Ellipsoidal Techniques for Closed-Loop Control of Oscillating System that Approximates the Telegraph Equation[1]

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1. Abstract

This paper describes constructive ellipsoidal algorithms for problems of closed-loop control in oscillating systems which approximate the telegraph equation.

2. The Problem

Consider the controlled linear ODE system:

$$
\begin{gathered}
m_1\ddot{y}_1(t) = -(k_1+k_2)y_1(t) + k_2y_2(t) - \mu_1\dot{y}_1(t) - \nu_1y_1(t) \\
m_2\ddot{y}_2(t) = k_2y_1(t) - (k_2+k_3)y_2(t) + k_3y_3(t) - \mu_2\dot{y}_2(t) - \nu_2y_2(t) \\
\dots \\
m_2\ddot{y}_{N_1}(t) = k_{N-1}y_{N-2}(t) - (k_{N-1}+k_N)y_{N-1}(t) + k_Ny_N(t) - \mu_{N-1}\dot{y}_{N-1}(t) - \nu_{N-1}y_{N-1}(t) \\
m_N\ddot{y}_N(t) = -k_Ny_{N-1}(t) + k_Ny_N(t) - \mu_N\dot{y}_N(t) - \nu_Ny_N(t) + u(t)
\end{gathered}
\tag{1}
$$

where u(t), $y_i(t)$, k_i, m_i, μ_i, ν_i, i=1,2,...,N, are positive real values.

The value y_i represents displacement of the ith weight from the equilibrium.

This system describes small oscillations of a chain of springs and weights in a viscous medium and thus approximates the **telegraph equation**:

$$
u_{tt} = v_0^2u_{xx} - (\sigma_1+\sigma_2)\frac{\partial u}{\partial t} - \sigma_1\sigma_2u + f. \tag{2}
$$

We subject the control $u(t)$ to constraints:

$$
u(t) \in \mathcal{P} = [u_{min},\ u_{max}] \tag{3}
$$

It is possible to rewrite the system (1) in the standard matrix form:

$$
\dot{x}(t) = Ax(t) + bu(t), \tag{4}
$$

where
$x(t) \in R^n$, $n = 2N$, $(x_1(t),\dots,x_N(t)) = (y_1(t),\dots,y_N(t))$, $(x_{N+1},\dots,x_{2N}) = (\dot{y}_{N+1},\dots,\dot{y}_{2N})$.

Let us give the definition of closed-loop control:

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Definition. A closed-loop control is a multimapping $\mathcal{U}(t,x) : [t_0,t_1] \times R^{2N} \to \mathcal{P}$, that is upper semicontinuous with respect to variable x, is measurable with respect to t and is with nonempty convex compact values in $\mathcal{P}$ The set of positional (feedback) strategies is denoted as $\mathcal{U}_{CL}$

The above mentioned class of controls provides the existence and continuity of solutions to the differential inclusion:

$$\dot{x}(t) \in Ax(t) + b\mathcal{U}(t,x), \ t \leq t_0 \tag{5}$$

Problem 1. Find the *solvability set* $\mathcal{W}[t_0] \subseteq R^n$ and the feedback control $\mathcal{U}(t,x)$ which ensures that all the trajectories of the system (4) starting from initial set $\mathcal{W}[t_0]$ would attain the equilibrium at finite time t_1 (i.e. $x(t_1) = 0$).

Consider the value function $V(t,x)$

$$V(t,x) = \min_{u \in \mathcal{U}_{CL}} \max_{x(\cdot) \in X_{\mathcal{U}}(\cdot)} \{d^2(x(t_1),0) | x(t) = x\} \tag{6}$$

Here $X_{\mathcal{U}}(\cdot)$ is the trajectory tube of solutions to the differential inclusion (5) with fixed closed-loop control $\mathcal{U}(t,x)$ and initial condition $x(t) = x$.

This function is a solution of the Hamilton-Jacoby-Bellman equation.

$$\frac{\partial V(t,x)}{\partial t} + \min_{u \in [u_{min},\ u_{max}]} < \frac{\partial V(t,x)}{\partial x}, Ax + bu >= 0 \tag{7}$$

$$V(t_1,x) = d^2(x,0) \tag{8}$$

The optimal closed loop control can be found from the following relation:

$$\mathcal{U}(t,x) = \underset{u \in [u_{min},\ u_{max}]}{Argmin} < \frac{\partial V(t,x)}{\partial x}, bu > \tag{9}$$

We note that the value function can be expressed via the solvability set $\mathcal{W}[t]$:

$$V(t,x) = d^2(e^{(t_1-t)A}x, e^{(t_1-t)A}\mathcal{W}[t]). \tag{10}$$

The optimal feedback control has the form:

$$\mathcal{U}^*(t,x) = \begin{cases} u_{\min}, & \ell_n^0 > 0; \\ u_{\max}, & \ell_n^0 < 0; \\ [u_{\min}, u_{\max}], & \ell_n^0 = 0, \end{cases} \tag{11}$$

where ℓ^0 is the maximizer in the next expression

$$\begin{aligned} d^2(e^{(t_1-t)A}x, e^{(t_1-t)A}\mathcal{W}[t]) = \max_{\ell \in R^n} < \ell, x > -\rho(\ell|\mathcal{W}[t]) - \|e^{(t-t_1)A'}\ell^0\|^2 = \\ =< \ell^0, x > -\rho(\ell^0|\mathcal{W}[t]) - \|e^{(t-t_1)A'}\ell^0\|^2. \end{aligned} \tag{12}$$

Here $\rho(\ell|\mathcal{W}[t])$ is support function of the solvability set $\mathcal{W}[t]$.

The most difficult computational part of the above solution is the optimization problem in (12). We can simplify the calculations if the solvability set $\mathcal{W}[t]$ is replaced by an internal ellipsoidal approximation:

$$\mathcal{E}(x^*(t), X_-(t)) = \cup\{x \in R^n | < x - x^*(t), X_-(t)^{-1}(x - x^*(t)) > \leq 1\},$$

which be found from the next system:

$$\dot{x}^*(t) = Ax^*(t) + bp, \quad x^*(t_1) = 0;$$
$$\dot{X}_-(t) = AX_-(t) + X_-(t)A' +$$
$$X_-^{1/2}(t)S(t)(bPb')^{1/2} + (bPb')^{1/2}S'(t)X_-^{1/2}(t), \quad X_-(t_1) = 0;$$
$$S(t)P^{1/2}B's(t) = \lambda(t)X_-^{1/2}s(t), \quad S'(t)S(t) = I.$$

Here $p = 1/2(u_{\min} + u_{\max})$, $P = 1/4\|(u_{\max} - u_{\min})\|^2$ are the parameters of ellipsoid

$$\mathcal{E}(p, P) = \mathcal{P} = [u_{\min}, u_{\max}].$$

In this case the maximizer ℓ^0 in (12) can be calculated from the next relation

$$\ell^0 = 2\lambda(X_- + \lambda F)^{-1}(x - x^*), \quad F = e^{(t-t_1)A}e^{(t-t_1)A'}, \tag{13}$$

where λ is a unique nonnegative root of the equation

$$< (X_- + \lambda F)^{-1}(x - x^*), X_-(X_- + \lambda F)^{-1}(x - x^*) >= 1. \tag{14}$$

and $\ell^0 = 0$ if there are no positive roots.

Thus the closed loop control may be calculated using the relations (13),(14).

References

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