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STATE ESTIMATION, ROBUST PROPERTIES AND STABILIZATION OF POSITIVE LINEAR SYSTEMS WITH SUPERSTABILITY CONSTRAINTS

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Abstract
The paper offers an approach to the analysis and stabilization of positive linear systems based on the use of additional constraints that provide superstability of the system. It is shown that meeting the superstability conditions singles out a specific class of positive linear systems for which we can get an effective state estimation with or without external bounded disturbances and also expand the understanding of robust properties. Two ways of solving stabilization problem are given. Each of them is reduced to linear programming and can be generalized to the case of bounded control. The paper contains examples demonstrating the peculiarities of dynamics of superstable positive systems and regulator synthesis that provides required behavior of the closed-loop system.

Key words
Positive systems, superstability, state estimation, robustness, stabilization.

1 Introduction

Natural peculiarity of dynamic systems that occur in practice is the nonnegativity of their state. These systems constitute an important class of positive systems [Farina, Rinaldi, 2000], [Kaczorek, 2002], and their main property is that their output and state during all the time of functioning are nonnegative for any nonnegative inputs and initial state. Systems with this property arise in network flows and communications (traffic, transport, control in TCP networks, distributed power control algorithms for cellular communications, job balancing in computer networks, consensus and synchronization problems), transportation (vehicle formation, electrical power transmission, a supply chain for fresh products), industrial and engineering systems (wind farm, irrigation channels, chemostats, bioreactors). The property of positiveness is also seen in biology, ecology, economics and sociology. A wide range of applications and improvement of technology explains the burst of attention to positive systems that lasts for more than a decade and motivates further studies.

By now two approaches to the analysis and stabilization of positive linear systems have evolved. The first approach is based on the stable positive system having diagonal quadratic Lyapunov function. The use of its existence conditions allows finding the feedback that ensures positiveness and asymptotic stability of the closed-loop system. The problem of designing the control law is reduced to solving of LMIs, complemented by the structural constraints, responsible for the positiveness of the closed-loop system [Gao, Lam, Wang, Xu, 2005]. The development of this approach led to important results in analysis and synthesis of positive systems (see ref. in [Tanaka, Langbort, 2011], [Ebihara, Peaucelle, and Arzelier, 2014]). The second approach to the study of positive linear systems drives of the fact that their stability can be connected to the existence of linear copositive Lyapunov functions [Fornasini, Valcher, 2010]. The corresponding conditions allowed developing an effective approach to the feedback synthesis for positive systems [Rami, Tadeo, 2007]. This approach provides necessary and sufficient stabilizability conditions and attractive for numerical computations as it is reduced to linear programming (LP). Development of this approach allowed for the advancement in robust stabilization of positive systems [Briat, 2012], [Ebihara, Peaucelle, and Arzelier, 2012].

Approaches to positive system control, using LMI or LP are united by the implementation of the properties of Metzler matrices characterized by nonnegativity of of-diagonal entries [Berman, 1994]. The Metzlerian character of the matrix A is a necessary and sufficient positiveness condition of the linear system $\dot{x} = Ax$. Hence most of the positive system theory results (stability, reachability, controllability, stabilization, robustness, optimization) are based on the peculiarities specific to this class of matrices.

In this paper we study what stability properties and stabilization possibilities are revealed with the use of superstability constraints. The idea of implementing superstability conditions to the analysis and synthesis of linear systems is offered in [Polyak, Shcherbakov, 2002(a)], [Polyak, Shcherbakov, 2002(b)]. Superstability is a sufficient stability condition. Superstable linear systems are characterized by the existence of a specific Lyapunov function. Just like positiveness, superstability is caused by structural constraints formulated by means of linear restrictions on the entries of the system matrices. Combination of these constraints allows defining a class of systems not studied before - superstable positive systems. Due to the performance of superstability conditions a positive system acquires additional practically important properties inaccessible by usual positive systems. The utility of implementing superstability conditions to positive systems also becomes apparent as we get a developed approach to solving a number of complex analysis and control problems. Some of them have not been studied relating to positive systems yet.

The paper is organized as follows. Section II contains the main properties of positive and superstable systems considered separately as well as methods of synthesis of stabilizing regulators. Section III studies the peculiarities of superstable positive linear systems, formulates the positive superstabilization problem and presents several ways of solving it.

Notation: for the vector $x \in R^n$ and matrix $A = (a_{ij}) \in R^{n\times n}$ the notations $x > 0$ ($x < 0$) and $A \geq 0$ ($A > 0$), correspondingly mean $x_i > 0$ ($x_i < 0$) and $a_{ij} \geq 0$ ($a_{ij} > 0$); $A \succ 0$ ($A \prec 0$) is positive (negative) definite matrix; $||x||_\infty = \max_i |x_i|$, $||A||_1 = \max_i (\sum_j |a_{ij}|)$.

2 Preliminaries

Consider a linear continuous-time system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \geq 0, \quad (1)$$

where $x(t) \in R^n$ is the system state, $u(t) \in R^m$ is input vector, $A = (a_{ij}) \in R^{n\times n}$, $B = (b_{is}) \in R^{n\times m}$.

It is supposed that the system (1) is unstable, i.e.

$$A \notin H = \{A \in R^{n\times n} : \quad \max_i\{Re\lambda_i(A)\} < 0\},$$

where H is the set of Hurwitz matrices. The pair (A, B) is controllable.

Let the stabilizing regulator be found in the form of the state feedback

$$u(t) = Kx(t), \quad K = (k_{sj}) \in R^{m\times n}. \quad (2)$$

Below we give definitions and statements used in the analysis and synthesis of positive and superstable systems.

2.1 Positive Systems

The following definition registers the distinctive peculiarity of positive systems. Having started up at any initial condition x_0 in R^n_+, the trajectories $x(t, x_0)$ further evolve in the positive orthant R^n_+.

Definition 1: The system (1) is called positive if for any initial condition $x_0 \in R^n_+$ and input $u(t) \in R^m_+$ the state of the system is $x(t) \in R^n_+$ for all $t \geq 0$.

Generally accepted approach to defining positive systems is meeting the constraints for system matrices.

Definition 2: Matrix A is called Metzler if $A \in M$, where

$$M = \{A = (a_{ij}) \in R^{n\times n} : \quad a_{ij} \geq 0, \ i \neq j\}.$$

Theorem 1 [Farina, Rinaldi, 2000]: The system (1) is positive if and only if $A \in M$ and $B \geq 0$.

The following theorem accumulates known properties of the stable positive systems [Farina, Rinaldi, 2000], [Berman, 1994].

Theorem 2: Let the system (1) be positive. Then it is asymptotically stable if and only if one of the following equivalent conditions is satisfied: 1) $A \in H$; 2) there exists a vector $v \in R^n$ such that $v > 0$ and $Av < 0$; 3) there exists a vector $\psi \in R^n$ such that $\psi > 0$ and $A^T\psi < 0$; 4) there exists a diagonal matrix $P \succ 0$ such that $A^TP + PA \prec 0$.

Positive system stabilization cannot be accomplished by traditional methods. The control law must be chosen so that the closed-loop system $\dot{x} = (A + BK)x$ was simultaneously positive and asymptotically stable. Thus the positive stabilization problem is in finding the regulator in the form (2) that will stabilize the closed-loop system so that $A + BK \in H \cap M$. It follows from conditions 3 and 4 of Theorem 2, that stable positive system possesses two Lyapunov functions: a linear copositive one $V(x) = \psi^Tx$ and a quadratic one $V(x) = x^TPx$, $P = diag(p_i) \succ 0$. Depending on which condition of Theorem 2 we use for the basis, alternative ways of finding the gain matrix K such that $A + BK \in H \cap M$ are possible. The use of the condition 4 gives us an efficient approach, which is reduced to the solution of LMI complete with constraints of closed-loop system positiveness [Gao, Lam, Wang, Xu, 2005]. As to numerical computation, the approach [Rami, Tadeo, 2007] based on the condition 2 is preferable. Unlike the first one, this approach provides necessary and sufficient conditions of positive system stabilization. The solution for this category of problems can be obtained via LP. The main result here is the following theorem.

Theorem 3 [Rami, Tadeo, 2007]: For the system (1) the following statements are equivalent: 1) there exists such feedback (2) that the closed-loop system is positive and asymptotically stable; 2) there exists the matrix K such, that $A+BK \in H \cap M$; 3) the following LP problem, in the variables $\xi = [\xi_1\, \xi_2\, \ldots\, \xi_n]^T \in R^n$ and $z_i \in R^m,\ i = 1, 2, ..., n$, is feasible:

$$A\xi + B\sum_{i=1}^{n} z_i < 0, \quad \xi > 0,$$

$$a_{ij}\xi_j + b_i z_j, \quad i \neq j,$$

with $B = [b_1\ b_2\ \ldots\ b_n]^T$. The gain matrix can be obtained as

$$K = \begin{bmatrix} \frac{z_1}{\xi_1} & \frac{z_2}{\xi_2} & \ldots & \frac{z_n}{\xi_n} \end{bmatrix}.$$

2.2 Superstable Systems

Superstability conditions are formulated as restrictions on the entries of the system matrix.

Let

$$\sigma(A) = \min_i(-a_{ii} - \sum_{j\neq i}|a_{ij}|)$$

be the superstability degree A.

Definition 3: Matrix A is called superstable, if $A \in S$, where

$$S = \{A = (a_{ij}) \in R^{n\times n} : \quad \sigma(A) > 0\}.$$

Superstable matrices that form the set S are characterized by the performance for all their rows the condition of negative diagonal dominance

$$-a_{ii} > \sum_{j\neq i}|a_{ij}|, \quad i = 1, 2, ..., n. \qquad (3)$$

Definition 4 : The system (1) is called superstable if $A \in S$.

Superstability provides a sufficient stability condition: if $A \in S$, then $A \in H$ (the inverse statement is not true).

Though the class of superstable systems is narrow, the interest in them is caused by their practically important properties [Polyak, Shcherbakov, 2002(a)]. The superstable system (1) possesses the Lyapunov function $V(x) = ||x||_\infty = \max_i |x_i|$ and at the zero input ($u(t) \equiv 0$) the estimation

$$||x(t)||_\infty \leq ||x_0||_\infty e^{-\sigma(A)\,t}, \quad t \geq 0 \qquad (4)$$

is true for the system state. It follows from (4) that unlike usual (including positive) stable linear systems for the superstable systems the possibility of appearance of the so called "peak" effect [Polyak, Tremba, Khlebnikov, Shcherbakov, Smirnov, 2015] is excluded. It consists in drastic increase of the values of the state vector components $x_i(t)$ at the initial stage of the transient process, and can result in undesirable consequences for the reliable functioning of the system.

If the input in (1) exists and is restricted $||u(t)||_\infty \leq 1$, then the state of the system satisfies

$$||x(t)||_\infty \leq \gamma + \eta e^{-\sigma(A)t}, \quad t \geq 0, \qquad (5)$$

where $\gamma = ||B||_1/\sigma(A)$, $\eta = \max\{0, ||x_0||_\infty - \gamma\}$. It follows from (5) at $||x_0||_\infty \leq \gamma$ that $||x(t)||_\infty \leq \gamma$, $t \geq 0$, meaning that for all admissible $u(t)$ the invariant set of the superstable system is

$$Q = \{x(t) \in R^n : \quad ||x(t)||_\infty \leq \gamma\}.$$

The problem of the system (1) superstabilization is in finding the superstabilizing matrix K, that provides the performance of $A + BK \in S$ for the closed-loop system. For the entries of matrix $A+BK$, we can write the condition (3) as

$$-a_{ii} - \sum_s b_{is}k_{si} - \sum_{j\neq i}|a_{ij} + \sum_s b_{is}k_{sj}| > 0 \quad (6)$$

and formulate the general existence condition of the superstabilizing feedback in the theorem below.

Theorem 4: For the system (1) with the state feedback control law (2) there exists the gain matrix K such that $A + BK \in S$, if the system of inequalities (6) has the solution $k_{sj},\ s = 1, 2, ..., m,\ j = 1, 2, ..., n$.

Note that the necessary condition for the existence of solution of linear inequalities (6) is the performance (the analysis of superstability achievement conditions is given in [Talagaev, Tarakanov, 2012])

$$-a_{ii} - \sum_s b_{is}k_{si} > 0, \quad i = 1, 2, ..., n.$$

The existence check of the matrix K, satisfying (6), can be reduces to LP [Polyak, Shcherbakov, 2002(b)]. If the solution of the LP problem is found, we get the superstabilizing regulator, that maximizes the superstability degree $\sigma(A + BK)$ of the closed-loop system. The approach remains efficient for the case when superstabilization must be performed by the static output feedback. Superstability persists at nonlinear disturbancies, that allows using the conditions corresponding to it for the analysis and control of non-linear systems with complex dynamics [Talagaev, Tarakanov, 2012], [Talagaev, 2014].

3 Superstable Positive Systems

The similarity of positive and superstable systems is that each class is characterized by special structural constraints on the system matrix entries (see Theorem 1 and Definition 4). We can combine these constraints in one general condition and study a new class of systems, which acquires unique properties from the types of the systems that constitute it.

The matrix $A \in R^{n\times n}$ can be called the superstable Metzler matrix, if

$$a_{ij} \geq 0, \quad i \neq j, \quad i,j = 1,2,...,n \tag{7}$$

$$-\sum_j a_{ij} > 0, \quad i = 1,2,...,n. \tag{8}$$

The inequalities (7)-(8) combine the conditions that provide the simultaneous membership of A to the classes of the Metzler and superstable matrices, i.e. $A \in M \cap S$. The constraint (7) means that $A \in M$. The constraint (8) ensures the performance of superstability conditions (3) written with regard to (7). Note that from $A \in M \cap S$ follows $A \in M \cap H$.

By performing the condition $A \in M \cap S$ for the matrix A we can separate the class of *superstable positive* linear systems

$$\dot{x}(t) = Ax(t) + Bu(t), A \in M \cap S, B \geq 0, x_0 \in R^n_+. \tag{9}$$

The study of positive systems with performed superstability conditions allows deepening our understanding of the peculiarities of their dynamics and discovering new properties.

3.1 The Analysis

3.1.1 State Estimation Along with the existence for the stable positive systems linear copositive and diagonal quadratic Lyapunov functions, the superstable positive system (9) at $u(t) \equiv 0$ also possesses the following Lyapunov function

$$V(x) = \max_i(x_i)$$

with the estimation $V(x(t)) = V(x_0)e^{-\sigma(A)t}$, by inheriting this property form superstable systems. The possibility of a positive system simultaneously possessing three Lyapunov functions was mentioned earlier (see [Rantzer, 2012]). Now it is clear that such situation really takes place and is a property of the class of superstable positive linear systems.

It is possible to get an effective state estimation for superstable positive systems with and without external bounded disturbances. Namely, unlike the usual stable systems, for the positive superstable system (9) at $u(t) \equiv 0$ the estimation (4) is performed. It means that the ∞-norm solution of the unperturbed system will monotonically decrease in R^n_+.

Rewrite the system (9) as

$$\dot{x}(t) = Ax(t) + Dw(t),$$

where $w(t) \in R^r$ is the external disturbance, satisfying the restriction

$$0 \leq ||w(t)||_\infty \leq 1, \quad t \geq 0,$$

$D \in R^{n\times r}_+$ is constant matrix. So, if $A \in M \cap S$, then for any $x_0 \in Q_M$ at all $t \geq 0$ it will be

$$x(t) \in Q_M = \{x(t) \in R^n_+ : \quad 0 \leq ||x(t)||_\infty \leq \lambda\},$$

where $\lambda = ||D||_1/\sigma(A)$.

Thus, at bounded perturbations ∞-norm solution remains restricted, and we get an easy way of estimating the invariant set Q_M of the positive system (9).

Example 1. Compare the dynamics of the stable and superstable positive systems given by the matrices

$$A_1 = \begin{bmatrix} -1 & 5 \\ 0 & -1 \end{bmatrix} \in M \cap H, A_2 = \begin{bmatrix} -6 & 5 \\ 0 & -1 \end{bmatrix} \in M \cap S$$

In the numerical experiment, the initial conditions were chosen at random from the set $X_0 = \{x_0 \in R^2_+ : 0 \leq x_{0i} \leq 1, \ i = 1,2\}$. The dynamics of both systems is shown in the Fig. 1. We can see that, unlike the stable one, the trajectories of the superstable positive system satisfy the estimation $||x(t)||_\infty \leq ||x_0||_\infty e^{-\sigma(A_2)\,t}$, where $\sigma(A_2) = 1$. The differences in transient processes of the systems are shown in Fig. 2 ($x_0 = (0.1, \ 1)$). While the components $x_2(t)$ of the state vector $x(t) = (x_1(t), x_2(t))$ of both systems behave the same way, the component $x_1(t)$ of the stable positive system undergoes the "peak" of $\max(x_1(t))/x_1(0)) \approx 18.8$.

3.1.2 Robustness Fulfillment of superstability conditions expands our understanding of robust properties of positive systems.

Consider a matrix family

$$A = DA_0,$$

where $A_0 = (a^0_{ij}) \in M$ is the nominal matrix, $D = diag(d_i) > 0$ is the strictly positive diagonal matrix. The first robust property of positive systems is known as *D-stability* [Farina, Rinaldi, 2000]: *let* $A_0 \in M \cap H$, *then* $DA_0 \in M \cap H$. It is easy to see that by replacing stability with superstability we don't

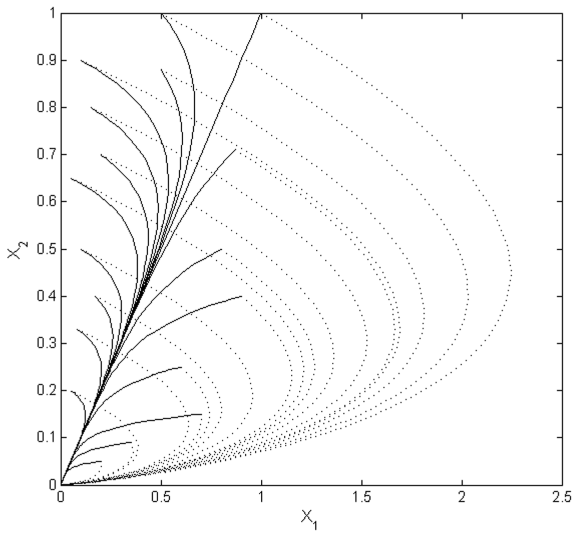

Figure 1. Phase trajectories of the stable positive system (dotted) and superstable positive system (solid).

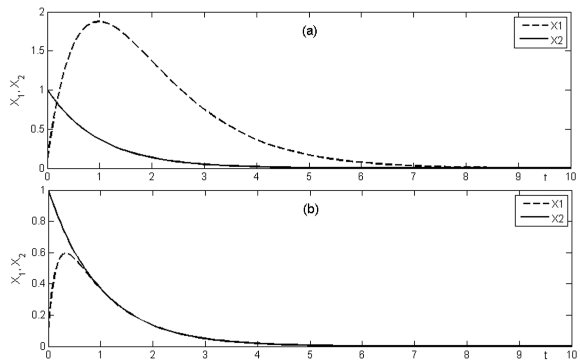

Figure 2. Transient responses of the stable (a) and superstable (b) positive systems.

break this property. For the matrix $A = DA_0 \in M$ the superstability condition $-\sum_j d_i a^0_{ij} > 0,\ i = 1, 2, ..., n$ is performed at any $d_i > 0$. Following the accepted terminology, we get that superstable positive systems are *D-superstable: if the positive system* $\dot{x} = A_0 x$ *is superstable, then all diagonally perturbed systems* $\dot{x} = DA_0 x$ *are superstable too.*

Consider the following matrix family

$$A = A_0 + \Delta,$$

where $A_0 \in M \cap H$, $\Delta = (\delta_{ij}) \in R^{n\times n}$ is the uncertainty. The second robust property of the positive systems is so called *connective stability* [Farina, Rinaldi, 2000]: *let* $A_0 \in M \cap H$*, then* $A_0 + \Delta \in M \cap H$*, if* $-a^0_{ij} \le \delta_{ij} \le 0,\ i \ne j,\ \delta_{ii} = 0$. Let's demonstrate how this property expands if we transfer from just stable to superstable positive systems. Let $A_0 \in M \cap S$. Obviously, any matrix from the $A_0 + \Delta$ family at $\delta_{ii} = 0$ and $-a^0_{ij} \le \delta_{ij} \le 0$ will also remain Metzler and superstable at all $i \ne j$. Thus superstable positive systems are *connectively superstable*. Now we demonstrate that fulfillment of superstability conditions allows expanding the restrictions on the elements δ_{ij} of the matrix Δ. The condition $A_0 + \Delta \in M$ is met for any $\delta_{ij} \ge 0,\ i \ne j$ and $\delta_{ii} = 0$. However at $A_0 + \Delta \in M$ the matrix $A_0 + \Delta$ will also be superstable, if $\delta_{ii} = 0$ and for every $i = 1, 2, ..., n$ there performed inequalities $-a^0_{ii} - \sum_{j\ne i}(a^0_{ij} + \delta_{ij}) > 0,\ i = 1, 2, ..., n$ or $\sum_{j\ne i}\delta_{ij} < \sum_j a^0_{ij},\ i = 1, 2, ..., n$. By combining we get the next robust property of superstable positive systems: *let* $A_0 \in M \cap S$*, then the uncertain system*

$$\dot{x}(t) = (A_0 + \Delta)x(t),$$

is superstable and positive if $\delta_{ii} = 0$ *and there performed*

$$-a^0_{ij} \le \delta_{ij} \le 0 \quad \forall\, i \ne j,$$

or

$$\sum_{j\ne i}\delta_{ij} < \sum_j a^0_{ij} \quad \forall\, i.$$

Robustness to the class of perturbations $A_0 \to A_0 + \Delta$ has useful practical applications (drift, "aging " of parameters, etc.). The first condition $(-a^0_{ij} \le \delta_{ij} \le 0, i \ne j)$ is simply inherited from the well-known property of positive systems. The second condition $(\sum_{j\ne i}\delta_{ij} < \sum_j a^0_{ij}, i = 1, 2, ..., n)$ is new. It arises only from combining the properties of Metzler and superstable matrices and applicable only to superstable positive systems. Notice that in a special case, when $\delta_{ii} = 0$ and for all $i \ne j$ there performed $\delta_{ij} = \delta$, the preserving condition of $A_0 + \Delta \in M \cap S$ will be

$$\max(-a^0_{ij}) \le \delta \le \sum_j a^0_{ij}/(n-1).$$

3.2 Stabilization

Implementation of superstability conditions to positive systems allows better understanding of their properties. Along with that there arises the possibility to develop several mutually complementary approaches to the solution of the stabilization problem.

The peculiarity of the problem under consideration is that the desired feedback should simultaneously provide both positiveness and superstability of the closed-loop system, i.e. the matrix K such that $A + BK \in M \cap S$ must be found. As the free system can be nonpositive, let's call the problem *positive superstabilization*. This problem can arise in cases when the dynamics of the closed-loop system must possess the features of transient processes peculiar to superstable systems, and positiveness is native for the system or an additional restriction.

The first approach is based on the modification of Theorem 3. For this to the existing theorem conditions we add a constraint, requiring the superstability of the matrix $A+BK$.

Theorem 5: There exists a state feedback control law (2) for the system (1) such that the closed-loop system becomes positive and superstable if the following LP has a feasible solution with respect to the variables $\xi = [\xi_1\, \xi_2\, \dots\, \xi_n]^T \in R^n$ and $z_1, ..., z_n \in R^m$

$$A\xi + B\sum_i z_i < 0, \quad \xi > 0, \tag{10}$$

$$a_{ij}\xi_j + b_i z_j \geq 0, \quad i \neq j, \tag{11}$$

$$-\sum_j (a_{ij}\xi_j + b_i z_j) > 0, \quad i = 1, 2, ..., n \tag{12}$$

with $B = [b_1\ b_2\ \dots\ b_n]^T$. The gain matrix can be obtained as

$$K = \begin{bmatrix} \frac{z_1}{\xi_1} & \frac{z_2}{\xi_2} & \dots & \frac{z_n}{\xi_n} \end{bmatrix}.$$

Proof. Constraints (10) and (11) come from Theorem 3 unchanged. They provide the performance of $A + BK \in H$ (10) and $A + BK \in M$ (11). The addition of (12) allows performing the condition $A + BK \in S$. The corresponding structural constraint (12) is constructed similarly to (11). For the matrix $A + BK$ we have $(A+BK)_{ij} = a_{ij} + b_i K_j = a_{ij} + b_i \frac{z_j}{\xi_j}$. Then from (8) we get (12).

Implementation of Theorem 5 opens a consistent approach to the synthesis of the superstable positive systems. Like Theorem 3, Theorem 5 can be complemented with the constraints caused by uncertainties in the matrices A, B or control boundedness (a kind of the arising constraints is given in [Rami, Tadeo, 2007]). For example, if the control law is sought with regard to the condition $0 \leq u(t) \leq \bar{u}$, then constraints $z_i \geq 0, \quad \sum_{i=1}^n z_i \leq \bar{u}$ are added to (10)-(12). Notice that an objective restriction to wide use of this approach is its application only to the systems (1) with matrices B being of size $n \times 1$. The approach below allows removing this restriction.

The second approach provides sufficient existence conditions of the positively superstabilizing regulator. It is based on Theorem 4.

Theorem 6: There exists a state feedback control law (2) for the system (1) such that the closed-loop system becomes positive and superstable, if there exists the solution $K = (k_{sj})$, σ for the LP problem

$$\max\ \sigma, \tag{13}$$

$$-(a_{ii} + \sum_s b_{is}k_{si}) - \sum_{j\neq i} p_{ij} \geq \sigma > 0, \quad i = 1, ..., n \tag{14}$$

$$0 \leq a_{ij} + \sum_s b_{is}k_{sj} \leq p_{ij}, \quad i \neq j, \tag{15}$$

and $\sigma > 0$.

Proof. First, we get superstability conditions for the matrix $A + BK$. It follows from Theorem 4 that $A + BK \in S$ if there exists the solution k_{sj}, $s = 1, 2, ..., m$, $j = 1, 2, ..., n$ of the system of inequalities (6). By introducing additional variables σ and p_{ij}, $i, j = 1, ..., n$, we can write condition (6) in the equivalent form [Polyak, Shcherbakov, 2002(b)]. For the diagonal entries of the matrix $A + BK$ we get constraints (14), and for all the others $-p_{ij} \leq a_{ij} + \sum_s b_{is}k_{sj} \leq p_{ij}$. To provide simultaneously $A + BK \in S$ and $A + BK \in M$, we must also take condition (7) into account. Thus we come to (15). Finally to check the existence of solution k_{sj}, we can use the LP problem (13).

Like the previous one, the presented approach to the stabilization of positive systems has its advantages and disadvantages. Unlike the first approach, the matrix B can be of any size ($B \in R_+^{n\times m}$). It is not restricted by the condition $B = [b_1\ b_2\ \dots\ b_n]^T$. Another advantage is the possibility to get the solution (if there exists one) of the problem output positive superstabilization. Suppose, that the output $y = Cx$, $C \in R_+^{p\times n}$ is given for the system (1), and control is sought in the form $u = Ky$. The performance check for $A+BK \in M\cap S$ is done similarly. We solve the problem (13), but in constraints (14)-(15) instead of $a_{ij} + \sum_s b_{is}k_{sj}$ it should be written $w_{ij}(K) = (BKC)_{ij} = b_i K c_j$, where b_i is the i-th line of the matrix B, c_j is the j-th column of the matrix C. Notice that within the restrictions of the first approach output stabilization of positive systems is a complex problem, and its solution can be obtained only for a specific case [Rami, 2011]. The second approach provides the complete solution of the problem. Still, it must be taken into consideration that superstability conditions are strict. If for this particular system Theorem 6 gives negative answer to the question about the existence of positively superstabilizing feedback (i.e. $\sigma \leq 0$), it doesn't mean that stabilization can't be performed another way.

Example 2. Let us have the system (1) with

$$A = \begin{bmatrix} 3 & -1 \\ 6 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{16}$$

Matrix A is such that $A \notin M$ and $A \notin H$, i.e. the system (16) is nonpositive and unstable. Let's demonsitate, how we can study the existence conditions of feedback, that satisfies $A + BK \in M \cap S$ and obtain the clear solution of the positive superstabilization problem.

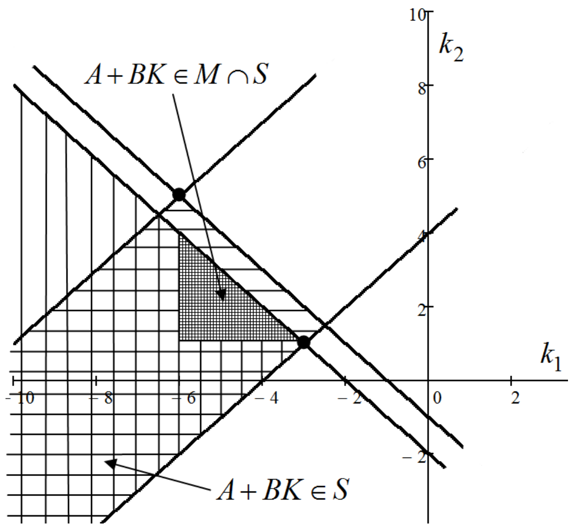

Figure 3. The existence area of the positivily suprstabilizable fuzzy regulator for the system (16).

In the first stage, for the pair A, B we check the existence of the matrix $K = [k_1 \quad k_2]$, that provides the solution of the superstabilization problem, i.e. $A + BK \in S$. To do this we use the superstabilizability condition obtained in [Polyak, Shcherbakov, 2002(b)]: $a_{11} - a_{21} + a_{22} - a_{12} < 0$. It is easy to check that for the given matrix A the inequality is performed. Hence, superstabilizing regulator exists.

Superstability conditions (6) for the matrix $A + BK$ look like $-(a_{11} + k_1) > |a_{12} + k_2|, -(a_{22} + k_2) > |a_{21} + k_1|$. As shown in Fig. 3, every inequality marks a right angle in the plane (k_1, k_2) (first inequality – horizontal lining, second – vertical lining). The existence area of K, such that $A + BK \in S$, lies in the common points of these angles. Additionally in Fig. 3 for the system (16) there was plotted a bounded area (a triangle), where for matrices $A + BK \in S$ the positiveness condition $a_{12} + k_2 \geq 0, a_{21} + k_1 \geq 0$ is performed. The resulting area allows choosing the matrix and stabilize the system (16) so, that $A+BK \in M \cap S$.

4 Conclusion

A new approach to the analysis stabilization of positive linear systems based on the use of superstability conditions is offered. Just like positiveness conditions, superstability conditions are formulated as constraints on the entries of the system matrix. By simultaneously performing the conditions of positiveness and superstability we separate a special class of systems, possessing practically useful properties. It is shown that for superstable positive systems we can get an efficient state estimation with and without external bounded disturbances. Robust properties (D-superstability, connective superstability) are studied. Positive superstabilization problem for both usual linear and positive linear systems is formulated. Two approaches to stabilization are presented. Each of them can be reduced to solving the linear programming problem and generalized for the case of bounded control. The examples demonstrating the dynamics peculiarities of the new class of systems, as well as the study of the existence conditions of the regulator, which provides both positiveness and superstability of the closed-loop system are given.

The subject of the study in this work are linear positive systems. However, in practice positive systems can be nonlinear. Expansion of the positive system theory to nonlinear control systems is an important problem (see [Churilova, 2010] where it is shown that the assumption of positiveness appreciably simplifies analysis of the absolute stability of a nonlinear Lurie system). It is known that superstability is preserved at nonlinear perturbations. That's why further investigation is seen in generalization of the achieved results to nonlinear systems.

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