{3.53}$$

with $\overline{\Pi}_1 \equiv \overline{P}_3 + \langle(\nabla\widetilde{\Phi}_1)^2/2\rangle$. We denote its solution as the action of the Navier-Stokes operator $\mathfrak{N}$ on the tangential velocity $\overline{\boldsymbol{U}}_1 = -\overline{\boldsymbol{u}}_1$ at $z = 0$

$$\overline{\boldsymbol{V}}_1 = \mathfrak{N}[-\overline{\boldsymbol{u}}_1] = \mathfrak{N}[-\frac{1}{\kappa}\mathfrak{A}^2\overline{\boldsymbol{Q}}] \quad \text{for} \quad z \geqslant 0 \tag{3.54}$$

where $\overline{\boldsymbol{u}}_1$ is taken from (3.43). For simplicity we choose that for an unsteady streaming $\overline{\boldsymbol{V}}_1 \equiv 0$ at $t = 0$; if it happens to be not true, then the initial data should be additionally incorporated into the definition of $\mathfrak{N}$. For the t-independent solutions (for the steady streaming) the initial data at $t = 0$ is irrelevant.

Step 7: The final form of solution. At this stage we have determined all unknown functions of the first two approximations. Now we gather all results of this section together and present the solution of (2.14)-(2.16) for $\alpha = 1$ as an approximation with an error $O(\varepsilon^2)$

$$\begin{aligned}
&\widehat{\boldsymbol{u}} = \widetilde{\boldsymbol{u}}_0 + \varepsilon(\overline{\boldsymbol{U}}_1 + \widetilde{\boldsymbol{U}}_1 + \overline{\boldsymbol{u}}_1 + \widetilde{\boldsymbol{u}}_1) + O(\varepsilon^2) \quad \text{for} \quad z \geqslant 0 \\
&\widehat{w} = \varepsilon(\widetilde{w}_0 + \overline{W}_1 + \widetilde{W}_1) + O(\varepsilon^2) \\
&\widehat{p} = \varepsilon \widetilde{P}_1 + O(\varepsilon^2)
\end{aligned} \tag{3.55}$$

where $\widetilde{\boldsymbol{u}}_0$, $\widetilde{w}_0$ are given by (3.19); $\overline{\boldsymbol{U}}_1$, $\overline{W}_1$ by (3.54); $\widetilde{\boldsymbol{U}}_1$, $\widetilde{W}_1$, and $\widetilde{P}_1$ by (3.50),(3.51); $\overline{\boldsymbol{u}}_1$ by (3.43),(3.36); and $\widetilde{\boldsymbol{u}}_1$ by (3.52),(3.36). One can see that non-oscillating (streaming) 'outer' fields $\overline{\boldsymbol{U}}_1, \overline{W}_1$ appear already in the first approximation, hence we stop our calculations here[20]. It must be emphasised that in (3.55) we replace $s \to z/\varepsilon, \tau \to \omega t$, so the whole expression appears as a function of x, y, z, t.

Step 8: The justification of the two-timing method. Now we can perform the following key operation. After replacements $s \to z/\varepsilon, \tau \to \omega t$ in (3.55) we substitute it into (2.1)-(2.3), (2.6). Then one can see that the use of the chain rule of the differentiation leads to the reconsidering of the whole above procedure. Hence the original governing equations are satisfied with the error (reminder) $O(\varepsilon^2)$[21]. It serves as *a posteriori* mathematical justification of the two-timing method. It is also worth to state that as soon as the analytical solution (3.55) has been explicitly written and its mathematical justification is performed, one can use it without any reference to the method of its deriving.

4 The Moderate $\alpha = 2$ and Weak $\alpha = 3$ Nonlinearities

Solving the cases $\alpha = 2$ and $\alpha = 3$ are somewhat similar to $\alpha = 1$. However in order to reach the first streaming terms we have to make extra steps in the successive approximations: one extra step for $\alpha = 2$ and two steps for $\alpha = 3$. Below we present these results briefly, avoiding all comments that have been already made. In this section we give the sequence of steps describing the solution for $\alpha = 2$ and by the end we present only one selected result for $\alpha = 3$.

[20] The entire next approximation (proportional to ε^2) has been also found by a similar procedure that is not presented here. Some parts of this higher approximation have already been calculated: for example $\widetilde{w}_1$ (3.43). Looking at the whole problem one can notice that the mathematical structure of the problems for the successive approximations indicates that any required (n-th) approximation can be found.

[21] The terms $\overline{\boldsymbol{U}}_1, \overline{W}_1$ from (3.55) are not fully required to satisfy the equations with this error (actually they fully appear only in the equations of order ε^3). Therefore two problems: (i) to calculate an approximate solution with a given order of the remainder $O(\varepsilon^n)$; and (ii) to satisfy the governing equations with the error of the same order $O(\varepsilon^n)$, are different. Clearly the first task requires more information than the second one.

4.1 The zero approximation for $\alpha = 2$

The terms of order $\varepsilon^0 = 1$ in (2.18)-(2.24) are:

$$\boldsymbol{V}_{0\tau} = -\nabla P_0, \quad \operatorname{div} \boldsymbol{V}_0 = 0 \qquad \text{for} \quad z > 0 \tag{4.1}$$

$$\boldsymbol{u}_{0\tau} - \kappa \boldsymbol{u}_{0ss} + \nabla_{\boldsymbol{y}} p_0 = 0, \quad p_{0s} = 0, \quad w_{0s} = -\nabla_{\boldsymbol{y}} \boldsymbol{u}_0 \qquad \text{for } s > 0 \tag{4.2}$$

$$\boldsymbol{U}_0 + \boldsymbol{u}_0 = \widetilde{\boldsymbol{a}}, \quad W_0 = 0, \qquad \text{at} \quad z = s = 0 \tag{4.3}$$

$$\boldsymbol{V}_0 \to 0, \ P_0 \to 0 \qquad \text{as } z \to \infty \tag{4.4}$$

$$\boldsymbol{v}_0 \to 0, \ p_0 \to 0 \qquad \text{as } s \to \infty \tag{4.5}$$

The only difference with (3.1)-(3.5) is: the term $-W_0 \boldsymbol{u}_{0s}$ that appears in the right hand side of the first equation (3.2) for $\alpha = 1$, in its $\alpha = 2$ counterpart (4.2) does not appear from the very beginning. However this term vanishes in (3.2) due to the boundary condition $W_0 = 0$ (3.3). Therefore the system of equations (3.1)-(3.5) and (4.1)-(4.5) are mathematically identical. Hence the results for (4.1)-(4.5) are the same as (3.18)-(3.20):

$$\overline{P}_0 \equiv 0, \ \widetilde{P}_0 \equiv 0, \ \widetilde{\boldsymbol{V}}_0 \equiv 0 \quad \text{for} \quad z \geqslant 0 \tag{4.6}$$

$$\overline{p}_0 \equiv 0, \ \widetilde{p}_0 \equiv 0, \ \overline{\boldsymbol{v}}_0 \equiv 0, \ \widetilde{\boldsymbol{u}}_0 = \mathfrak{D}[\widetilde{\boldsymbol{a}}], \ \widetilde{w}_0 = \mathfrak{A}\mathfrak{D}[\nabla_{\boldsymbol{y}} \widetilde{\boldsymbol{a}}] \quad \text{for} \quad s \geqslant 0 \tag{4.7}$$

$$\operatorname{div} \overline{\boldsymbol{V}}_0 = 0 \text{ for } z > 0, \quad \overline{\boldsymbol{V}}_0 = 0 \text{ at } z = 0, \ \overline{\boldsymbol{V}}_0 \to 0 \text{ as } z \to \infty \tag{4.8}$$

4.2 The first approximation for $\alpha = 2$

The terms of order $\varepsilon^1 = \varepsilon$ in (2.18)-(2.24) are:

$$\boldsymbol{V}_{1\tau} = -\nabla P_1, \quad \operatorname{div} \boldsymbol{V}_1 = 0 \qquad \text{for} \quad z \geqslant 0 \tag{4.9}$$

$$\boldsymbol{u}_{1\tau} - \kappa \boldsymbol{u}_{1ss} + \nabla_{\boldsymbol{y}} p_1 = -W_0 \boldsymbol{u}_{0s} \qquad \text{for} \quad s > 0 \tag{4.10}$$

$$p_{1s} = 0, \quad w_{1s} = -\nabla_{\boldsymbol{y}} \cdot \boldsymbol{u}_1 \qquad \text{for} \quad s \geqslant 0 \tag{4.11}$$

$$\boldsymbol{U}_1 + \boldsymbol{u}_1 = 0, \quad W_1 + w_0 = 0, \qquad \text{at} \quad z = s = 0 \tag{4.12}$$

$$\boldsymbol{V}_1 \to 0, \ P_1 \to 0 \qquad \text{as } z \to \infty \tag{4.13}$$

$$\boldsymbol{v}_1 \to 0, \ p_1 \to 0 \qquad \text{as } s \to \infty \tag{4.14}$$

Step 1: The integration of the equation $p_{1s} = 0$ (4.11) gives $p_1 = p_1(\boldsymbol{y}, t, \tau)$ which along with the condition for pressure in (4.14) yields $p_1 \equiv 0$. Also $W_0 = 0$ in (4.10) by virtue of the latter condition (4.3).

Step 2: The 'bar' part of the remaining unsolved part of the system (4.9)-(4.14) is:

$$0 = -\nabla \overline{P}_1, \quad \operatorname{div} \overline{\boldsymbol{V}}_1 = 0 \qquad \text{for} \quad z > 0$$

$$\overline{\boldsymbol{u}}_{1ss} = 0 \quad \overline{w}_{1s} = -\nabla_{\boldsymbol{y}} \cdot \overline{\boldsymbol{u}}_1 \qquad \text{for} \quad s > 0$$

$$\overline{\boldsymbol{U}}_1 + \overline{\boldsymbol{u}}_1 = 0, \quad \overline{W}_1 + \overline{w}_0 = 0, \qquad \text{at} \quad z = s = 0$$

$$\overline{\boldsymbol{V}}_1 \to 0, \ \overline{P}_1 \to 0 \qquad \text{as } z \to \infty$$

$$\overline{\boldsymbol{v}}_1 \to 0, \qquad \text{as } s \to \infty$$

The consideration similar to (3.6)-(3.10) (with the use of $\overline{\boldsymbol{v}}_0 \equiv 0$ (4.7)) yields:

$$\overline{P}_1 \equiv 0,\ \operatorname{div} \overline{\boldsymbol{V}}_1 = 0 \quad \text{for } z > 0 \tag{4.15}$$
$$\overline{\boldsymbol{V}}_1 = 0 \quad \text{at } z = 0 \tag{4.16}$$
$$\overline{\boldsymbol{v}}_1 \equiv 0 \quad \text{for } s \geqslant 0 \tag{4.17}$$

Step 3: The 'tilde' part of (4.9)-(4.14) is:

$$\widetilde{\boldsymbol{V}}_{1\tau} = -\nabla \widetilde{P}_1, \quad \operatorname{div} \widetilde{\boldsymbol{V}}_1 = 0 \quad \text{for} \quad z > 0 \tag{4.18}$$
$$\widetilde{\boldsymbol{u}}_{1\tau} - \kappa \widetilde{\boldsymbol{u}}_{1ss} = 0 \quad \text{for} \quad s > 0 \tag{4.19}$$
$$\widetilde{w}_{1s} = -\nabla_{\boldsymbol{y}} \widetilde{\boldsymbol{u}}_1 \quad \text{for} \quad s > 0 \tag{4.20}$$
$$\widetilde{\boldsymbol{U}}_1 + \widetilde{\boldsymbol{u}}_1 = 0, \quad \widetilde{W}_1 = -\widetilde{w}_0 \quad \text{at} \quad z = s = 0 \tag{4.21}$$
$$\widetilde{\boldsymbol{V}}_1 \to 0,\ \widetilde{P}_1 \to 0 \quad \text{as } z \to \infty \tag{4.22}$$
$$\widetilde{\boldsymbol{v}}_1 \to 0 \quad \text{as } s \to \infty \tag{4.23}$$

It coincides with (3.44)-(3.49) where we have to take $\widetilde{\boldsymbol{Q}} \equiv 0$. Therefore we can rewrite the results as

$$\widetilde{\Phi}_1(\boldsymbol{x}, t, \tau) = -\mathfrak{P}\mathfrak{A}\mathfrak{D}[\nabla_{\boldsymbol{y}} \widetilde{\boldsymbol{a}}], \quad \widetilde{\boldsymbol{V}}_1 = \nabla \widetilde{\Phi}_1, \quad \widetilde{\Phi}_{1\tau} = \widetilde{P}_1 \quad \text{for } z \geqslant 0 \tag{4.24}$$

that coincides with (3.50), (3.51) and

$$\widetilde{\boldsymbol{u}}_1 = \mathfrak{D}\mathfrak{P}\mathfrak{A}\mathfrak{D}[\nabla_{\boldsymbol{y}}(\nabla_{\boldsymbol{y}} \widetilde{\boldsymbol{a}})], \quad \widetilde{w}_1 = \mathfrak{A}\nabla_{\boldsymbol{y}} \widetilde{\boldsymbol{u}}_1 = \mathfrak{A}\mathfrak{D}\mathfrak{P}\mathfrak{A}\mathfrak{D}[\nabla^2_{\boldsymbol{y}}(\nabla_{\boldsymbol{y}} \widetilde{\boldsymbol{a}})] \quad \text{for } s \geqslant 0 \tag{4.25}$$

that coincides (3.52) with additional simplification $\widetilde{\boldsymbol{Q}} \equiv 0$. Here we have also used the fact that the operator $\nabla_{\boldsymbol{y}}$ commutes with any other operators involved into (4.25).

4.3 The second approximation for $\alpha = 2$

The terms of order ε^2 in (2.18)-(2.24) are:

$$\boldsymbol{V}_{2\tau} = -\nabla P_2 - \{\boldsymbol{V}_{0t} + (\boldsymbol{V}_0 \cdot \nabla)\boldsymbol{V}_0 - \kappa \Delta \boldsymbol{V}_0\}, \quad \operatorname{div} \boldsymbol{V}_2 = 0 \quad \text{for} \quad z > 0 \tag{4.26}$$
$$\begin{aligned} \boldsymbol{u}_{2\tau} - \kappa \boldsymbol{u}_{2ss} + \nabla_{\boldsymbol{y}} p_2 = {} & -W_0 \boldsymbol{u}_{1s} - (W_1 + sW_{0z})\boldsymbol{u}_{0s} - \\ & -(\boldsymbol{U}_0 \nabla_{\boldsymbol{y}})\boldsymbol{u}_0 - (\boldsymbol{u}_0 \nabla_{\boldsymbol{y}})\boldsymbol{U}_0 - (\boldsymbol{u}_0 \cdot \nabla_{\boldsymbol{y}} + w_0 \nabla_s)\boldsymbol{u}_0 - \\ & -(\boldsymbol{u}_{0t} - \kappa \Delta_{\boldsymbol{y}} \boldsymbol{u}_0) \end{aligned} \quad \text{for} \quad s > 0 \tag{4.27}$$
$$p_{2s} = -w_{0\tau} + \kappa w_{0ss}, \quad w_{2s} = -\nabla_{\boldsymbol{y}} \cdot \boldsymbol{u}_2 \quad \text{for} \quad s > 0 \tag{4.28}$$
$$\boldsymbol{U}_2 + \boldsymbol{u}_2 = 0, \quad W_2 + w_1 = 0, \quad \text{at} \quad z = s = 0 \tag{4.29}$$
$$\boldsymbol{V}_2 \to 0,\ P_2 \to 0 \quad \text{as } z \to \infty \tag{4.30}$$
$$\boldsymbol{v}_2 \to 0,\ p_2 \to 0 \quad \text{as } s \to \infty \tag{4.31}$$

Step 1: In the former equation (4.28) $p_{2s} = -w_{0\tau} + \kappa w_{0ss}$ the value $\overline{w}_0 \equiv 0$ by (4.7). Hence with the use of (4.31) we derive $\overline{p}_2 \equiv 0$. Next, the direct calculations based on the solution $\widetilde{w}_0 = \mathfrak{AD}[\nabla_{\boldsymbol{y}}\widetilde{\boldsymbol{a}}]$ (4.7) show that $\widetilde{w}_{0\tau} - \kappa\widetilde{w}_{0ss} \equiv 0$. Hence from the former equation (4.28) we also derive that $\widetilde{p}_2 \equiv 0$, so the full function $p_2 \equiv 0$ for $s \geqslant 0$.

Step 2: With the use of $\widetilde{\boldsymbol{V}}_0 \equiv 0$ (4.6) the bar part of the former equation (4.26) combined with (4.8) gives a system

$$\begin{aligned} &\overline{\boldsymbol{V}}_{0t} + (\overline{\boldsymbol{V}}_0 \cdot \nabla)\overline{\boldsymbol{V}}_0 - \kappa\Delta\overline{\boldsymbol{V}}_0 = -\nabla\overline{P}_2 \quad \operatorname{div}\overline{\boldsymbol{V}}_0 = 0 \ \text{ for } \quad z > 0 \\ &\overline{\boldsymbol{V}}_0 = 0 \ \text{ at } z = 0, \quad \overline{\boldsymbol{V}}_0 \to 0 \text{ as } z \to \infty \end{aligned} \tag{4.32}$$

that represents the exact Navier-Stokes equations with zero boundary conditions. The only available solution is $\overline{\boldsymbol{V}}_0 \equiv 0, \overline{P}_2 \equiv 0$. Combining is with $\widetilde{\boldsymbol{V}}_0 \equiv 0$ (4.6) we conclude that

$$\boldsymbol{V}_0 \equiv 0 \text{ for } z \geqslant 0 \tag{4.33}$$

Step 3: After taking into account $p_2 \equiv 0$, $\boldsymbol{V}_0 \equiv 0$ the system (4.26)-(4.31) simplifies to:

$$\boldsymbol{V}_{2\tau} = -\nabla P_2, \quad \operatorname{div}\boldsymbol{V}_2 = 0 \qquad \text{for} \quad z > 0 \tag{4.34}$$

$$\boldsymbol{u}_{2\tau} - \kappa\boldsymbol{u}_{2ss} = -\boldsymbol{q}_2, \quad w_{2s} = -\nabla_{\boldsymbol{y}} \cdot \boldsymbol{u}_2 \ \text{ for } \quad s > 0 \tag{4.35}$$

$$\boldsymbol{q}_2 = \boldsymbol{Q} + \boldsymbol{u}_{0t} - \kappa\Delta_{\boldsymbol{y}}\boldsymbol{u}_0 \tag{4.36}$$

$$\boldsymbol{U}_2 + \boldsymbol{u}_2 = 0, \quad W_2 + w_1 = 0, \qquad \text{at} \quad z = s = 0 \tag{4.37}$$

$$\boldsymbol{V}_2 \to 0, \ P_2 \to 0 \qquad \text{as } z \to \infty \tag{4.38}$$

$$\boldsymbol{v}_2 \to 0 \qquad \text{as } s \to \infty \tag{4.39}$$

where $\boldsymbol{Q}$ is given in (3.35),(3.36); in the transformations of $\boldsymbol{q}_2$ into the form (4.36) we used $\overline{\boldsymbol{u}}_0 \equiv 0$ (4.7) as well as $\overline{W}_1 = 0$ and $\widetilde{W}_1 = -\widetilde{w}_0$ at $z = 0$ (4.16), (4.21).

Step 4: The 'bar' part of (4.34)-(4.39) is:

$$\overline{P}_2 \equiv 0, \quad \operatorname{div}\overline{\boldsymbol{V}}_2 = 0 \qquad \text{for} \quad z > 0 \tag{4.40}$$

$$\kappa\overline{\boldsymbol{u}}_{2ss} = \overline{\boldsymbol{q}}_2 \qquad \text{for} \quad s > 0 \tag{4.41}$$

$$\overline{w}_{2s} = -\nabla_{\boldsymbol{y}}\overline{\boldsymbol{u}}_2 \qquad \text{for} \quad s > 0 \tag{4.42}$$

$$\overline{\boldsymbol{U}}_2 + \overline{\boldsymbol{u}}_2 = 0, \quad \overline{W}_2 + \overline{w}_1 = 0 \ \text{ at } \quad z = s = 0 \tag{4.43}$$

$$\overline{\boldsymbol{V}}_2 \to 0, \ \overline{P}_2 \to 0 \qquad \text{as } z \to \infty \tag{4.44}$$

$$\overline{\boldsymbol{v}}_2 \to 0 \qquad \text{as } s \to \infty \tag{4.45}$$

It follows from (4.36) and $\overline{\boldsymbol{u}}_0 \equiv 0$(4.7) that $\overline{\boldsymbol{q}}_2 = \overline{\boldsymbol{Q}}$. Then the integration of (4.41),(4.42),(4.45) gives $\overline{\boldsymbol{v}}_2$ as

$$\overline{\boldsymbol{u}}_2 = \frac{1}{\kappa}\mathfrak{A}^2\overline{\boldsymbol{Q}}, \quad \overline{w}_2 = \frac{1}{\kappa}\nabla_{\boldsymbol{y}}\mathfrak{A}^3\overline{\boldsymbol{Q}} \quad \text{for} \quad s \geqslant 0 \tag{4.46}$$

that coincides with (3.43), except it is valid in the second approximation.

Step 5: The 'tilde' part of a subsystem from (4.34)-(4.39) is:

$$\begin{aligned} &\widetilde{\boldsymbol{V}}_{2\tau} = -\nabla\widetilde{P}_2, \quad \operatorname{div}\widetilde{\boldsymbol{V}}_2 = 0 && \text{for} \quad z > 0 \\ &\widetilde{W}_2 = -\widetilde{w}_1 = -\mathfrak{ADPAD}[\nabla^2_{\boldsymbol{y}}(\nabla_{\boldsymbol{y}}\widetilde{\boldsymbol{a}})] && \text{at} \quad z = s = 0 \\ &\widetilde{\boldsymbol{V}}_2 \to 0 && \text{as } z \to \infty \end{aligned}$$

where $\widetilde{w}_1$ is taken from (4.25). This system (in agreement with (2.29))immediately produces the potential solution:

$$\widetilde{\Phi}_2 = -(\mathfrak{P}\mathfrak{A}\mathfrak{D})^2[\nabla^2_{\boldsymbol{y}}(\nabla_{\boldsymbol{y}}\widetilde{\boldsymbol{a}})], \quad \widetilde{\boldsymbol{V}}_2 = -\nabla\widetilde{\Phi}_2, \quad \widetilde{P}_2 = -\widetilde{\Phi}_{2\tau} \quad \text{for} \quad z \geqslant 0 \tag{4.47}$$

The 'tilde' part of another subsystem from (4.34)-(4.39) is:

$$\begin{aligned} &\widetilde{\boldsymbol{u}}_{2\tau} - \kappa\widetilde{\boldsymbol{u}}_{2ss} = -\widetilde{\boldsymbol{q}}_2, \quad \widetilde{\boldsymbol{q}}_2 = \widetilde{\boldsymbol{Q}} + \widetilde{\boldsymbol{u}}_{0t} - \kappa\Delta_{\boldsymbol{y}}\widetilde{\boldsymbol{u}}_0 \ \text{ for } \quad s > 0 \\ &\widetilde{\boldsymbol{u}}_2 = -\widetilde{\boldsymbol{U}}_2 \text{ at } z = s = 0, \quad \widetilde{\boldsymbol{u}}_2 \to 0 \text{ as } s \to \infty \end{aligned} \tag{4.48}$$

With the use of (2.25) its solution is presented as

$$\widetilde{\boldsymbol{u}}_2 = -\mathfrak{D}[\widetilde{\boldsymbol{U}}_2, \widetilde{\boldsymbol{q}}_2] \quad \text{for} \quad s \geqslant 0 \tag{4.49}$$

with $\widetilde{\boldsymbol{U}}_2$, $\widetilde{\boldsymbol{q}}_2$ taken from (4.47),(4.48),(3.36),(3.16).

Step 6: At this stage we have found all unknown functions of the first and second approximations except $\overline{\boldsymbol{V}}_1$ and $\overline{\boldsymbol{V}}_2$. In order to find them we consider the $\boldsymbol{V}$-equation (2.18) for the next two approximations:

$$\boldsymbol{V}_{3\tau} = -\nabla P_3 - \{\boldsymbol{V}_{1t} - \kappa\Delta\boldsymbol{V}_1 + (\boldsymbol{V}_0\nabla)\boldsymbol{V}_1 + (\boldsymbol{V}_1\nabla)\boldsymbol{V}_0\} \tag{4.50}$$

$$\boldsymbol{V}_{4\tau} = -\nabla P_4 - \{\boldsymbol{V}_{2t} - \kappa\Delta\boldsymbol{V}_2 + (\boldsymbol{V}_0\nabla)\boldsymbol{V}_2 + (\boldsymbol{V}_2\nabla)\boldsymbol{V}_0 + (\boldsymbol{V}_1\nabla)\boldsymbol{V}_1\} \tag{4.51}$$

With $\boldsymbol{V}_0 \equiv 0$ (4.33) these equations are reduced to:

$$\boldsymbol{V}_{3\tau} = -\nabla P_3 - \boldsymbol{V}_{1t} + \kappa\Delta\boldsymbol{V}_1, \quad \boldsymbol{V}_{4\tau} = -\nabla P_4 - \boldsymbol{V}_{2t} + \kappa\Delta\boldsymbol{V}_2 - (\boldsymbol{V}_1\nabla)\boldsymbol{V}_1 \tag{4.52}$$

The 'bar' part of the former equation together with (4.15),(4.16),(4.22) gives us the following system of equation for $\overline{\boldsymbol{V}}_1$

$$\begin{aligned} &\overline{\boldsymbol{V}}_{1t} - \kappa\Delta\overline{\boldsymbol{V}}_1 = -\nabla\overline{P}_3, \quad \operatorname{div}\overline{\boldsymbol{V}}_1 = 0 \ \text{ for } \quad z > 0 \\ &\overline{\boldsymbol{V}}_1 = 0 \text{ at } z = 0, \ \overline{\boldsymbol{V}}_1 \to 0 \text{ as } z \to \infty \end{aligned} \tag{4.53}$$

that represents the unsteady Stokes equations with zero boundary conditions. Due to the uniqueness its solution is:

$$\overline{\boldsymbol{V}}_1 \equiv 0 \quad \overline{P}_3 \equiv 0 \quad \text{for} \quad z \geqslant 0 \tag{4.54}$$

The 'bar' part of the latter equation (4.52) together with (4.54),(4.40),(4.43),(4.44),(4.17) gives us the following system of equation for $\overline{\boldsymbol{V}}_2$

$$\begin{aligned} \overline{\boldsymbol{V}}_{2t} - \kappa\Delta\overline{\boldsymbol{V}}_2 = -\nabla\overline{\Pi}_2, \quad &\operatorname{div}\overline{\boldsymbol{V}}_2 = 0 \ \text{ for } \quad z > 0 \\ \overline{\boldsymbol{U}}_2 = -\overline{\boldsymbol{u}}_2, \quad &\overline{W}_2 = 0 \qquad\quad \text{at} \quad z = 0 \\ \overline{\boldsymbol{V}}_2 \to 0, \quad &\qquad\qquad\qquad \text{as } z \to \infty \end{aligned} \tag{4.55}$$

where $\overline{\Pi}_2 \equiv \overline{P}_4 + \langle(\nabla\Phi_1)^2\rangle/2$. One can see that it represents the unsteady Stokes equations with the well defined by (4.46) boundary conditions. We denote its solution as the action of the Stokes operator $\mathfrak{S}$ on the tangential velocity $\overline{\boldsymbol{U}}_2 = -\overline{\boldsymbol{u}}_2$ (4.46) at the plane $z = 0$

$$\overline{\boldsymbol{V}}_2 = -\mathfrak{S}[\overline{\boldsymbol{u}}_2] = -\frac{1}{\kappa}\mathfrak{S}[\mathfrak{A}^2\overline{\boldsymbol{Q}}] \quad \text{for} \quad z \geqslant 0 \tag{4.56}$$

Similar to (3.54) for simplicity we choose that for an unsteady streaming $\overline{\boldsymbol{V}}_2 \equiv 0$ at $t = 0$; if it happens to be not true, then the initial data should be additionally incorporated into the definition of $\mathfrak{S}$. For the t-independent solutions (for the steady streaming) the initial data at $t = 0$ is irrelevant.

Step 7: The final form of solution. At this stage we have determined all unknown functions of the first three approximations. Now we gather all results of this section together and present the solution of (2.14)-(2.16) for $\alpha = 2$ as an approximation with an error of order $O(\varepsilon^3)$:

$$\begin{aligned}
\widehat{\boldsymbol{u}} &= \widetilde{\boldsymbol{u}}_0 + \varepsilon(\widetilde{\boldsymbol{U}}_1 + \widetilde{\boldsymbol{u}}_1) + \varepsilon^2(\overline{\boldsymbol{U}}_2 + \widetilde{\boldsymbol{U}}_2 + \overline{\boldsymbol{u}}_2 + \widetilde{\boldsymbol{u}}_2) + O(\varepsilon^3) \\
\widehat{w} &= \varepsilon(\widetilde{w}_0 + \widetilde{W}_1) + \varepsilon^2(\widetilde{w}_1 + \overline{W}_2 + \widetilde{W}_2) + O(\varepsilon^3) \\
\widehat{p} &= \varepsilon\widetilde{P}_1 + \varepsilon^2\widetilde{P}_2 + O(\varepsilon^3)
\end{aligned} \tag{4.57}$$

where $\widetilde{\boldsymbol{u}}_0$, $\widetilde{w}_0$ are given by (4.7); $\widetilde{\boldsymbol{U}}_1$, $\widetilde{W}_1$, and $\widetilde{P}_1$ by (4.24); $\widetilde{\boldsymbol{u}}_1$, $\widetilde{w}_1$ by (4.25), $\overline{\boldsymbol{U}}_2$, $\overline{W}_2$ by (4.56); $\widetilde{P}_2$, $\widetilde{\boldsymbol{U}}_2$, $\widetilde{W}_2$ by (4.47); $\overline{\boldsymbol{u}}_2$ by (4.46),(3.36); and $\widetilde{\boldsymbol{u}}_2$ by (4.49),(3.36). One can see that non-oscillating (streaming) 'outer' fields $\overline{\boldsymbol{U}}_2, \overline{W}_2$ appear in the second approximation, hence we stop our calculations here.

Step 8: The justification of the two-timing method. In the right hand sides of (4.57) one should take $\tau = \omega t, s = z/\varepsilon$. Then similar to the case $\alpha = 1$ it can be shown that (4.57) satisfies the original equations (2.1)-(2.3), (2.6) with the error (reminder) $O(\varepsilon^3)$.

4.4 The Weak Nonlinearity: $\alpha = 3$

The case of $\alpha = 3$ requires more cumbersome calculations that should be carried forward considering the terms up to ε^3, so that the remainders in the final expressions similar to (3.55), (4.57) are of order $O(\varepsilon^4)$. For brevity we do not present these calculations here, since further we need only the equations for $\overline{\boldsymbol{V}}_3$, that appears identical to (4.55),(4.56), except that all subscripts '2' must be changed to '3'. Correspondingly, the final answer for $\overline{\boldsymbol{V}}_3$ is similar to (4.56):

$$\overline{\boldsymbol{V}}_3 = -\mathfrak{S}[\overline{\boldsymbol{u}}_3] = -\frac{1}{\kappa}\mathfrak{S}[\mathfrak{A}^2\overline{\boldsymbol{Q}}] \quad \text{for} \quad z \geqslant 0 \tag{4.58}$$

5 Brief Examples

The most striking result of the above considerations is the deriving of the exact Navier-Stokes equations (3.53) or unsteady Stokes equations (4.55) (both with the unit viscosity coefficients

$\kappa = 1$) along with precisely defined boundary conditions for the streaming velocity fields $\overline{\boldsymbol{V}}_1$ (3.54), $\overline{\boldsymbol{V}}_2$ (4.56), and $\overline{\boldsymbol{V}}_3$ (4.58). For the universality in notations we denote these streaming fields as $\overline{\boldsymbol{V}}_{(\alpha)}$ with $\alpha = 1, 2, 3$, where the subscript is taken into the parentheses in order to emphasise that it simultaneously represents the value of α identifying one of tree considered problems and (according to our solutions) the order of the first non-oscillating (streaming) terms. It is remarkable and surprising that the the boundary conditions at $z = 0$ are the same for all three fields $\overline{\boldsymbol{V}}_{(\alpha)}$; they are given by the expression:

$$\overline{\boldsymbol{V}}_{(\alpha)} = (\overline{\boldsymbol{U}}_{(\alpha)}, \overline{W}_{(\alpha)}) \quad \text{where} \quad \overline{\boldsymbol{U}}_{(\alpha)} = -\mathfrak{A}^2\overline{\boldsymbol{Q}}/\kappa, \; \overline{W}_{(\alpha)} = 0 \tag{5.1}$$

Such a coincidence between the effective boundary conditions demonstrates a strong tendency of the streaming velocity field to the universality[22]. In the special case of the t-independent boundary conditions ($\widetilde{\boldsymbol{a}}_t = 0$ in (2.2),(2.4)) functions $\overline{\boldsymbol{V}}_{(\alpha)}(\boldsymbol{x}, t)$ describe a *steady streaming*, otherwise they will give us an *unsteady streaming*. In particular, one can consider the time-dependent establishing of a steady streaming: a 'start-up' problem with initially zero velocity of the fluid and the boundary with a slow increasing of the boundary velocities up to a purely oscillating field.

Using the integration by parts one can represent $\overline{\boldsymbol{u}}_{(\alpha)}$ (3.43),(4.46),(3.36) as

$$\begin{aligned}
&\overline{\boldsymbol{u}}_{(\alpha)} = \frac{1}{\kappa}\mathfrak{A}^2\overline{\boldsymbol{Q}} = \frac{1}{\kappa}(\mathfrak{A}^2\overline{\boldsymbol{R}} + \mathfrak{A}\overline{\boldsymbol{S}}) \quad \text{for} \quad s \geqslant 0 \\
&\boldsymbol{R} \equiv (\widetilde{\boldsymbol{u}}_0\nabla_{\boldsymbol{y}})\widetilde{\boldsymbol{u}}_0 + \widetilde{\boldsymbol{u}}_0(\nabla_{\boldsymbol{y}}\widetilde{\boldsymbol{u}}_0), \quad \boldsymbol{S} \equiv \widetilde{\boldsymbol{u}}_0 \int_0^s (\nabla_{\boldsymbol{y}}\widetilde{\boldsymbol{u}}_0)ds' = \widetilde{\boldsymbol{u}}_0\mathfrak{A}^*[(\nabla_{\boldsymbol{y}}\widetilde{\boldsymbol{u}}_0)]
\end{aligned} \tag{5.2}$$

where $\widetilde{\boldsymbol{u}}_0 = \mathfrak{D}[\widetilde{\boldsymbol{a}}]$ (4.7),(3.16),(2.26). In order to study an example of the steady or unsteady streaming one should choose a particular boundary velocity $\widetilde{\boldsymbol{a}}$ (2.2),(2.4)), calculate $\widetilde{\boldsymbol{u}}_0 = \mathfrak{D}[\widetilde{\boldsymbol{a}}]$, then calculate $\overline{\boldsymbol{u}}_{(\alpha)}$ in (5.1) (5.2), and finally solve the Navier-Stokes or Stokes equations (3.54),(4.56),(4.58) for $z \geqslant 0$:

$$\overline{\boldsymbol{V}}_1 = \mathfrak{N}[-\frac{1}{\kappa}\mathfrak{A}^2\overline{\boldsymbol{Q}}] \text{ for } \alpha = 1, \quad \overline{\boldsymbol{V}}_2, \overline{\boldsymbol{V}}_3 = -\frac{1}{\kappa}\mathfrak{S}[\mathfrak{A}^2\overline{\boldsymbol{Q}}] \text{ for } \alpha = 2, 3 \tag{5.3}$$

There are some examples below.

5.1 The harmonic oscillations of the boundary

A General Wave: Let us take

$$\widetilde{\boldsymbol{a}} = \widetilde{\boldsymbol{a}}(\boldsymbol{y}, t, \tau) = \boldsymbol{A}(\boldsymbol{y}, t)\sin\tau + \boldsymbol{B}(\boldsymbol{y}, t)\cos\tau \tag{5.4}$$

that represents a sum of two standing (in the 'fast time' τ) harmonic waves with the given amplitude functions $\boldsymbol{A}$ and $\boldsymbol{B}$. The correspondent expression for $\widetilde{\boldsymbol{u}}_0 = \mathfrak{D}[\widetilde{\boldsymbol{a}}]$ is well known as the solution of the Stokes' second problem [30, 2, 1]:

$$\widetilde{\boldsymbol{u}}_0 = \boldsymbol{A}e^{-\gamma s}\sin(\tau - \gamma s) + \boldsymbol{B}e^{-\gamma s}\cos(\tau - \gamma s), \qquad \gamma \equiv 1/\sqrt{2\kappa}$$

[22]In contrast, one can see that the oscillating parts of velocity fields for different α are different

Calculations of (5.2) at $s = 0$ yield

$$\overline{\boldsymbol{u}}_{(\alpha)} = \frac{1}{4}\{(\boldsymbol{A}\nabla_{\boldsymbol{y}})\boldsymbol{A} + (\boldsymbol{B}\nabla_{\boldsymbol{y}})\boldsymbol{B} + 2\boldsymbol{A}(\nabla_{\boldsymbol{y}}\boldsymbol{A}) + 2\boldsymbol{B}(\nabla_{\boldsymbol{y}}\boldsymbol{B}) + \\ +\boldsymbol{B}(\nabla_{\boldsymbol{y}}\boldsymbol{A}) - \boldsymbol{A}(\nabla_{\boldsymbol{y}}\boldsymbol{B})\} \tag{5.5}$$

where the parameter κ has been eliminated during the transformations. Expression (5.5) does not represent a divergent form in variables (x, y), therefore even the spatially periodic in these variables $\boldsymbol{A}$ and $\boldsymbol{B}$ produce the nonzero averaged (over (x, y)) boundary value for $\overline{\boldsymbol{U}}_{(\alpha)}$. For a single harmonic $\boldsymbol{B} = 0$ it simplifies to

$$\overline{\boldsymbol{u}}_{(\alpha)} = \frac{1}{4}\{(\boldsymbol{A}\nabla_{\boldsymbol{y}})\boldsymbol{A} + 2\boldsymbol{A}(\nabla_{\boldsymbol{y}}\boldsymbol{A})\} \tag{5.6}$$

A Standing Wave: The simplest harmonic standing wave boundary conditions $\boldsymbol{A} = \boldsymbol{i}A(t)\sin kx$ gives

$$\overline{\boldsymbol{U}}_{(\alpha)} = -\boldsymbol{i}\frac{3kA^2}{8}\sin 2kx,\ \overline{W}_{(\alpha)} = 0 \text{ at } z = 0 \tag{5.7}$$

where $A(t)$ is a scalar function and $\boldsymbol{i}$ is the unit vector in the x-direction. For the steady Stokes equation $A = \text{const}$ corresponding to $\alpha = 2, 3$ the steady streaming flows coincide with the creeping ones and satisfy to the bi-harmonic equation $\nabla^4\Psi_{(\alpha)} = 0$ for the streamfunction $\Psi_{(\alpha)}(x, z)$ [2, 38, 45]. The explicit and exact solution is:

$$\Psi_{(\alpha)} = -\frac{3kA^2}{8}ze^{-2kz}\sin 2kx, \quad \overline{U}_{(\alpha)} = \frac{\partial\Psi_{(\alpha)}}{\partial z},\ \overline{W}_{(\alpha)} = -\frac{\partial\Psi_{(\alpha)}}{\partial x} \tag{5.8}$$

For $\alpha = 1$ the correspondent steady exact solution of the Navier-Stokes equations (the former case in (5.3)) can be calculated as a series in the powers of small amplitude A, where the zero approximation is given by (5.8). We do not present this solution here.

A Travelling Wave: As the next example we consider a travelling wave

$$\widetilde{\boldsymbol{a}} = \boldsymbol{i}A(t)\cos(kx - \tau) \tag{5.9}$$

It corresponds to the field (5.4) with $\boldsymbol{A} = \boldsymbol{i}A\cos kx$, $\boldsymbol{B} = \boldsymbol{i}A\sin kx$. Their substitution into (5.5) gives the boundary conditions

$$\overline{\boldsymbol{U}}_{(\alpha)} = -\boldsymbol{i}\frac{kA^2(t)}{4},\ \overline{W}_{(\alpha)} = 0 \text{ at } z = 0 \tag{5.10}$$

that correspond to the constant effective motion of the boundary in the direction opposite to the propagation of the wave (5.9)[23]. For any function $A(t)$ these boundary conditions produce the same unsteady unidirectional exact solutions (satisfying the 1+1 dimensional diffusion equation) for all three cases $\alpha = 1, 2, 3$. These solutions are well known, therefore they are also not presented here[24]. At the same time all such steady solutions (for $A = \text{const}$) correspond to the motion of all media with the same velocity (5.10)[25].

---

[23]It is interesting to see that in a similar problem for the Stokes flows (modelling a self-swimming of a 'plane microorganism' [18]) the direction of the streaming is also opposite to the velocity of wave propagation.

[24]The exact analytical solution of this problem in the most general case can be found in [29].

[25]The required changing of the boundary conditions at infinity does not create any difficulties here; this question will be addressed later.

5.2 Divergence-free motions of the boundary. An oscillating disc.

All previous results can be greatly simplified when $\nabla_{\boldsymbol{y}}\widetilde{\boldsymbol{a}} = 0$. This restriction leads to the vanishing of all oscillating fields (except $\widetilde{\boldsymbol{u}}_0$) preceding to the first non-oscillating (streaming) terms as well as the 'outer' oscillating term $\widetilde{\boldsymbol{V}}_{(\alpha)}$ of the same order as the streaming one. For example, for $\alpha = 1$ it is easy to check that in (3.55) the oscillating terms $\widetilde{\boldsymbol{w}}_0 \equiv 0$, $\widetilde{\boldsymbol{U}}_1 \equiv 0$, $\widetilde{W}_1 \equiv 0$, and $\widetilde{P}_1 \equiv 0$, so the solution takes form

$$
\begin{aligned}
&\widehat{\boldsymbol{u}} = \widetilde{\boldsymbol{u}}_0 + \varepsilon(\overline{\boldsymbol{U}}_1 + \overline{\boldsymbol{u}}_1 + \widetilde{\boldsymbol{u}}_1) + O(\varepsilon^2)\\
&\widehat{w} = \varepsilon\overline{W}_1 + O(\varepsilon^2)\\
&\widehat{p} = O(\varepsilon^2)
\end{aligned}
\tag{5.11}
$$

where the expression for $\widetilde{\boldsymbol{u}}_0$ stays the same (3.16), (2.25), while $\overline{\boldsymbol{V}}_1$, $\overline{\boldsymbol{u}}_1$, and $\widetilde{\boldsymbol{u}}_1$ are

$$
\overline{\boldsymbol{V}}_1 = \mathfrak{N}[-\frac{1}{\kappa}\mathfrak{A}^2\overline{\boldsymbol{Q}}], \quad \widetilde{\boldsymbol{u}}_1 = -\mathfrak{D}[0, \widetilde{\boldsymbol{Q}}], \quad \overline{\boldsymbol{u}}_1 = \frac{1}{\kappa}\mathfrak{A}^2\overline{\boldsymbol{Q}},
\tag{5.12}
$$

where the expression for $\boldsymbol{Q}$ is also much simpler then (3.36):

$$
\boldsymbol{Q} \equiv (\widetilde{\boldsymbol{u}}_0\nabla_{\boldsymbol{y}})\widetilde{\boldsymbol{u}}_0 = (\mathfrak{D}[\widetilde{\boldsymbol{a}}]\nabla_{\boldsymbol{y}})\mathfrak{D}[\widetilde{\boldsymbol{a}}] \quad \text{for} \quad s \geqslant 0
\tag{5.13}
$$

The solutions for $\alpha = 2, 3$ take similar to (5.11) forms:

$$
\begin{aligned}
&\widehat{\boldsymbol{u}} = \widetilde{\boldsymbol{u}}_0 + \varepsilon^\alpha(\overline{\boldsymbol{U}}_{(\alpha)} + \overline{\boldsymbol{u}}_{(\alpha)} + \widetilde{\boldsymbol{u}}_{(\alpha)}) + O(\varepsilon^{\alpha+1})\\
&\widehat{w} = \varepsilon\overline{W}_{(\alpha)} + O(\varepsilon^{\alpha+1})\\
&\widehat{p} = O(\varepsilon^{\alpha+1})
\end{aligned}
\tag{5.14}
$$

where the expression for $\overline{\boldsymbol{u}}_{(\alpha)}$ is the same as for $\overline{\boldsymbol{u}}_1$ in (5.12), while $\widetilde{\boldsymbol{u}}_{(\alpha)} = \mathfrak{D}[0, -\widetilde{\boldsymbol{u}}_0 + \kappa\Delta_{\boldsymbol{y}}\widetilde{\boldsymbol{u}}_0 - \widetilde{\boldsymbol{Q}}]$, and in $\overline{\boldsymbol{V}}_{(\alpha)}$ the Navier-Stokes operator is replaced with the Stokes one (5.3).

For the general wave boundary condition (5.4) expression (5.5) simplifies to

$$
\overline{\boldsymbol{u}}_{(\alpha)} = \frac{1}{4}\{(\boldsymbol{A}\nabla_{\boldsymbol{y}})\boldsymbol{A} + (\boldsymbol{B}\nabla_{\boldsymbol{y}})\boldsymbol{B}\}
\tag{5.15}
$$

which shows that these two harmonics are almost independent (they still interact for $\alpha = 1$, since the Navier-Stokes equations are nonlinear).

Expressions (5.11)-(5.15) allow us to build many interesting solutions. We restrict ourselves only with the torsional oscillations of an infinite rigid disc with the boundary condition:

$$
\widetilde{\boldsymbol{a}} = \Omega(-y, x)\cos\tau = \Omega(-y, x)\cos\omega t
\tag{5.16}
$$

There is a possibility to consider an arbitrary function $\Omega = \Omega(t)$, but we take only $\Omega = \text{const}$. In this case we have a degeneration in the list of parameters (2.11)-(2.13) since the set of U^*, L^*, T^*, ω, and ν is replaced with Ω, ω, and ν. It means that the introduction of the dimensionless variables should be performed independently. In all cases $\alpha = 1, 2, 3$ one can take

$$
T^* = 1/\Omega, \quad L^* = \sqrt{\nu/\Omega}, \quad U^* = \sqrt{\nu\Omega\varepsilon^{\alpha-2}}
\tag{5.17}
$$

If one accepts (5.17) then it follows that $\Omega/\omega = \varepsilon^2$ and the dependencies of the scaling parameters $S^{(1)}, S^{(2)}$ and R on ε are the same as in (2.13). Therefore all three considered cases are applicable to the flows driven by an oscillating disc.

The substitution of (5.16) into (5.15) with $\boldsymbol{B} = 0$ produces the boundary conditions:

$$\overline{\boldsymbol{U}}_{(\alpha)} = -\frac{1}{4}(\boldsymbol{A}\nabla_{\boldsymbol{y}})\boldsymbol{A} = k\boldsymbol{y} = k(x, y); \quad k \equiv \frac{1}{4}\Omega^2 \tag{5.18}$$

that is easy to derive since $(\boldsymbol{A}\nabla_{\boldsymbol{y}})\boldsymbol{A}$ here represents the classical centrifugal acceleration for a 'velocity field' $\boldsymbol{A}(\boldsymbol{y}) = \Omega(-y, x)$ of the rigid-body rotation. One can see that (5.18) gives the linear stretching of the wall $z = 0$. The corresponding solutions are well known [5, 23, 24, 25]. They have the form

$$\overline{U}_{(\alpha)} = kxf_\eta(\eta, t),\ \overline{V}_{(\alpha)} = kyf_\eta(\eta, t),\ \overline{W}_{(\alpha)} = -2\sqrt{\kappa}kf(\eta, t);\ \eta \equiv \sqrt{k/\kappa}z \tag{5.19}$$

with an unknown function $f(\eta, t)$. The arising problem for the Navier-Stokes equations ($\alpha = 1$) is:

$$\frac{1}{\kappa}f_{t\eta} + f_\eta^2 - 2ff_{\eta\eta} - f_{\eta\eta\eta} = 0 \tag{5.20}$$

$$f(0, t) = 0, \quad f_\eta(0, t) = 1, \quad f_\eta(\eta, t) \to 0 \text{ as } \eta \to \infty \tag{5.21}$$

$$f(\eta, 0) = 0 \tag{5.22}$$

The last condition in (5.21) shows that we require only the vanishing of $\overline{U}_{(\alpha)}$ and $\overline{V}_{(\alpha)}$ at $z \to \infty$ but leave $\overline{W}_{(\alpha)}$ free to make the problem solvable. This change of the boundary condition is justified at the end of this subsection.

For the steady solution of (5.20)-(5.22) $f_t \equiv 0$ and the initial data (5.22) should be abolished; the exact solution of this problem in a closed analytical form is given in [24][26]. For $\alpha = 2, 3$ the similar problem for the Stokes equation is:

$$f_{t\eta} - \kappa f_{\eta\eta\eta} = 0 \tag{5.23}$$

$$f(0, t) = 0, \quad f_\eta(0, t) = 1, \quad f_\eta(\eta, t) \to 0 \text{ as } \eta \to \infty \tag{5.24}$$

$$f(\eta, 0) = 0 \tag{5.25}$$

This system can be solved by the integration of (5.23) once over η

$$f_t - \kappa f_{\eta\eta} = F_t(t) \tag{5.26}$$

where $F_t(t)$ is an arbitrary function of integration. Then without the introducing of any additional restrictions we accept that

$$f(\eta, t) \to F(t) \text{ as } \eta \to \infty, \quad F(0) = 0 \tag{5.27}$$

[26]This solution is too cumbersome to quote here.

where the former condition is the same as $f_{\eta\eta} \to 0$ as $\eta \to \infty$. Then the introduction of a new function $g \equiv f - F$ transforms (5.23)-(5.27) into the system

$$g_t - \kappa g_{\eta\eta} = 0 \tag{5.28}$$

$$g_\eta(0,t) = 1; \quad g(\eta,t) \to 0, \text{ as } \eta \to \infty \tag{5.29}$$

$$g(\eta,0) = 0 \tag{5.30}$$

with the additional conditions

$$g(0,t) = -F(t), \quad g_\eta(\eta,t) \to 0, \text{ as } \eta \to \infty \tag{5.31}$$

The solution of (5.28)-(5.30) coincides with one for the unidirectional flow driven by the constant surface stress applied at $z = 0$ [31]:

$$g(\eta,t) = (4\kappa t)^{1/2}[\xi \mathrm{erfc}\{\xi\} - \pi^{-1/2}\exp(-\xi^2)], \quad \xi \equiv \eta/\sqrt{4\kappa t} \tag{5.32}$$

where $\mathrm{erfc}\{\xi\}$ is the complimentary error function. This solution automatically satisfies the latter condition in (5.31). At the same time the former condition in (5.31) gives $F(t) = -g(0,t) = (4\kappa t/\pi)^{1/2}$ that immediately produces the required function $f(\eta,t) = g(\eta,t)+F(t)$ for (5.19). In particular, the flow at $z \to \infty$ is purely vertical and uniform with the velocity

$$\overline{W}_{(\alpha)} = -4\kappa k\sqrt{t/\pi} \tag{5.33}$$

It shows that the vertical velocity is growing in its amplitude $\overline{W}_{(\alpha)} \to -\infty$ as $t \to \infty$. This result can be anticipated, since the governing equation (5.23) for a steady flow degenerates to $f_{\eta\eta\eta} = 0$ that has a quadratic polynomial in η as its general solution. Taking into account only the boundary condition $f_\eta(0,t) = 1$ (5.24) leaves out of this polynomial only $f = \eta$, that also shows that $\overline{W}_{(\alpha)} \to -\infty$ as $z \to \infty$ for the steady problem. Such kind of a secular behaviour at infinity is typical and well known for the steady Stokes equations [2, 38, 45].

Justification of the Changed Boundary Condition at $z \to \infty$: The accepted boundary condition at $z \to \infty$ in (5.21),(5.24) differs from the considered in the bulk of the paper by the allowing a nonzero limit for $\overline{W}_{(\alpha)}$. Here we state that all considered in this paper problems allow the introduction of that kind of change. It is easy to show by inspection of our asymptotic solutions that: (i) one can keep the requirement that the oscillating parts of both velocity and pressure are vanishing at infinity, while (ii) the non-oscillating parts $\overline{\boldsymbol{V}}_{(\alpha)}$ at infinity can be prescribed arbitrary. Physically, it corresponds to the placing at infinity some external devices that maintain a non-oscillating streaming flow at infinity. The considered case, when the vertical non-oscillating velocity at infinity appears as an eigenvalue is a special case of this statement. The related simplest approach is to keep (in the procedure of the successive approximations) all boundary conditions at infinity vanishing until it becomes necessary to abolish a part of these conditions in order to keep the problem solvable.

6 Discussion

The aim of this paper is to develop a general theory of vibrational boundary layers by the V-L method. For this reason we have chosen the simplest geometry but studied the most general oscillating tangential velocity field. Our choice of purely tangential vibrations is not enforced by any mathematical difficulties or restrictions. If one considers the nonzero normal vibrations of a plane (deformations of a plane) then the using of the V-L method remains straightforward and the obtaining of solutions is simpler than the described in this paper. The simplification reveals itself as the appearance of the streaming flows one step earlier in the successive approximations. However the systematic comparison of such new results (obtained by the V-L method, but not presented here) and the old ones (obtained from the classical boundary layer theory in [32, 33, 34, 18, 16, 5, 35]) requires significant efforts that are beyond the scope of this paper.

There are at least two papers [19, 20] devoted to a torsionally oscillating infinite rigid disc: [19] uses the method of matching asymptotic expansions, while [20] gives the global solution. It is likely that the author of [20] independently discovered a special form of the V-L method. It is also likely that he considered our case $\alpha = 1$ and presented similar results in a different form. At the same time there are some other interesting flows caused by purely tangential vibrations; these flows can have either zero or nonzero tangential velocity divergence at the boundary. For example, all papers on the torsional oscillations of axisymmetric solids *e.g.* [36] belong to the first class (of zero tangential divergence). As an interesting flow of the second class (of non-zero tangential divergence) we can mention Batchelor's flow caused by a rotating cylinder with a fixed sleeve [37, 38, 39]. There are also potential interesting problems in biology and medicine where the purely tangential oscillations of boundaries may have applications [40, 41].

Going back to the introducing of small parameters (2.13) one can check that the accepted relation $1/\omega T_* = \lambda\varepsilon^2$ can be relaxed by allowing $1/\omega T_* \sim \varepsilon^\beta$ with $\beta \geqslant 2$ where $\beta = 2$. The straightforward inspection of our solutions shows that the taking of $\beta > 2$ causes the vanishing of the derivatives $\partial/\partial t$ in the leading order equations for a streaming. In this case, instead of the full Navier-Stokes (3.53) or the unsteady Stokes (4.55) equations, the streaming will be governed by their 'steady' versions, while the time t will be allowed to play a part of a parameter.

One can also consider $\alpha > 3$; the apparent result will be the 'moving' of nonlinear terms to the higher approximations, in such a way that the number of successive approximations required to obtain the leading term for the streaming will be equal to α.

In connection to the multiplicity of the asymptotic models (with different α's and β's) one can try to identify the 'most valuable practically' or the 'most important theoretically' scaling. This interesting question remains open, however the choice of $\alpha = 1$ and $\beta = 2$ likely gives the 'strongest nonlinearity' and the 'strongest unsteadiness' of the considered problem (2.8)-(2.13). Our attempts to lower either or both of these two numbers have resulted in the failing to derive self-consistent equations for successive approximations[27]. However the case $\alpha = 0$ (or $\alpha = 1$ for vibrations with normal component) may require further studies aimed to discover the phenomena of the 'high streaming Reynolds number' and the 'double boundary layers' [3, 5] within the V-L method.

The choosing of the general periodical boundary conditions for velocity (2.2) has one hidden danger: they can give the Stokes drift [42, 2] of the material points of the boundary plane itself. We do not discuss this interesting question here; among our examples only the travelling wave (5.9),(5.10) causes such a drift that can be calculated by the method [28]. However, as it is well known the Stokes drift of the boundary particles is a phenomenon different from the steady streaming [5]. At the same time it is interesting to see that for the travelling wave the directions of the the wave propagation (5.9) and the Stokes drift [42, 2] are always the same, while the calculated streaming (5.10) has the opposite direction.

We deliberately avoided almost all considerations of the stresses and forces exerted on the boundary. Their studies are certainly important and have numerous applications *e.g.* to the streaming flows generated by surface waves [43, 44]. The used technique is certainly applicable to this case.

The surprising fact that the streaming (the non-oscillating part of a flow) can be governed by the steady or unsteady versions of Stokes equations with the unit viscosity shows a new area of using the numerous known results for the creeping flows [38, 45]. The further exploiting of the Navier-Stokes equations is also likely, since the classes of boundary conditions that have earlier been considered as being 'too artificial' and 'unrealistic', now may attract interest. One actually can consider an 'inverse problem' of starting from an arbitrary chosen boundary data (for the Stokes or Navier-Stokes equations) and obtaining such vibrations of boundary that produce such boundary data for a streaming flow.

Finally, we emphasise the key items obtained in the paper: (i) all derived solutions are global (valid in the whole flow domain), uniform (satisfy the governing equations with a given error), and do not contain any secular terms; (ii) there are many solutions corresponding to the different ways of the introduction of small parameter; and (iii) the streaming flows (either steady or unsteady) are described by the Navier-Stokes or by Stokes equations with unit viscosity and with explicitly defined effective boundary conditions (data); strikingly these conditions are the same for solutions with different scalings.

References

[1] H. SCHLICHTING, *Boundary Layer Theory*, McGraw-Hill book company, 1960.

[2] G. K. BATCHELOR, *An Introduction to Fluid Dynamics*, CUP, Cambridge, 1967.

[27] If we allow the normal boundary vibrations then our calculations (that are not presented here) show that the 'strongest available' nonlinearity is $\alpha = 2$.

[3] J. T. STUART. Double boundary layers in oscillatory viscous flow, *Journal of Fluid Mechanics*, 24:673-687 (1966).

[4] J. T. STUART. Unsteady boundary layers, In: *Laminar Boundary Layers.* Oxford, Clarendon Press, 349-408 (1963).

[5] N. RILEY. Steady streaming, *Ann. Rev. Fluid Mech.* 33:43-65 (2001).

[6] T. von KARMAN. The engineer grapples with non-linear problems, *Bull. Amer. Math. Soc.* 6, 8:615-683 (1940).

[7] M. I. VISHIK and L.A. LYUSTERNIK. Regular degeneration and boundary layer for linear differential equations with small parameter (in Russian), *Uspehi Mat. Nauk*, 12:3-122 (1957). *Amer. Math. Soc. Transl.* ser.2, 20:239-364 (1962).

[8] V. A. TRENOGIN. Development and applications of Vishik-Lyusternik's asymptotic method (in Russian), *Uspekhi Math. Nauk,* 25:123-156 (1970).

[9] W. WASOW, *Asymptotic Expansions for Ordinary Differential Equations,* Interscience Publishers, 1965.

[10] A.H. NAYFEH, *Perturbation Methods,* John Wiley & Sons, 1973.

[11] D.R. SMITH, *Singular-perturbation Theory. An Introduction with Applications,* CUP, Cambridge, 1985.

[12] L.A. CHUDOV. On some shortfalls of the classical boundary layer theory (in Russian), *Computational Methods in Gas Dynamics,* 2:98-109, Moscow University Press (1963).

[13] I. Yu. BRAILOVSKAJA and L.A. CHUDOV. Two methods of construction of higher approximations with respect to a small viscosity parameter (in Russian), *Computing Methods and Programming,* 8:173-185, Moscow University Press (1967).

[14] V. P. STULOV. On the laminar boundary layer equations in a two-phase medium (in Russian), *Izv. Acad. Nauk SSSR Mech. Zhidkosti i Gasa,* 1:21-25 (1979).

[15] V. I. YUDOVICH. On the asymptotic integration of the Navier-Stokes equations for a fluid under impact forces (in Russian), *Izvestiya VUZov Severo-Kavkazskii region, Series of Natural Sciences,* Special issue:126-132 (2004).

[16] M. ARGENTINA, , J. SKOTHEIM, and L. MAHADEVAN. Settling and swimming of flexible fluid-lubricated foils, *Physical Review Letters,* 99, 224503 (2007).

[17] A. E. HOSOI and L. MAHADEVAN. Peeling, healing, and bursting in a lubricated elastic sheet. *Physical Review Letters,* 93, 137802 (2004).

[18] S. CHILDRESS, *Mechanics of swimming and flying,* CUP, Cambridge, 1981.

[19] S. ROSENBLAT. Torsional oscillations of a plane in a viscous fluid, *Journal of Fluid Mechanics,* 6:206-220 (1958).

[20] D. J. BENNEY. The flow induced by a disk oscillating in its own plane, *Journal of Fluid Mechanics,* 18:385-391 (1964).

[21] D. J. BENNEY. The flow induced by a disk oscillating about a state of steady rotation, *Journ. Mech. and Applied Math.* XVIII:333-345 (1965).

[22] P. G. DRAZIN and N. RILEY. *The Navier-Stokes equations: a classification of flows and exact solutions. London Mathematical Society Lecture Note Series,* 334, CUP, Cambridge, 2006.

[23] C. Y. WANG. Exact solutions of the steady-state Navier-Stokes equations, *Ann. Rev. Fluid Mech.* 33:43-65 (1991).

[24] C. Y. WANG. Effect of spreading of material on the surface of a fluid – an exact solution, *Int. J. Non-Linear Mechanics,* 6:255-262 (1971).

[25] C. Y. WANG. The three-dimensional flow due to a stretching flat surface, *Phys. Fluids,* 27:1915-1917 (1984).

[26] H. P. GREENSPAN, *The Theory of Rotating Fluids,* CUP, Cambridge, 1968.

[27] V. A. VLADIMIROV. On vibrodynamics of pendulum and submerged solid. *Journal of Mathematical Fluid Dynamics* 7:S226-259 (2005).

[28] V. I. YUDOVICH. Vibrodynamics and Vibrogeometry of Mechanical Systems with Constrains, Parts 1-3, (in Russian) *Uspehi Mekhaniki,* 4:3-158 (2006).

[29] A. N. TIKHONOV and A. A. SAMARSKY, *Equations of Mathematical Physics,* Interscience Publishers, 1965.

[30] G. G. STOKES. On the effect of the internal friction of fluids on the motion of pendulums, *Trans. Camb. Phil. Soc.,* 9:8-106 (1851).

[31] F. S. SHERMAN, *Viscous Flow,* McGraw-Hill book company, 1990.

[32] W. H. LYNE. Unsteady viscous flow over a wavy wall, *Journal of Fluid Mechanics,* 50:33-48 (1971).

[33] F. CARLSSON, M. SEN and L. LOFDAHL. Fluid mixing induced by vibrating walls, *European Journal of Mechanics B/Fluids,* 24:366-378 (2005).

[34] F. CARLSSON, M. SEN and L. LOFDAHL. Fluid mixing due to vibrating walls, *Physics of Fluids,* 16:1822-1825 (2004).

[35] C. ATKINSON and M. MANRIQUE de LARA. Fluid reaction on a vibrating disc in a viscous medium, *Int. J. Eng. Sci.* 44:973-995 (2006).

[36] A. GOPINATH. Steady streaming due to small-amplitude torsional oscillations of a sphere in a viscous fluid, *Q. J. Mech. appl. Math.* 46:504-520 (1993).

[37] G. K. BATCHELOR. On steady laminar flow with closed streamlines at large Reynolds number, *Journal of Fluid Mechanics,* 1:177-190 (1956).

[38] H. K. MOFFATT, *Microhydrodynamics,* Ecole Polytechnique, Palaiseau, 1996.

[39] V. A. VLADIMIROV and V. F. TARASOV. Structure of flow of a viscous liquid with closed streamlines. *Sov. Phys. Dokl,* 27, 1:17-21 (1982).

[40] T. J. PEDLEY, *The Fluid Mechanics of Large Blood Vessels.* CUP, Cambridge, 1980.

[41] P. W. CARPENTER and T. J. PEDLEY (Eds.), *Flows past Highly Compliant Boundaries and in Collapsible Tubes.* Kluwer Academic Publishers, 2001.

[42] G. G. STOKES. On the theory of oscillatory waves, *Trans. Camb. Phil. Soc.,* 8:314-326. Reprinted in Math. Phys. Pap. 1:314-26 (1847).

[43] M. S. LONGUET-HIGGINS. Mass transport in water waves, *Phil. Trans. A,* 245:535-581 (1953).

[44] M. S. LONGUET-HIGGINS. Mass transport in the boundary layer at a free oscillating surface, *Journal of Fluid Mechanics,* 8, 1:293-306 (1960).

[45] J. HAPPEL and H. BRENNER, *Low Reynolds Number Hydrodynamics,* Prentice-Hall Inc., 1965.