On Fractional Fourier Analysis in Ultra-distributional set-up and Image processing

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ABSTRACT:*The fractional Fourier analysis is used for investigations of fractal structures; which in turn are used to analyze different physical phenomena. With the advent of Fractional Fourier Transform (FrFT) and related concepts, it is seen that the properties and applications of the conventional Fourier Transform are special cases of those FrFT. The intimate relationship of FrFT to time-frequency representation leads to many applications in signal analysis and processing for which the Fourier transform fails to work.*

In this paper, the fractional Fourier analysis is carried out in Ultra-distributional set-up. Its important connection with Radon-Wigner transform and Wigner Distribution of Ambiguity functions in the context of its applicability in image processing is discussed. Analogous results to that of Paley-Wiener theorems are obtained for ultra-differentiable functions and ultra-distributions.

1 Introduction:

The Fractional Fourier Transform (FrFT) is a generalization of Fourier transform (FT) and depends on a parameter α that is associated with the angle in phase plane.

This leads to the generalization of notion of space (or time) and frequency domain which are central concepts of signal processing [7]. A signal is uniquely defined in the position domain, f(x) ($\alpha = 0$), or as its FT in frequency domain, $F_{\pi/2}$ (y).

It follows from the additive property that if one produces the FrFT, F_α (ξ) of a signal f(x), then its FT, $F_{\alpha+\pi/2}$ (k_ξ) is the FrFT for the parameter α of $F_{\pi/2}$ (y).

Thus the FrFT is a signal representation along an axis ξ rotated at an angle α in the phase plane:

$$\begin{pmatrix} \xi \\ k_\xi \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ k_x \end{pmatrix} \qquad (1.1)$$

In this paper, FrFT is extended to the spaces of ultra-distributions and studied it in ultra-distributional set up. Its relationships with the Ambiguity Function (AF), Wigner Distribution (WD) and Radon-Wigner Transform (RWT) are discussed and in the end the potent applications are indicated.

For this purpose we consider the test function space E (M_p; Ω) and define the fractional FT under suitable conditions.

2. The spaces E(M_p; Ω), E' (M_p; Ω):

Let $\{M_p\}_{p\in \mathrm{IN}0}$ be an increasing sequence of positive numbers. Then a

C^{∞}- function ϕ on an open set Ω in IR^n is called as ultra-differentiable function of class M_p (of Roumieu type) if on each compact set K in Ω its derivatives are estimated as follows: there are positive constants C and A such that

$$\sup_{x\in K} \left| D^{\beta} \phi(x) \right| \leq C A^{|\beta|} M_{|\beta|}, \ |\beta| = 0,1,2... \qquad (2.1)$$

where $D^{\beta} = D_1^{\beta},\ldots D_n^{\beta}$ and $|\beta| = \beta_1+\beta_2+\ldots+\beta_n$,

The space of ultra-differentiable functions is denoted by $E(M_p; \Omega)$.

Here the increasing sequence $\{M_p\}_{p\in IN0}$ is imposed of certain conditions (M.1)-(M.3): for the constants R> 0 and H> 1,

(M.1) $M_p^2 \leq M_{p-1} M_{p+1}$, $p\in IN$ (Logarithmic convexity),

(M.2) $M_p \leq RH^p \min_{0\leq q\leq p} M_q M_{p-q}$, $p\in IN_0$ (Stability under differentiable operators),

(M.3) $\sum_{j=0}^{\infty} M_j/M_{j+1} < \infty$ (Non-quasi-analyticity).

In some problems (M.2) may be replaced by the weaker condition (M.2)':

(M.2)' $M_{p+1} \leq RH^p M_p$, $p\in IN_0$, $R>0$ (Stability under differentiation).

The conditions (M.1) and (M.2) ensure that the products of ultra-differentiable functions and the derivatives of ultra-differentiable functions belong to the same class; and the condition (M.3) guarantees that there exists an infinitely differentiable function $\phi(x) \neq 0$, of compact support that satisfies the inequalities

$$\sup_{x\in K} \left| D^{\beta} \phi(x) \right| \leq C A^{|\beta|} M_{|\beta|}, \ |\beta| = 0,1,2...$$

3. The fractional FT:

The one dimensional fractional FT with parameter α of a signal f(x) denoted by R^{α} f(x) [1], performs a linear operation, given by the integral transform

$$[R^{\alpha} f(x)] (\xi) = F_{\alpha} (\xi) = \int_{-\infty}^{\infty} K_{\alpha} (x, \xi) f(x) dx \qquad (3.1)$$

where the kernel

$$K_{\alpha} (x,\xi) = (2\pi i \ \mathrm{Sin}\alpha)^{-1/2} \exp (i\alpha/2) \exp (i/2\mathrm{Sin}\alpha ((x^2+\xi^2) \mathrm{Cos}\alpha - 2x\xi))$$

K_{α} (x,ξ) is the propagator of the non-stationary Schrodinger equation for a harmonic oscillator, which is well known in quantum mechanics (where $\alpha = \omega t$ relates to time t and classical frequency ω, and ξ is a position at the moment t). Changing gradually the angle α the fractional FT permits to look at the continuous transformation of an input function f(x) to its Fourier image $F_{\pi/2}$ (ξ) for $\alpha= \pi/2$, then to f(-x) for $\alpha= \pi$ and to $F_{\pi/2}$ ($-\xi$) for $\alpha= 3\pi/2$ Thus, the FrFT for $\alpha= \pi/2$ and $\alpha= -\pi/2$ reduces to ordinary and inverse Fourier transform.

For $\alpha = 0$ the transform kernel reduces to the identity operator:

$$[R^0 f(x)] (\xi) = F_0 (\xi) = \int \delta(\xi - x) f(x) dx f(x).$$

With respect to parameter α, the FrFT is continuous, periodic

$[R^{\alpha+2\pi n} f(x)] = [R^{\alpha} f(x)]$ with n an integerand additive $R^{\alpha+\beta} = R^{\alpha} + R^{\beta}$.

It is possible to recover the function f by means of the inversion formula:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_{\alpha}(\xi) \; \overline{K_{\alpha}}(x, \xi)\, d\xi \qquad (3.2)$$

where

$\overline{K_{\alpha}}(x, \xi) = (1/\mathrm{Sin}\alpha)(2\pi i \,\mathrm{Sin}\alpha)^{1/2} \exp(-i\alpha/2) \exp(-i/2\mathrm{Sin}\alpha\,((x^2+\xi^2)\,\mathrm{Cos}\alpha$

$- 2x\xi))$

We eventually have

$$f(x) = R^{-\alpha}[F_{\alpha}(\xi)](x) = \int_{-\infty}^{\infty} F_{\alpha}(\xi) \; K_{-\alpha}(x, \xi)\, d\xi$$

We also establish the following relationship of FrFT with that of Fourier transform:

$R^{\alpha}[f(x)] = \exp(iC_{2\alpha}\xi^2\mathrm{Cos}\,\alpha)\mathcal{F}\,[f(x)C_{1\alpha}\exp(iC_{2\alpha}x^2\mathrm{Cos}\,\alpha)]\,(2\,C_{2\alpha}\,\xi) \quad (3.3)$

where $C_{1\alpha} = (2\pi i\,\mathrm{Sin}\alpha)^{-1/2} \exp(i\alpha/2)$ and $C_{2\alpha} = 1/2\mathrm{Sin}\alpha$.

We note here that defining FrFT via this integrand at (3.1), we can say that FrFT exists for $f \in L^1(IR)$ (and hence in $L^2(IR)$) or when it is a generalized function[3].

Indeed, in that case the integrand is also in $L^1(IR)$ or in $L^2(IR)$) or is a generalized function. Thus the FrFT exists under exactly the same conditions as under which the Fourier transform exists and analogous results can be obtained.

4. The fractional Fourier transform of ultra-differentiable functions:

The fractional Fourier transform (fractional FT) R^α is an extension of the Ordinary Fourier transform and depends on a parameter α. For $\alpha = \pi/2$, fractional FT reduces to the ordinary Fourier transform; changing α from 0 to $\pi/2$, we get a continuous transformation of a function to its Fourier image. Thus, using analogy, we can state the Paley-Wiener theorem [5, p.82] for ultra-differentiable functions, as follows:

Theorem 4.1: Suppose that M_p satisfies (M.1) and that K is a compact convex set in IR^n. For x restricted to compact set K, an entire function $\phi^\wedge(\zeta)$ on C^n is the fractional FT for parameter α of an ultra-differentiable functions $\phi(x)$ in $E(M_p;K)$, i.e.

$$\phi^\wedge(\zeta) = R^\alpha[\phi(x)] = \int_{IR^n} \phi(x) K_\alpha(x, \zeta)\, dx \tag{4.1}$$

where the kernel

$K_\alpha(x, \zeta) = (2\pi i \, Sin\alpha)^{-1/2} \exp(i\alpha/2) \exp(i/2Sin\alpha((x^2+\zeta^2) Cos\alpha - 2x\zeta))$

if and only if there are constants A and C such that

$$|\phi\hat{}(\zeta)| \leq (1/(2\pi \operatorname{Sin} \alpha))^{1/2} C \exp\{-M(|\eta|/A) + H_K(\eta)\} \qquad (4.2)$$

where $H_K(\zeta) = \sup_{x \in K} (-\mathrm{Im}(x, \eta))$, $\eta = 2 C_{2\alpha} \zeta$ is the support function.

A subset B of $\mathcal{D}(M_p; K)$ is bounded if and only if we can choose constants A and C independent of $\phi \in B$ such that (4.2) holds.

We have the following definition:

Definition 4.1: The space of fractional FT of elements of ϕ in $\mathcal{D}(M_p, A; IR^n)$ is the space $Z(M_p, A; IR^n)$ of functions satisfying the inequality (4.2).

5. Ultra-distributions:

Suppose that $\{M_p\}$ satisfies (M.1) and (M.3) and that Ω is an open set in IR^n. We denote by $\mathcal{D}'(M_p; \Omega)$, the strong dual of $\mathcal{D}(M_p; \Omega)$ and call its elements as ultra-distributions of Roumieu type [6] or of class M_p.

The dual space $\mathcal{E}'(M_p; \Omega)$ of $\mathcal{E}(M_p; \Omega)$ is identified with the subspace composed of all $f \in \mathcal{D}'(M_p; \Omega)$ with compact support.

We note the following observations in respect of the space $\mathcal{D}'(M_p; \Omega)$ from [5, p.84].

$\mathcal{D}(M_p; \Omega)$ is a dense subspace of $\mathcal{D}(\Omega)$ and the injection is continuous. Therefore, $\mathcal{D}'(M_p; \Omega)$ contains the distribution space $\mathcal{D}'(\Omega)$ as a dense subspace.

The definitions of product, derivative, support and convolutions of the ultra-distributions are natural extensions of the corresponding ones for distributions.

6. The fractional FT of ultra-distributions:

We recall that the Fourier transform of $U \in D'(M_p; IR^n)$ is defined to be the element $V \in Z'(M_p, A; C^n)$ such that the generalized Parseval relation:

$$< V, \psi > = (2\pi)^n < U, \phi^\vee > \text{ holds,}$$

where $\phi \in D(M_p; IR^n)$ and $\psi = \phi^\wedge \in Z'(M_p, A; C^n)$

Suppose that f is an ultra-distribution with compact support in IR^n. For each $\zeta \in C^n$, α being parameter, the function $K_\alpha(x, \zeta)$ in x; x restricted to the compact subset K, belongs to $E(M_p; IR^n)$ and it can be seen that $K_\alpha(x, \zeta)$ depends on ζ holomorphically in the topology of $E(M_p;K)$,.

Hence $[R^\alpha f(x)](\zeta) = f^\wedge(\zeta) = F_\alpha(\zeta) = < f(x), K_\alpha(x, \zeta) >$,

defines an entire function on C^n, which we call the fractional FT of f.

Using the analogy of the Paley-Wiener theorem for ultra-distributions due to Komastu [4], we state the following theorem in the context of the fractional FT.

Theorem 6.1: Suppose M_p satisfies (M.1), (M.2)' and (M.3)' and that K is a compact set in IR^n. Then the following conditions are equivalent for an entire function $f^\wedge(\zeta)$:

(a) For the parameter α, x restricted to the compact subset K,

$f\,\hat{}(\zeta)$ (= $[R^{\alpha} f(x)](\zeta)$) is the fractional FT of an ultra-distribution

$f \in D'(M_p;K)$ with support in K;

(b) For each L>0 there is a constant C_α such that

$$|f\,\hat{}(\xi)| \le C_\alpha \exp M(|\xi|), \quad \xi \in IR^n \tag{6.1}$$

for each $\varepsilon > 0$ there is a constant $C_{\alpha\varepsilon}$ such that

$$|f\,\hat{}(\zeta)| \le C_{\alpha\varepsilon} \exp\{M(\varepsilon|\eta|) + H_k(\zeta)\}, \zeta \in C^n \tag{6.2}$$

where $H_K(\zeta) = \sup_{x \in K} (-\mathrm{Im}(x, \eta))$, $\eta = 2\,C_{2\alpha}\,\zeta$ is the support function

A subset B of $D'(M_p;K)$ is bounded in $D'(M_p;K)$ if and only if for each L>0 there exists a constant C independent of $f \in B$ such that (6.1) holds.

A sequence $\{f_j\} \in D'(M_p;K)$ converges if and only if for any L>0, $\exp\{-M(L|\xi|)\}\, f_j\hat{}\,(\xi)$ converges uniformly on IR^n.

7. Important connection between FrFT, Wigner Distribution (WD) and Ambiguity Function (AF):

The FrFT depends on a parameter α that is associated with the rotation angle in phase space. Thus, the FrFT produces the rotation of the AF and WD in the phase space. These functions can be reconstructed from the knowledge of only squared moduli of FrFT related to the intensity distribution.

7.1 FrFT, AF and WD:

We define the AF of a signal f(x) as,

$$A_f(x, \xi) = \int f(x'+x/2)\, f^*(x'-x/2)\, \exp(-i\pi\xi x')\, dx'$$

where f^* is a complex conjugate of f.

The ambiguity function is closely related to Wigner distribution. AF is like the WD except the integral is over the other variable.

Wigner distribution function is a time frequency representation that maps one dimensional time (or space in optics) –varying signal into two-dimensional signal representation of both time and frequency.

The WD function can be interpreted as a joint time-frequency power spectrum distribution function under the restriction of the uncertainty principle.

Wigner function belongs to a class of bilinear distribution and gives a representation of a function f(x) in a phase space:

$$W_f(x, k_x) = \int f(x+x'/2)\, f^*(x-x'/2)\, \exp(-i\pi x' k_x)\, dx'$$

$$= \int F_{\pi/2}(x+\nu/2)\, F^*_{\pi/2}(x-\nu/2)\, \exp(i\pi\nu k_x)\, d\nu$$

where $F_{\pi/2}(x) = \int f(x') \exp(ix' k_x)\, dx'$ (Fourier transform).

This equation shows a simple relationship between the FrFT and the WD of a given function through Fourier transform.

The inverse transform can be written as

$f(x) = (1/(2\pi f^*(0))) \int W_f (x/2, k_x) \exp(ix\, k_x)\, dk_x$

Note here that the frequency and time integrals of WD:

$\int W_f (x, k_x)$ and $\int W_f (x, k_x)\, dx$ correspond to the signal's instantaneous power $| f(x)|^2$ and its spectral energy density $| F_\alpha(x) |^2$ resp.

The important property of FrFT, which allows us to establish a connection between it and WD, and AF, is that a FrFT produces a rotation of these functions in the time-frequency plane:

$$\begin{array}{lcl} f(x) & & W_f (x, k_x) \text{ and } A_f (x, \xi) \\ \updownarrow \quad \text{FrFT} & & \updownarrow \quad \text{rotation of WD and AF} \\ [R^\alpha f(x)] (\xi) = F_\alpha (\xi) & & W_{F\alpha}(\xi, k_\xi) = W_f(\xi \cos \alpha - k_\xi \sin \alpha, \\ & & \qquad \xi \sin \alpha + k_\xi \cos \alpha) \\ & & A_{F\alpha}(x, \xi) = A_f(x \cos \alpha - \xi \sin \alpha, \\ & & \qquad x \sin \alpha + \xi \cos \alpha) \end{array}$$

Hence, we conclude that the WD of the FrFT for a parameter α of f(x) is the WF of f(x) rotated at the angle α in the phase space where the coordinates (ξ, k_ξ) in the rotated frame are related to (x, k_x) via matrix relation (1.1).

Note further that a similar relation holds for AF.

7.2 The Fractional Power Spectrum and Radon-Wigner Transformation (RWT)

If we introduce the fractional power spectrum $| F_\alpha (x) |^2$ as a squared modulus of the corresponding FrFT, we obtain that the fractional power spectrum are the projection of WD upon the direction at an angle α in the time-frequency plane:

$$| F_\alpha (\xi) |^2 = \int W_{F\alpha}(\xi, k_\xi)\, dk_\xi = \int W_f(\xi \cos\alpha - k_\xi \sin\alpha, \xi \sin\alpha + k_\xi \cos\alpha)\, dk_\xi,$$

and that they are related to the AF by a Fourier transform:

$$| F_\alpha (\xi) |^2 = \int A_f(k_\xi \sin\alpha, - k_\xi \cos\alpha)\ \exp(-i\pi\xi\, k_\xi)\, dk_\xi$$

The set of power spectra for the angle $\alpha \in [0, \pi)$ is called the Radon-Wigner transform because it defines the Radon transform of the WD.

Thus, RWT is the squared modulus of its FrFT: $| R^\alpha[f(x)] (\xi) |^2$ with $\alpha \in [0, \pi)$ and therefore corresponds to the intensity distributions at different planes of a first order system that provides an optical regularization of the FrFT

The WD can be obtained by applying an inverse Radon transform.

Note that the AF can also be reconstructed from RWT by a simple inverse Fourier transform. At the angle $\alpha = 2\pi$ the RWT of a complex field amplitude corresponds to its intensity distribution and at $\alpha = \pi/2$ to its Fourier power spectrum.

8. Applications of FrFT, AF, WD, RWT

1. FrFT has itself established as the powerful tool for analysis of time-varying signals for which the Fourier transform fails to work, signal processing, and

optics. It also leads to generalization of notion of space (or time) and frequency domain, which are central concepts in signal processing.

2. The intimate relationship between the AF in quasi-polar coordinates system and the Fractional Power Sptectra:

$$A_f(k_\xi \sin\alpha, -\ k_\xi \cos\alpha) = \int |\ F_\alpha\ (\xi)\ |^2\ \exp(-i\pi\xi\ k_\xi)\ dk_\xi$$

implies that the fractional power spectrum is the Fourier transform of the AF. This relationship is very important for the experimental determination of the AF in optics where the fractional power spectrum related to intensity distribution can be measured by a simple optical set up.

3. The FrFT depends on a parameter that is associated with a rotation angle in phase space. Thus, the FrFT produces a rotation of the AF and WD in the phase space. These functions can be reconstructed from the knowledge of only squared moduli of FrFT related to the intensity distribution.
4. The important property of the FrFT, which allows to establish connection between it and AF, WD and other members of the Cohen's class of time-frequency representations, is that a FrFT produces a rotation of these functions in the time-frequency plane. Moreover, the projection of WF upon the direction rotated at angle α is squared modulus of the corresponding FrFT, named the Radon-Wigner transform (RWT). Note that RWT relates

with the easily measured physical parameters like probability in quantum mechanics and intensity distribution in optics and signal processing.

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