CYBERNETICS AND PHYSICS, VOL. 3, NO. 4, 2014 , 157–160

STABILIZATION OF LINEAR SYSTEMS VIA A TWO-WAY CHANNEL UNDER INFORMATION CONSTRAINTS

Vladimir Bondarko
Department of Mathematics and Mechanics
Saint Petersburg State University
Russia
vbondarko@gmail.com

Abstract
The rate of informational exchange between the subsystems is limited for a lot of physical systems. For this reason the problems of information management in the presence of constraints are of interest to study. This paper is devoted to feedback stabilization problems for linear time-invariant control systems with saturating quantized measurements and control. This quantization occurs due to the finite capacity of the discrete-time two-way channel which connects the plant and the controller. The main result of the article is a very simple controller and a rough estimate of the upper bound for the minimum capacity sufficient to stabilize the system.

Key words
Feedback stabilization, discrete-time quantization of measurements and control.

1 Introduction

Assume that a control system consists of a linear time-invariant SISO (single input/ single output) plant, a digital two-way discrete-time communication channel, and a remote feedback controller. Any digital channel distorts transmitted signals. The values of this distortion depend on the channel word length and saturation level. Given a fixed word length we will try to construct a control law for stabilizing the closed system. This control law should include a description of controller and a rule to choose a saturation level. Finally, we should specify a word length sufficient to yield stability.

Our setting is close to the concept of measurements quantization [Brockett and Liberzon, 2000], but the current article treats quantization of the control input too. Besides, the quantization is executed in discrete time. This allows us to stabilize both discrete and continuous sampled-time systems.

Let us now introduce some notation. Denote the set of integer numbers as $\mathbb{Z}$. The following floor function $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k < x\}$ is convenient for defining the *quantizer*

$$Q_\Delta(x) = \\ = \begin{cases} -M & \text{if } \frac{x}{\Delta} \leqslant -M - 1/2, \\ \left\lfloor \frac{x}{\Delta} + \frac{1}{2} \right\rfloor, & \text{if } \frac{x}{\Delta} \in (-M - 1/2, M + 1/2], \\ M & \text{if } \frac{x}{\Delta} > M + 1/2 \end{cases}$$

for a given positive integer *saturation value* M and real *sensitivity* Δ. We will use $M = 2^\nu$ where $\nu + 2$ is a channel binary word length that is sufficient to transmit a quantized signal.

Having a quantized signal $\tilde{x} = q(x)$ one can use $\hat{x} = \Delta\tilde{x}$ to approximate the original value x. Approximation of the error $e = \hat{x} - x$ depends on the band where x lies. If $|x| \leqslant \Delta(M + 1/2)$, then $|e|$ is bounded by $\Delta/2$. Otherwise $|e|$ may be arbitrarily large. For this reason, the strategy of control will consist of two stages. First, since the initial state is unknown, we will have to zoom out, i.e., increase Δ until the state of the system can be adequately measured, and control value can be adequately transmitted. Second, we will zoom in, i.e., decrease Δ in to drive the state to 0.

2 Stabilization of Discrete-time Systems

Let us consider the output feedback stabilization problem for the discrete-time linear SISO system (plant) with input u_k and observable output y_k:

$$\alpha(q)y_k = q\beta(q)\hat{u}_k,\ k = 0,1,\dots \tag{1}$$

with initial data $y_{-1}, y_{-2}, \dots y_{-n}, u_{-1}, u_{-2}, \dots u_{-n}$, where q denotes backshift operator: $q^m y_k = y_{k-m}$. The polynomials

$$\alpha(\lambda) = 1 + \alpha_1\lambda + \cdots + \alpha_n\lambda^n,$$
$$\beta(\lambda) = \beta_0 + \beta_1\lambda + \cdots + \beta_{n-1}\lambda^{n-1}$$

are coprime.

The input of the plant is denoted as $\hat{u}_k$ to emphasize that this is a reconstructed version of the control signal u_k produced by the remote controller. A channel transmits a quantized signal

$$\tilde{u}_k = Q_{\Delta_k}(u_k), \tag{2}$$

and a decoder generates

$$\hat{u}_k = \begin{cases} \tilde{u}_k\Delta_k, & \text{if } \varkappa_k = 2n-1, \\ 0, & \text{if } \varkappa_k < 2n-1, \end{cases} \tag{3}$$

using a counter

$$\varkappa_k = \begin{cases} 0, \text{ if } |\tilde{y}_k| = M \text{ or } |\tilde{u}_{k-1}| = M, \\ \min\{2n-1, \varkappa_{k-1}+1\}, \\ \qquad \text{if } |\tilde{y}_k| < M \text{ and } |\tilde{u}_{k-1}| < M, \end{cases} \tag{4}$$
$$k = 1,2,\dots,\ \varkappa_0 = 0.$$

Next, $\hat{u}_k$ comes to the plant (1) and produces the observable output y_{k+1}. The channel transmits the quantized signal

$$\tilde{y}_k = Q_{\Delta_k}(y_k) \tag{5}$$

to the decoder; this results in

$$\hat{y}_k = \tilde{y}_k\Delta_k. \tag{6}$$

Let us describe a control law. It is based on the idea of a deadbeat controller

$$u_k + \gamma(q)u_{k-1} = \delta(q)y_k, \tag{7}$$

where the polynomials $\gamma(\lambda)$ and $\delta(\lambda)$ give a solution of the Diophantine equation

$$\alpha(\lambda)[1 + \lambda\gamma(\lambda)] - \beta(\lambda)\lambda\delta(\lambda) \equiv 1. \tag{8}$$

This identity is equivalent to the system of linear equations for the vector of coefficients of $\delta(\lambda)$ and $\gamma(\lambda)$. The determinant of this system is equal to the resultant [van der Waerden, 2003] of $\alpha(\lambda)$ and $\lambda\beta(\lambda)$. Since these polynomials are coprime, the Diophantine equation (8) has a unique solution.

Unfortunately, the controller (7) is not realizable, since y_k are unknown. For this reason, we will use $\hat{y}_k$ instead of y_k. Besides, we can take into account that our plant (1) receives input values $\hat{u}_k$ instead of u_k. Thus a reasonable controller may be described as

$$u_k = \delta(q)\hat{y}_k - \gamma(q)\hat{u}_{k-1}. \tag{9}$$

At last, let us fix $\rho \in (0,1)$. We describe a rule for adjusting the channel sensitivity as follows:

$$\Delta_k = \begin{cases} 2^{k^2}, \text{ if } \varkappa_{k-1} < 2n-1 \\ \rho\max_i \Delta_{k-i},\ i \in [1, 2n-1], \\ \qquad \text{if } \varkappa_{k-1} = 2n-1, \end{cases} \tag{10}$$
$$k = 1,2,\dots,\ \Delta_0 = 1.$$

There is no need to transmit $\varkappa_k$ and Δ_k through the channel. These values may be computed independently at both ends of the channel.

Define the polynomial norm as the sum of the absolute values of its coefficients. Denote

$$N_{11} = \|\beta(\lambda)\|,\ N_{12} = \|\beta(\lambda)\delta(\lambda)\|,$$
$$N_{22} = \|\delta(\lambda)\alpha(\lambda)\|. \tag{11}$$

Theorem 1. *Let*

$$M > (N_{12} + \max\{N_{11}, N_{22}\})/\rho + 1. \tag{12}$$

Then the system (1),(2),(3),(4),(5),(6),(9),(10) is asymptotically stable.

Proof. Assume that $\varkappa_k < 2n-1$ for all k. Then (3) implies $\hat{u}_k \equiv 0$. Hence $|y_k|$ and $|u_k|$ grow exponentially by virtue of (1) and (9), whereas Δ_k grows much faster due to (10). Thus $y_k/\Delta_k \to 0$ and $u_k/\Delta_k \to 0$ as $k \to \infty$. Therefore $\tilde{y}_k$ and $\tilde{u}_k$ will be less then M for all sufficiently large k, i.e. $\varkappa_k = 2n-1$. This contradicts our assumption.

Consider the disturbances

$$v_k = \hat{u}_k - u_k,\ e_k = \hat{y}_k - y_k.$$

If $\max\{|\tilde{y}_k|, |\tilde{u}_k|\} < M$, then $|v_k| < \Delta_k/2$ and $|e_k| < \Delta_k/2$.

Let k be a moment when $\varkappa_{k-1} = 2n-1$. Then $\Delta_k = \rho \max_{i\in[1,2n-1]} \Delta_{k-i}$ and

$$|v_{k-i-1}| < \Delta_{k-i-1}/2,\ |e_{k-i}| < \Delta_{k-i}/2 \quad (13)$$

for $i = 1, 2, \ldots, 2n-1$. Furthermore,

$$\begin{aligned} y_k &= \{\alpha(q)[1+q\gamma(q)] - \beta(q)q\delta(q)\}\, y_k = \\ &=[1+q\gamma(q)][\beta(q)q\hat{u}_k] - \beta(q)q\delta(q)y_k = \\ &=\beta(q)q\,\{u_k + v_k + q\gamma(q)\hat{u}_k - \delta(q)\hat{y}_k + \delta(q)e_k\} \\ &=\beta(q)q[v_k + \delta(q)e_k], \end{aligned} \quad (14)$$

$$\begin{aligned} u_{k-1} &= q\,\{\alpha(q)[1+q\gamma(q)] - \beta(q)q\delta(q)\}\, u_k = \\ &=q\alpha(q)[1+q\gamma(q)]u_k - q\beta(q)q\delta(q)(\hat{u}_k - v_k) = \\ &=q\alpha(q)[u_k + q\gamma(q)(\hat{u}_k - v_k)] - q\delta(q)\alpha(q)y_k + \\ &\quad + q^2\beta(q)\delta(q)v_k = \\ &=q\alpha(q)[u_k + q\gamma(q)(\hat{u}_k - v_k)] - \\ &\quad - q\delta(q)\alpha(q)(\hat{y}_k - e_k) + q^2\beta(q)\delta(q)v_k = \\ &=q\alpha(q)[u_k + q\gamma(q)\hat{u}_k - \delta(q)\hat{y}_k] + q\delta(q)\alpha(q)e_k + \\ &\quad + q^2\beta(q)\delta(q)v_k = \\ &=q\delta(q)\alpha(q)e_k + q^2\beta(q)\delta(q)v_k. \end{aligned} \quad (15)$$

Inequalities (13) and equations (14),(15) imply

$$|y_k| \leqslant (N_{11} + N_{12}) \max_{i\in[1,2n-1]} \Delta_{k-i}/2, \quad (16)$$

$$|u_{k-1}| \leqslant (N_{12} + N_{22}) \max_{i\in[1,2n-1]} \Delta_{k-i}/2. \quad (17)$$

Hence,

$$\begin{aligned} &\left|\frac{y_k}{\Delta_k}\right| = \left|y_k \left(\rho \max_{i\in[1,2n-1]} \Delta_{k-i}\right)^{-1}\right| \leqslant \\ &\qquad \leqslant (N_{11} + N_{12})/\rho < M - 1, \\ &\left|\frac{u_{k-1}}{\Delta_{k-1}}\right| = \left|u_{k-1} \left(\rho \max_{i\in[1,2n-1]} \Delta_{k-i}\right)^{-1}\right| \leqslant \\ &\qquad \leqslant (N_{12} + N_{22})/\rho < M - 1 \end{aligned}$$

by virtue of (12).

Thus we obtain two points. Firstly, there exists a moment k_0 such that $\varkappa_{k_0} = 2n-1$. Secondly, if $\varkappa_{k-1} = 2n-1$ then $\varkappa_k = 2n-1$, too. Therefore

$$|y_k| < M\Delta_k,\ |u_k| < M\Delta_k,\ \Delta_k = \rho \max_{i\in[1,2n-1]} \Delta_{k-i}$$

for all $k > k_0$. This yields the asymptotic stability of the system under consideration. The theorem is proved.

3 Stabilization of Continuous-time Systems

Let us now fix a sampling period $h > 0$. Suppose that our digital discrete-time two-way channel can transmit in each direction a bounded signed integer number every h seconds. Using this channel we will stabilize the continuous-time linear SISO system (plant)

$$a(d/dt)y(t) = b(d/dt)\hat{u}(t),\ t \in [0, \infty) \quad (18)$$

with coprime polynomials

$$\begin{aligned} a(\lambda) &= (\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n), \\ b(\lambda) &= b_1\lambda^{n-1} + b_2\lambda^{n-2} + \ldots + b_n. \end{aligned} \quad (19)$$

Here $y(t)$ is the observable output, $\hat{u}(t)$ is a zero-order hold (ZOH) control signal reconstruction given by the following digital-to-analog converter (DAC):

$$\hat{u}(kh + \varepsilon) = \hat{u}_k \text{ for } \varepsilon \in [0, h),\ k = 0, 1, \ldots, \quad (20)$$

where $\hat{u}_k$ are obtained from (3) and (4),

$$y_k = y(kh), \quad (21)$$

Furthermore, the quantized discrete-times signals $\tilde{y}_k$ and $\tilde{u}_k$ should be given by (2),(5), and the reconstructed version $\hat{y}_k$ of the output samples (21) should be computed by (6).

In order to finish the description of our control system, consider the sampled system (18), (20), (21). The differential equation (18) may be equivalently rewritten in the state-space form as

$$dx(t)/dt = Ax(t) + Bu(t),\ y(t) = Cx(t), \quad (22)$$

where $x(t) \in \mathbb{R}^n$, and the matrices $A \in \mathbb{R}^{(n\times n)}$, $B \in \mathbb{R}^n$, $C \in \mathbb{R}^{1\times n)}$ satisfy the equation $C(\lambda I - A)^{-1}B = b(\lambda)/a(\lambda)$ for all λ. Denote $x(kh)$ by x_k and use the Cauchy formula to solve (22) for $t \in [kh, (k+1)h)$:

$$\begin{aligned} x_{k+1} =& Px_k + Qu_k,\ y_k = Cx_k, \\ & P = e^{hA}, Q = \int_0^h e^{sA} ds B. \end{aligned} \quad (23)$$

This is the state-space form of discrete-time linear time-invariant system. It obviously implies the input-output equation (1) with polynomials

$$\alpha(\lambda) = \prod_{i=1}^{n}(1 - \lambda e^{h\lambda_j}),\ \beta(\lambda) = \alpha(\lambda)C(I - \lambda P)^{-1}Q. \quad (24)$$

If the polynomials (19) are coprime and

$$\lambda_i \neq \lambda_j \;\Rightarrow\; e^{h\lambda_i} \neq e^{h\lambda_j}, \tag{25}$$

then the polynomials (24) are coprime also. This fact seems to be well known; its proof may be found, for example, in the paper [Bondarko, 1994]. The inequality (25) is equivalent to the following condition for the sampling period h: if two distinct zeros of $a(\lambda)$ have equal real parts, then the difference between their imaginary parts should not be a multiple of $2\pi/h$.

Putting all of this together, we will consider the system (18), (20), (21), (2), (3), (4), (5), (6), (9). The polynomials $\gamma(\lambda)$ and $\delta(\lambda)$ in the equation (9) should be determined by the equation (8) with polynomials (24). An appropriate choice of the sampling period h ensures the solvability of (8). The constants N_{11}, N_{12}, N_{22} should be computed via the formulas (11).

Theorem 2. *Let the sampling period h satisfy the inequalities (25). Let M satisfy the inequality (12) with some $\rho \in (0,1)$. Then the system (18), (20), (21), (2), (3), (4), (5), (6), (9) is asymptotically stable.*

Proof. Using the ZOH input (20) we ensure the equation (1) with polynomials (24). They are coprime by (25). Therefore $y_k \to 0$ and $u_k \to 0$ as $k \to \infty$ by virtue of Theorem 1.

Furthermore, the matrices A and C in (22) may be chosen to form an observable pair. Hence $Cz \neq 0$ for all $z \neq 0$, $Az = \lambda z$. Then the pair $\{P, C\}$ will be observable also, since the condition (25) implies that P has the same set of eigenvectors as A does. So as $x_k \to 0$ as $k \to \infty$ together with y_k and u_k. This is enough to ensure that $x(t) \to 0$ as $t \to \infty$. The theorem is proved.

4 Simulation Results

Let us consider the plant

$$y_k - 3y_{k-1} + 2y_{k-2} = u_{k-1} - 3u_{k-2},$$
$$y_0 = 10,\; y_{-1} = -12,\; u_{-1} = 30.$$

So $\alpha(\lambda) = 1 - 3\lambda + 2\lambda^2$, $\beta(\lambda) = 1 - 3\lambda$. The equation (8) yields $\gamma(\lambda) = 10.5$, $\delta(\lambda) = 7.5 - 7\lambda$, since $\beta(\lambda)\delta(\lambda) = 21\lambda^2 - 59\lambda/2 + 15/2$, $\alpha(\lambda)\delta(\lambda) = -14\lambda^3 + 36\lambda^2 - 59\lambda/2 + 15/2$, $N_{11} = 4$, $N_{12} = 58$, $N_{22} = 87$. Hence the values $M = 127$ and $\rho = 0.6$ satisfy the condition (12).

The simulation results are shown at the figure 1.

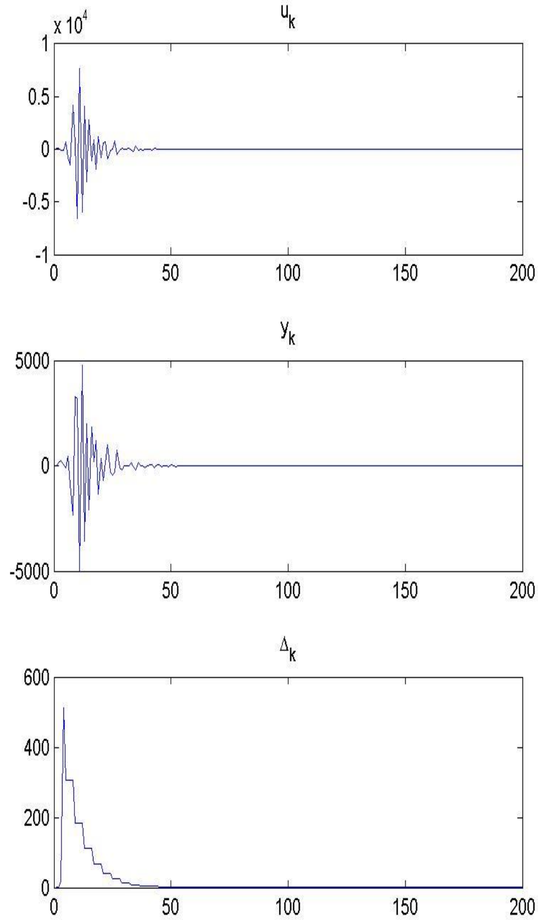

Figure 1. Simulation results.

5 Conclusion

A very simple feedback controller yields asymptotic stabilization of LTI systems through discrete-time channels with limited capacity. This result may be easily generalized to the case of disturbed systems, systems with time delay, and so on. It seems to be possible to extend this approach to the problem of adaptive control, where the system parameters are uncertain and unknown.

References

Bondarko, V.A. (1994) Adaptive stabilization of nonminimum-phase objects with unknown lag. *Journ. of Computer and Systems Sciences International, **32**(4), pp. 77–82.*

Brockett, R.W., and Liberzon, D (2000) Quantized feedback stabilization of linear systems. *IEEE Trans. on Automatic Control, **AC45**(7), pp. 1279–1289.*

van der Waerden, B.L.(2003) *Algebra*. Springer-Verlag. New York.