On the Controlled Synchronization of Dynamical Networks with Non Identical Nodes

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***Abstract*— The problem of chaos synchronization on networks of structurally different dynamical systems is investigated. Synchronization of dynamical networks is usually defined in terms of identical accordance on the evolution of each individual node in the network. However, for a network consisting of strictly different nodes, this type of synchronization should be redefined. In this case, a generalized definition of synchronization can be considered, where the evolution of each node can be related to others in terms of a map. In this study, the case of systems that can be expressed in canonical form by an appropriately chosen coordinate transformation. In order to achieve generalized synchronization on a network of strictly different nodes, local robust controllers are designed which force the network to synchronize in terms of their transformed coordinates. The main results of this study are illustrated by numerical simulations of a network of well-known chaotic benchmark systems.**

I. Introduction

Recently, the synchronization of complex dynamical systems has received increasing attention from the scientific community. In particular, the synchronization of chaotic systems coupled in complex small-world and scale-free topologies [7], [8]. The main concern of these investigations has been oriented towards the understanding of the synchronization phenomenon in real-world complex networks, such as the Internet, bio-molecular networks and even social interactions. However, most of the researchers have focus their attention on networks consisting of identical n-dimensional dynamical systems with linear and diffusive couplings, where full knowledge about both the node dynamics and coupling structure are available [2], [5]. This is a highly unlikely situation under real-world conditions, where the nodes may not be identical and their connectivity can be partially or fully unknown, or even change over time. Yet, even in these situations, real-world systems present synchronization phenomena; like interdependence and collaboration, which can be thought in terms of generalized synchronization.

Synchronization on dynamical networks is usually defined in terms of identical dynamical evolution of the state variables at every node in the network. Thus is usually called complete or identical synchronization. For a network with non identical nodes this type of synchronization can't be expected. An alternative form of synchronization is considered, in which the relation between the nodes is defined in terms of a mapping between the state variables of the nodes in the network. In this way, generalized synchronization can be achieved [3]. Different types of generalized synchronization can be defined, depending on how the state space of one node is mapped to the others. The simplest form of generalized synchronization is define the relation between nodes by way of a coordinate transformation, for example a change of coordinates defined by a feedback linealization [4].

Synchronization is a phenomenon that can occur spontaneously. But, in certain circumstances it may be necessary to add interconnections or controllers to the system in order to achieve or improve the characteristics of the synchronization. In this contribution, the latter case is considered. The proposed approach consists on designing a robust controller such that generalized synchronization is achieved. This type of network synchronization is call controlled synchronization [1].

The remainder of the paper is organized as follows. On Section 2, the synchronization problem for a network of non identical dynamical nodes is stated. In Section 3, the main result of this contribution is presented. The numerical simulations, presented in Section 4, are used to illustrate the effectiveness of the proposed approach. In closing, some comments and conclusions are presented in Section 5.

II. Problem Description

Consider a network of N nodes, with each one being a dynamical system described by

$$\dot{x}_i = f_i(x_i) + u_i \tag{1}$$

where $x_i = [x_{i1}, x_{i2}, ..., x_{in}]^\top \in \mathbf{R}^n$ are the state variables of the ith node; $f_i : \mathbf{R}^n \to \mathbf{R}^n$ are known nonlinear functions describing the dynamics of node i; and $u_i \in \mathbf{R}^n$ is a local controller to be designed. Then, the state equations of the entire network are given by

$$\dot{x}_i = f_i(x_i) + \gamma \sum_{j=1}^{N} c_{ij}\Gamma x_j + u_i \tag{2}$$

where $\gamma > 0$ is a fixed uniform coupling strength, and $C = \{c_{ij}\} \in \mathbf{R}^{N\times N}$ is a zero-one constant matrix describing the connection structure of the network, if there is a connection between node i and j, then $c_{ij} = c_{ji} = 1$, otherwise $c_{ij} = c_{ji} = 0$ ($i \neq j$). The diagonal elements of C satisfy the diffusive coupling restrictions ($c_{ii} = -\sum_{j=1,j\neq i}^{N} c_{ij}$, for $\forall i$). The inner coupling matrix $\Gamma \in \mathbf{R}^{n\times n}$ is a zero-one constant matrix describing which state variables are connected from node to node.

Let all the nodes in the network be n-dimensional dynamical systems and assume there is a coordinate transformation

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$\mathcal{T}_i$, such that (1) can be rewritten as:

$$\dot{z}_i = A_i z_i + B\psi_i + \nu_i \tag{3}$$

where $z_i = \mathcal{T}_i x_i$ are the transformed state variables of the i-th node; the constant matrices A_i and B have the controller-type canonical form; with ψ_i a nonlinear function, possibly a linearizing feedback controller; and ν_i the local controller u_i express in terms of the transformed coordinates. The expression (3) is called the normal form for the affine system (1), with the nodes so described, the state equations for the entire network (2) become

$$\dot{z}_i = A_i z_i + B\psi_i + \gamma \sum_{j=1}^{N} c_{ij}\Gamma z_j + \nu_i \tag{4}$$

for $i = 1, 2, ..., N$.

Note that if the network is connected such that there are no isolated clusters, the coupling matrix C, will be irreducible and symmetric ($C = C^\top$), with its eigenvalues ordered as [8]

$$0 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq ... \geq \lambda_N \tag{5}$$

A dynamical network is said to be identically synchronized if the state solutions of its nodes satisfy

$$\lim_{t\to\infty} \|x_i - x_j\| = 0$$

for $i, j = 1, 2, ..., N$. Then, the network in (2) is synchronized in a generalized sense with respect to the coordinate transformations $\phi_{ij} = \mathcal{T}_i^{-1}\mathcal{T}_j$, if for $i, j = 1, 2, ..., N$ the following condition is satisfied:

$$\lim_{t\to\infty} \|z_i - z_j\| = 0 \tag{6}$$

Suppose the control objective is to synchronize the network, in the generalized sense of (6), to a reference node:

$$\dot{z}_r = A_r z_r + B\psi_r \tag{7}$$

By defining the synchronization error as, $e_i = z_i - z_r$, from (4) and (7), the error dynamics are found to be

$$\dot{e}_i = \mathcal{A}_i + \gamma \sum_{j=1}^{N} c_{ij}\Gamma e_j + \nu_i \tag{8}$$

for $i = 1, 2, ..., N$, where $\mathcal{A}_i = A_i z_i + B\psi_i - A_r z_r - B\psi_r$, which is a measure of the difference between the reference node and the current node. Assuming that the trajectories of both nodes are bounded, it follows that their difference is also bounded. Then, the following inequalities can be defined

$$\|\mathcal{A}_i\| \leq \beta_i \tag{9}$$

for $i = 1, 2, ..., N$ where β_i are nonnegative constants.

In order to achieve the generalized synchronization of the network (2) in terms of criterion (6), the local controllers ν_i, have to be designed such that e_i becomes asymptotically stable about its zero fixed point.

III. Generalized Synchronization Design

The main result of this contribution can be stated as follows:

Theorem 1: The dynamical network on (2) will achieve a generalized synchronization in terms of (6), if the local controllers ν_i are constructed as

$$\nu_i = -\gamma k \Gamma e_i - \delta sgn(e_i) \tag{10}$$

for $i = 1, 2, ..., N$, where $sgn(e_i) = [sgn(e_{i1}), sgn(e_{i2}), ..., sgn(e_{in})]^\top$, with $sgn(\cdot)$ the signum function, and furthermore, the smooth $k > 0$ and discontinuous $\delta > 0$ controller gains are designed to satisfy the bounds

$$k > \lambda_i + \frac{1}{\gamma} \tag{11}$$

$$\delta > \frac{\sum_{j=1}^{N} \beta_j \|\omega_{ji}\|}{\sum_{j=1}^{N} \|\omega_{ji}\|} \tag{12}$$

for any i, where λ_i is the i-th eigenvalue of the matrix C, and $\omega_i \in \mathbf{R}^N$ its associated eigenvector.

`Proof:` Applying the local controllers in (10) to the error dynamics (8), and rewriting in vector form one gets:

$$\dot{\bar{e}} = \bar{\mathcal{A}} + \gamma\Gamma\bar{e}(C^\top - K) - \delta sgn(\bar{e}) \tag{13}$$

where $\bar{e} = [e_1, e_2, ..., e_N] \in \mathbf{R}^{n\times N}$, $\bar{\mathcal{A}}(\cdot) = [\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_N] \in \mathbf{R}^{n\times N}$, $K = diag(k, ..., k) \in \mathbf{R}^{N\times N}$, and $sgn(\bar{e}) = [sgn(e_1), ..., sgn(e_N)] \in \mathbf{R}^{n\times N}$.

Given that the connectivity matrix satisfies (5), there are two matrices, $\Omega = (\omega_1, \omega_2, ..., \omega_N) \in \mathbf{R}^{N\times N}$ and $\Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_N) \in \mathbf{R}^{N\times N}$ such that:

$$C = \Omega^\top \Lambda \Omega$$

where λ_i and ω_i are the i-th eigenvalue and associated eigenvector of C, respectively. With $\Omega^\top\Omega = I_N$, the N-dimensional identity matrix.

Using a change of variables $\bar{\eta} = \bar{e}\,\Omega^\top$, the error dynamics become

$$\dot{\bar{\eta}} = [\bar{\mathcal{A}} - \delta sgn(\bar{\eta}\Omega)]\Omega^\top + \gamma\Gamma\bar{\eta}\,(\Lambda - K)$$

where $\bar{\eta} = (\eta_1, \eta_2, ..., \eta_N)$, with $\eta_i = \bar{e}\,\omega_i^* \in \mathbf{R}^n$ and $\omega_i^* = [\omega_{1i}, \omega_{2i}, ..., \omega_{Ni}]^\top \in \mathbf{R}^{N\times 1}$; or equivalently,

$$\dot{\eta}_i = (\bar{\mathcal{A}} - \delta sgn(\bar{\eta}\Omega))\omega_i^* + \gamma\Gamma\eta_i\,(\lambda_i - k) \tag{14}$$

for $i = 1, 2, ..., N$.

The stability of the error dynamics (13) around the zero fixed point can be determine using the Lyapunov candidate function:

$$V = \frac{1}{2}\sum_{i=1}^{N} \eta_i^\top \eta_i,$$

The time derivative of V along the trajectories of the error dynamics in (14) is given by

$$\dot{V} = \sum_{i=1}^{N} \begin{pmatrix} \eta_i^\top \bar{\mathcal{A}}\omega_i^* - \eta_i^\top \delta sgn(\bar{\eta}\Omega)\omega_i^* \\ +\eta_i^\top \gamma(\lambda_i - k)\Gamma\eta_i \end{pmatrix}$$
$$\leq \sum_{i=1}^{N} \begin{pmatrix} \|\eta_i\|^\top \|\bar{\mathcal{A}}\omega_i^*\| \\ -\delta\|\eta_i\|^\top \|sgn(\bar{\eta}\Omega)\omega_i^*\| \\ +\gamma(\lambda_i - k)\|\eta_i\|^\top \Gamma \|\eta_i\| \end{pmatrix}$$

Considering the bounds of each term of $\dot{V}$. From (9) one has the bound of the first term:

$$\|\bar{\mathcal{A}}\omega_i^*\| \leq \|\sum_{j=1}^{N} \mathcal{A}_j \omega_{ji}\| \leq \sum_{j=1}^{N} \beta_j \|\omega_{ji}\|$$

The bound for the second term is given by:

$$\|sgn(\bar{\eta}\Omega)\omega_i^*\| \leq \sum_{j=1}^{N} \|sgn(\bar{\eta}\omega_j)\|\|\omega_{ji}\| \leq \sum_{j=1}^{N} \|\omega_{ji}\|$$

The third term is quadratic and will be negative if the coefficient is negative ($\gamma(\lambda_i - k) < 0$) for any i. The bound on the third term can be expressed as

$$\gamma(\lambda_i - k)\|\eta_i\|^\top \Gamma \|\eta_i\| \leq -\|\eta_i\|^\top \|\eta_i\|$$

from this observation one get condition (11) in *Theorem 1* by algebraic manipulation.

From the above results the time derivative V is bounded by

$$\dot{V} \leq \sum_{i=1}^{N} \begin{pmatrix} \|\eta_i\|^\top \begin{pmatrix} \sum_{j=1}^{N} \beta_j \|\omega_{ji}\| \\ -\delta \sum_{j=1}^{N} \|\omega_{ji}\| \end{pmatrix} \\ -\|\eta_i\|^\top \|\eta_i\| \end{pmatrix}$$

For $\dot{V}$ to be negative, the discontinuous gain must satisfy

$$\delta > \frac{\sum_{j=1}^{N} \beta_j \|\omega_{ji}\|}{\sum_{j=1}^{N} \|\omega_{ji}\|}$$

for any i, resulting on the condition (12) in *Theorem 1*. Then, the error dynamics in (14) are globally uniformly asymptotically stable about the zero fixed point ($\bar{\eta} = 0$), which implies that

$$\lim_{t\to\infty} \bar{e} = \lim_{t\to\infty} \{z_1 - z_r, ..., z_N - z_r\} = 0$$

In consequence, the dynamical network (2) under the controller (10), achieves generalized synchronization in the sense of (6).

Q.E.D.

IV. Numerical Experiments

Example 1: To illustrate the main result of this contribution, consider a network constructed with two different types of Sprott circuits [6], which are defined by:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -0.6x_3 - x_2 + G_i(x_1) + u \end{aligned} \tag{15}$$

where the term $G_i(x_1)$ can take one of the following forms:

$$G_1(x_1) = |x_1| - 2 \tag{16}$$
$$G_2(x_1) = -1.2x_1 + 2sgn(x_1) \tag{17}$$

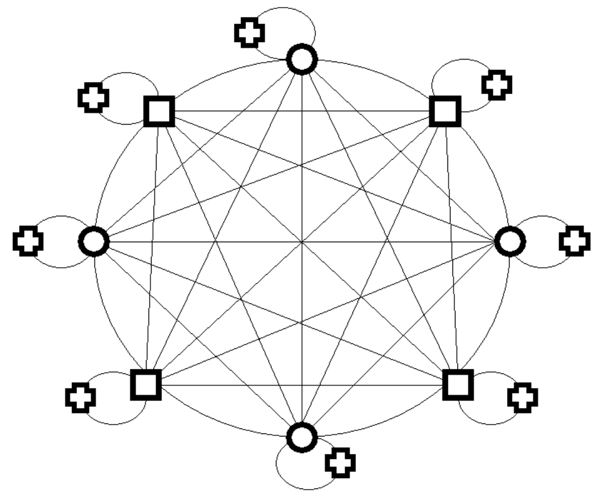

Fig. 1. Network of non-identical nodes. "circles" are Sprott circuits with $G_1(x_1)$; "squares" are Sprott circuits with $G_2(x_1)$ and "crosses" are local controllers appropriately designed

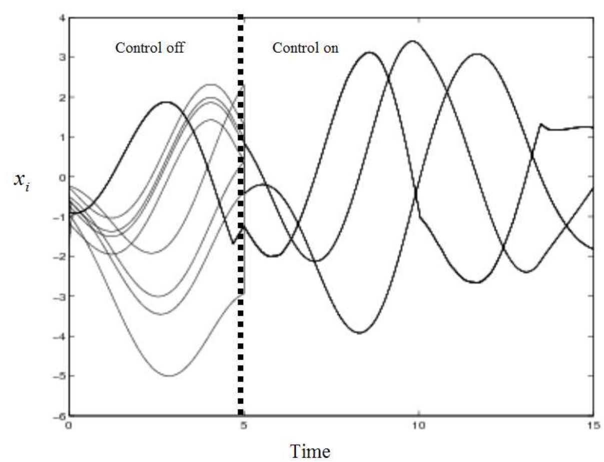

Fig. 2. Synchronization of eight Sprott circuits of two different structures with the local controllers activated at $t = 5$

A network is constructed by coupling together eight Sprott circuits in a fully connected structure as shown in Figure 1; where half of the nodes are Sproot circuits with $G_1(x_1)$ (represented by circles) and the other half are Sproot circuits with $G_2(x_1)$ (represented by squares), with a local controller in each node (represented by crosses). To synchronize this network the local controller are designed according to the specification of *Theorem 1*. In Figure 2 the dynamical evolution of the entire network when the controllers are activated at $t = 5$ are presented. As can be seen the nodes become synchronized to the reference node, a Sproot circuit with $G_1(x_1)$ in this case.

Example 2: To further illustrate the proposed generalized synchronization the network shown in Figure 3 is constructed coupling together Sprott circuits with $G_1(x_1)$ ("circles"), Sprott circuits with $G_2(x_1)$ ("squares") and Rössler circuits

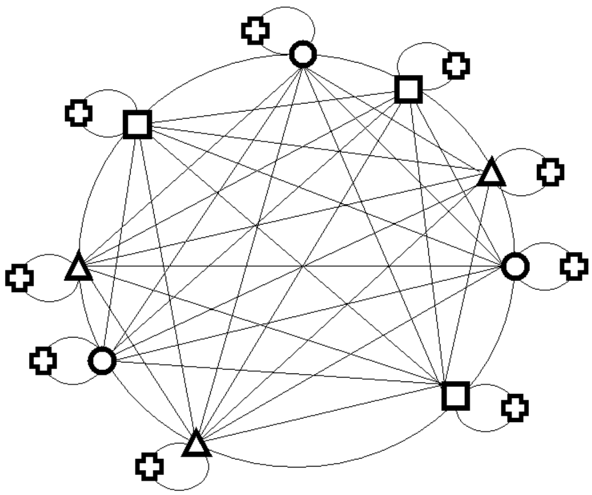

Fig. 3. Network of non-identical nodes. “circles” are Sprott circuits with $G_1(x_1)$; “squares” are Sprott circuits with $G_2(x_1)$; “triangles” are Rössler circuits; and “crosses” are local controllers appropriately designed

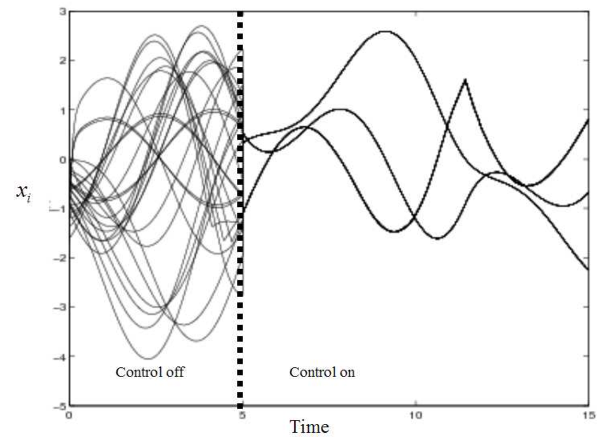

Fig. 4. Synchronization of six Sproot circuits of two different structures and three Rössler circuits in their transformed coordinates, with the local controllers activated at $t = 5$

(“triagles”), which are given by

$$\begin{array}{rcl} \dot{x}_1 &=& -x_2 - x_3 \\ \dot{x}_2 &=& x_1 + ax_2 \\ \dot{x}_3 &=& x_3(x_1 - b) + a + u \end{array} \tag{18}$$

For such a network to achieve generalized synchronization a coordinate transformation is used to take the Rössler system into its normal form. Assuming that the output of (18) is $y = x_2$, the following coordinate transformation takes the Rössler system to the canonical form (3) in the transform variables $z = \phi(x)$ [3]:

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 + ax_2 \\ ax_1 + (a^2 - 1)x_2 - x_3 \end{pmatrix} \tag{19}$$

This coodinate transformation is a diffemorphism and its inverse is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} az_1 + z_2 \\ z_1 \\ (2a^2 - 1)z_1 + az_2 - z_3 \end{pmatrix} \tag{20}$$

The dynamical evolution of the globally coupled network of three Sproot circuits with $G_1(x_1)$, three with $G_2(x_1)$ and three Rössler circuits in their normal coordinates is shown in Figure 4. The trajectories of the Rössler and Sprott circuits in their original coordinates are shown in Figure 5. The Rössler circuit (18) is synchronized in the generalized sense of the composition of the coordinate transformations (19) and (20) to the evolution of the reference Sprott circuit (15)-(16).

V. Comments and Discussion

Different approaches can be consider for the synchronization of complex dynamical networks. In this contribution, the nodes are considered to be non identical, but with the restriction of having a coordinate transformation that can make them relatively similar in the transform coordinates,

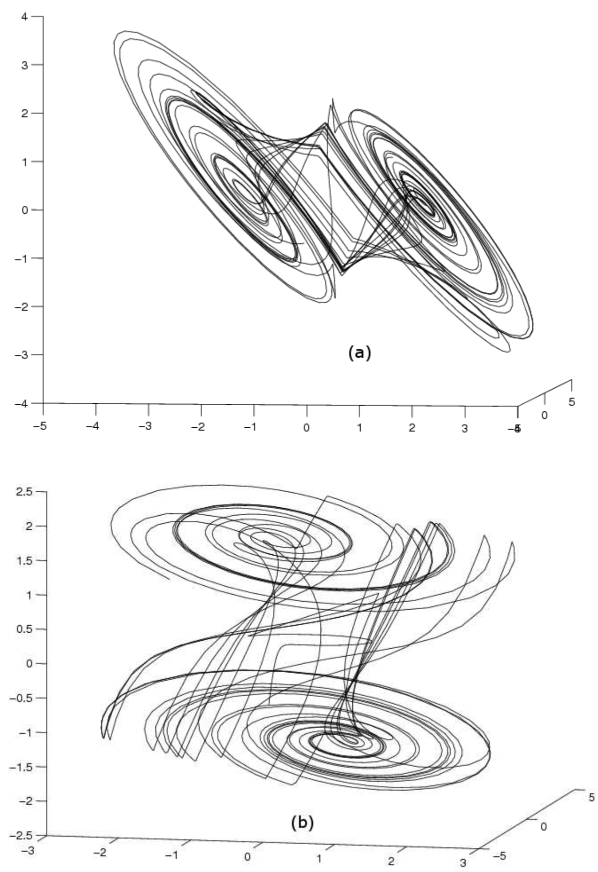

Fig. 5. Trajectories of the (a) Sproot circuit and (b) Rössler circuits in their original coordinates

that is, they have the same dimension and have a controller type canonical form. Then, the difference between them can be seen as a perturbation, which is eliminated by a robust controller properly designed. With these controller the nodes of the network can be made to identically synchronize on their transform coordinates, which in their original coordinates becomes a form of generalized synchronization. An obvious limitation of the proposed method is the fact the requires the same number of controllers than nodes, in a work to be reported elsewhere this robust controlled synchronization design is combined with a pinning control strategy, providing a reduction on the number of nodes where controlled action in taken. Yet another aspect of interest to considered as future work is determining conditions for the existence of an appropriate coordinate transformation such that this result is applicable for a more general class of oscillators.

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