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A DISCRETE-TIME HYBRID LURIE TYPE SYSTEM WITH STRANGE HYPERBOLIC NONSTATIONARY ATTRACTOR

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Abstract

In this paper we present sufficient conditions for existence of strange hyperbolic nonstationary attractor of hybrid continuous piecewise smooth discrete-time dynamical system.

Key words

Piecewise smooth maps, hyperbolic attractors, hybrid systems.

1 Introduction

In the field of dynamical chaos hyperbolic strange attractors play a central role as one of the basic units linking dynamical and ergodic theories. Hyperbolic strange attractors generate random (in terms of mixing property) stationary (in terms of Sinai-Bowen-Ruelle measure (SBR-measure)) processes [Anosov and Sinai, 1967; Bowen,1977; Katok and Hasselblatt, 1995; Afraimovich, Chernov and Sataev, 1995]. In the case of ODEs there are several examples showing the possible existence of a hyperbolic attractor [Belykh, Belykh and Mosekilde, 2005; Kuznetsov, 2005; Kuznetsov and Pikovsky 2007]. Unfortunately all these examples do not lend itself so far to mathematical verification. Contrary, in the case of maps (discrete-time dynamical systems) there are several well defined examples (Smale-Williams attractor, Lozi map, Belykh map, etc) [Katok and Hasselblatt, 1995; Lozi, 1978; Belykh, 1995; Belykh, Komrakov and Ukrainsky, 2002] for which the hyperbolicity, the existence of invariant measure and mixing property are proved [Pesin 1992; Sataev, 1999; Schmeling and Troubetzkoy, 1998]. In particular, the first example of hyperbolic attractor in Lurie discrete-time system with continuous nonlinearity, having a bounded away from zero discontinuous derivative, was presented in [Belykh, Komrakov and Ukrainsky, 2002]. This system serves as a model of electromechanical control systems. In the map representation it takes the form

$$\overline{u} = Au + p\varphi(x), \quad x = c^T u, \tag{1}$$

where $u = u(i)$, $\overline{u} = u(i+1)$, $u \in \mathbb{R}^m$; A is constant $m \times m$-matrix; p, c are constant $m \times 1$-vectors; "T" is transpose and $\varphi : \mathbb{R}^1 \to \mathbb{R}^1$ is a continuous piecewise smooth function, $|\varphi'(x)| > K$, $x \in \mathbb{R}^1$.

For the parameter domain [Belykh, Komrakov and Ukrainsky, 2002] where the map (1) is hyperbolic from papers [Sataev, 1999; Schmeling, 1998] it follows that the hyperbolic attractor of (1) is stationary in the sense of SBR-measure.

In the present paper we consider the hybrid system of the form

$$\begin{cases} u(i+1) = Au(i)u + p\varphi(x(i), z(i)) \\ z(i+1) = \psi(x(i), z(i)), \quad x = c^T u, \end{cases} \tag{2}$$

where the integer $z \in \mathbb{Z}$, the function $\psi : \mathbb{R}^1 \times \mathbb{Z}_N \to \mathbb{Z}_N$ is bounded: $[1, N] = \mathbb{Z}_N$.

Our main purpose is to obtain sufficient conditions such that the hybrid map (2) has a strange hyperbolic nonstationary attractor.

Note, that in the case of a periodic nonautonomous system (2) when $\psi = z(\mod N)$, the system (2) is a composition of N maps (1) with $\varphi(x, i)$, $i = 0, 1, ..., N-1$ standing for $\varphi(x)$, and the hyperbolicity of each such map from N sequential maps does not imply that the map (2) is also hyperbolic. We consider an arbitrary (even random) sequence of $z(i)$ generated by the second equation in (2). The proof of hyperbolicity is based on a comparison principle for multidimensional maps, and the construction of cones that are invariant with respect to a linearization of the map (2), and are independent of the phase coordinates.

2 Reduction to a normal form

We introduce the normal form of the Lurie system as the following map F

$$\begin{cases} \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{1} \\ 0 & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix} g(x,z) \\ \overline{z} = \psi(x,z), \end{cases} \tag{3}$$

where the overbar denotes the forward shift in time, $g(x,z) \equiv kx + \varphi(x,z)$; B is $n \times n$-matrix ($n = m - 1$), $y = column(y_1, y_2, \dots, y_n)$, $\mathbf{1} = (1, 1, \dots, 1)$, $b = column(b_1, b_2, \dots, b_n)$ is n-vector of parameters, k and a are scalar parameters. Denoting $v - (x, y^T)^T$, and introducing the transformation $v = Su$, where S is a nonsingular $m \times m$-matrix, we obtain that the system (2) takes the form (3) as long as the following system of equations has a solution:

$$\begin{cases} SAS^{-1} = \begin{pmatrix} 1 - ka & \mathbf{1} \\ -kb & B \end{pmatrix} \\ c^T S^{-1} = e^T \\ Sp = \begin{pmatrix} -a \\ -b \end{pmatrix}, \end{cases} \tag{4}$$

where $e^T = (1, 0, \dots, 0)$. Note, that the following system of equations

$$\begin{cases} kp + (E - SAS^{-1})e = 0 \\ c^T S^{-1} = e^T \\ e^T SAS^{-1} = \begin{pmatrix} 1 - ka & \mathbf{1} \end{pmatrix} \end{cases} \tag{5}$$

takes a form of necessary conditions for resolving the system (4). We assume that the system (5) can be resolved with respect to the parameter k and matrix S (an example is shown in [Belykh, Komrakov and Ukrainsky, 2002]). Hence, the system (1) is reduced to the map (3) which we consider below.

3 Existence of invariant domain

Consider an arbitrary map $\Phi : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ of the form $(x, y) \to (P(x, y), Q(x, y))$, and reduced map $\Phi_0 : (x, y) \to (P(x, y), y)$, where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$ and $y = const$ for Φ_0. Our problem is to derive the conditions for the map Φ as well as for the boundaries of a domain D such that 1) $\Phi D \subset D$; 2) $D = D_x \times D_y$ (direct product).

Comparison principle. Assume that there exist some compacts D_x and D_y such that:

1. $\Phi_0 D_x \subset D_x$ for any $y = const$ from compact D_y
2. $Q(x, y) \in D_y$ for any $x \in D_x$ and $y \in D_y$.

Then D is invariant under the map Φ.

From this principle it follows that the map has an attractor $A = \Phi A, A \subset D$.

Remark. The variables separation in this obvious principle is immediately directed to the finding of the compacts D_x and D_y for certain maps. The map, which we consider in the paper is the case.

As the main example we consider the following class of nonlinear functions. For a natural $n > 1$, from the interval $[c, d]$ consider two sets of real numbers $S_a = (a_0 = c < a_1 < a_2 < \dots < a_n = d)$ and $S_b = (b_0 = 0, b_1, \dots, b_n = 0) | b_{i-1} b_i < 0$, $i = \overline{2, n-1}$, and consider a set of functions $S_\eta = (\eta_1(\xi), \eta_2(\xi), \dots, \eta_n(\xi))$, where $\eta_1(\xi)$ is given in the interval $(-\infty, a_1]$, $\eta_n(\xi)$ — in the interval $[0, \infty)$, and for $i = \overline{2, n-1}$ the functions $\eta_i(\xi)$ are given in the intervals $[0, a_i - a_{i-1}]$. Assume that each function from S_η is continuous, smooth; $\eta_i(0) = 0$, $\eta_i'(\xi) > k > 0$. Introduce the following function $\eta(x, n) = b_{i-1} + \dfrac{b_i - b_{i-1}}{\eta_i(a_i - a_{i-1})} \eta_i(x - a_{i-1})$, where index $i = 1$ for $x \in (-\infty, a_1]$, index $i = n$ for $x \in [a_n, \infty)$ and for $x \in (a_{i'-1}, a_{i'}]$ index $i = i'$, $i = \overline{2, n-1}$. This function has singularities at critical points a_i and $f(a_i) = b_i$ for $i = \overline{1, n-1}$. An example of such function for $n = 5$, $a_i = c + (i-1)\dfrac{d-c}{n}$, $b_i = (-1)^i b$, $\eta_i(x) = x$ for all i is depicted in Fig.1.

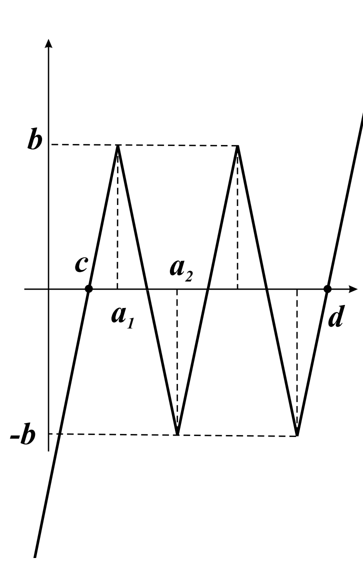

Figure 1. Example of $\eta(x, 4)$ with 4 critical points

Now the map (3) is defined with the function $g(x, z) = \eta(x, z)$ and an arbitrary function $\psi(x, z)$. We consider a set of functions $\Re(h)$: $f(x, z) = x - ag(x, z)$, such that:

1. For even z $M = \max\limits_{x \in [c,d]} f(x, z) < d$; $m = \min\limits_{x \in [c,d]} f(x, z) > c$. For odd z $f(M, z) > c$, $f(m, z) > c$
2. $\max\limits_{x \in [c,d]} f'(x, z) > h$.

Figure 2 illustrates functions $f(x,z) \in \Re(h)$ for even and odd z.

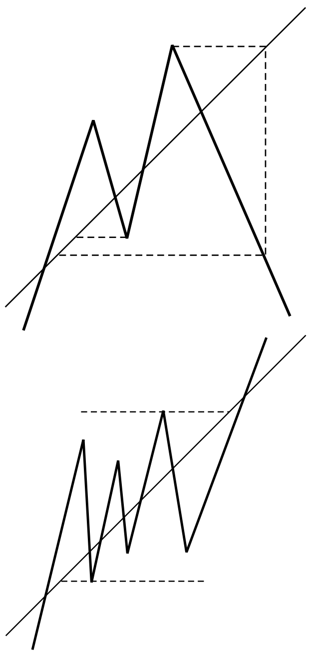

Figure 2. An example $f(x,z) \in \Re(h)$ for even and odd z.

First we consider reduced system (3), that is a one-parameter z family of maps F_1:

$$\begin{cases} \overline{x} = x + \mathbf{1}y - ag(x,z) \\ \overline{y} = By - bg(x,z), \end{cases} \tag{6}$$

where $z \in \mathbb{Z}_N$ is a constant parameter. Under a nonsingular linear transformation the map (6) can be reduced to the form

$$\begin{cases} \overline{x} = x + \mathbf{1}y - ag(x,z) \\ \overline{y}_i = \lambda_i (y_i - b_i g(x,z)), \end{cases} \tag{7}$$

where λ_i denotes either a real eigenvalue of matrix B or $\alpha \pm \beta$ for complex eigenvalues $\alpha \pm \beta i$ of matrix B. Note that in the case of multiple eigenvalues of matrix B some of the such values of λ_i must be increased by some quantity ε from a Jordan form of the matrix B.

Applying the comparison principle for the map F_1 as the auxiliary map Φ_0 we consider the map F_0 in the form $(x,y) \rightarrow (x - ag(x,z) + Y, y)$, where $Y = \mathbf{1}y$ a parameter. This map is a two parameter $y = const, z = const$ family of one dimensional maps $f_1 : \overline{x} = x - ag(x,z) + Y$. As for each $f_1 \in \Re(h)$ the first condition of the comparison principle is fulfilled, so the interval $[c,d] = D_x$. This condition is illustrated in Fig. 2 for our main example. It is easy to verify that there exists an interval $[y^-, y^+] = D_y$, satisfying the condition 2 of the comparison principle. In fact this condition becomes valid for small enough eigenvalues of the matrix B.

These conditions provide a simple technical rule for the system (3): from $x \in [c,d]$ and $Y \in [y^-, y^+]$ it follows that $\overline{x} \in [c,d]$. Under the condition $F_1 D \subset D$ the next theorem holds.

Theorem 1. *There exists a number λ_0, such that for any $\lambda^+ < \lambda_0$ ($\lambda^+ \triangleq \max_i \{|\lambda_i|\}$) and any $z \in \mathbb{Z}_N$ the map F_1 has an invariant domain $D = \{(x,y) : c < x < d, y_i^- < y_i < y_i^+, i = \overline{1,n}\}$ and, therefore this map has an attractor $A \subset D$.*

4 Conditions of hyperbolicity

Assume, that $g(x,z)$ is a continuous piecewise smooth function, i.e. smooth in each interval of monotonicity. Denote $h \triangleq \inf_{x \in A, z \in \mathbb{Z}_N} |g'_x(x,z)|$.

Definition 1. *We call a cone in R^{n+1} with one-dimensional axes being a set of vectors of the form $K_1 = \{(\xi,\eta) \in R^{n+1} : \xi \in R^1, \eta \triangleq column(\eta_1, \eta_2, ...\eta_n) \in R^n, \frac{\eta_i}{\xi} = \alpha_i, \alpha_i \in (\alpha_i^-, \alpha_i^+), i = \overline{1,n}\}$, and we call a cone with n-dimensional axial space $K_n = \{(\xi,\eta) \in R^{n+1} : \xi + \sum_i \beta_i \eta_i = 0, \beta_i \in (\beta_i^-, \beta_i^+), i = \overline{1,n}\}$.*

The cone K_1 is a set of vectors being parallel to those vectors, which have one unit coordinate and all the others are bounded, the cone K_n is a set of vectors from n-dimensional plane and its vectors $column(1, \beta_1, ..., \beta_n)$ are similar to those as for K_1 (see figure 3).

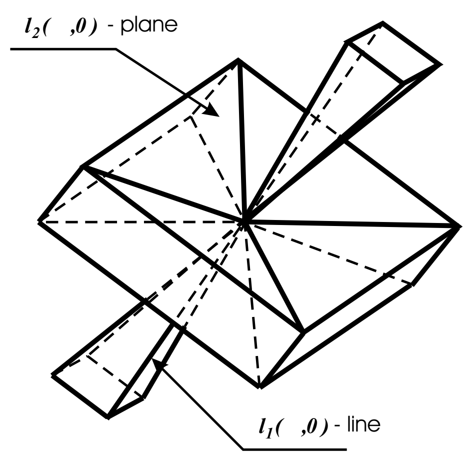

Figure 3. Cones K_1 and K_2.

For each $z = const$ consider the linearization of the map F_1 in a point (x,y) of the phase space resulting in the linear map T of the form

$$\begin{cases} \overline{\xi} = (1 - s(x,z))\xi + \sum \eta_i \\ \overline{\eta}_i = -t_i(x,z)\xi + \lambda_i \eta_i \quad i = \overline{1,n}, \end{cases} \tag{8}$$

where $s(x,z) \triangleq a g'_x(x,z), \quad t_i(x,z) \triangleq \lambda_i b_i g'_x(x,z)$. We consider the cones in the space (ξ,η), which are independent of the points (x,y) in the phase space of F_1.

Introduce two families of linear manifolds

$$l_1(\alpha,\sigma) \triangleq \left\{ \begin{matrix} \eta_i - \alpha_i\xi = \sigma_i \\ i = \overline{1,n}, \end{matrix} : \begin{matrix} \sigma_i \in R^1 \\ \alpha_i \in (\alpha_i^-, \alpha_i^+) \end{matrix} \right\},$$

$l_2(\beta,\rho) \triangleq \{\xi + \sum_i \beta_i\eta_i = \rho : \rho \in R^1, \beta_i \in (\beta_i^-, \beta_i^+)\}$.

The images of these manifolds Tl_1 and Tl_2 are also linear manifolds with new values $\overline{\alpha}_i$, $\overline{\beta}_i$, $\overline{\sigma}_i$ and $\overline{\rho}$.

Definition 2. *An attractor A is called a hyperbolic attractor, if there exist cones K_1^u and K_n^s, such that $\forall (x,y) \in A$ the following conditions hold:*

h1. $clos(K_1^u) \cap clos(K_n^s) = \{0\}$;
h2. $clos(TK_1^u) \subset K_1^u, \quad clos(T^{-1}K_n^s) \subset K_n^s$;
h3. There exist a constant $l, 0 < l < 1$ such, that
a) If $(\xi,\eta) \in K_1^u$ and $\beta_i \in (\beta_i^-, \beta_i^+)$ then $\overline{\beta_i} \in (\beta_i^-, \beta_i^+)$ and $|\overline{\rho}| > l^{-1}|\rho|$,
b) If $(\xi,\eta) \in K_n^s$ and $\alpha_i \in (\alpha_i^-, \alpha_i^+)$ then $\overline{\alpha_i} \in (\alpha_i^-, \alpha_i^+)$ and $|\overline{\sigma_i}| < l|\sigma_i|$.

We pay special attention at the condition h3, which implies that the image of a line and pre-image of a plane tend to the origin due to decrease of values ρ and $\overline{\sigma_i}$ in a geometrical progression with the factor l. One iterate of $l_1(\alpha,\sigma)$ and $l_2(\beta,\rho)$ in the cones is schematically shown in the Fig. 4.

Under the above conditions on function $g(x,z)$ the following theorem holds.

Theorem 2. *There exist numbers $h_0 > 1$ and λ_0, $0 < \lambda_0 < 1$, such that the attractor A of the map F_1 is a hyperbolic attractor for $h > h_0$, $\lambda^+ < \lambda_0$ and any $z \in \mathbb{Z}_N$.*

Proof (in the case of K_1^u) is based on:

1. Invariance:
   (a) We prove, that the images of (α_i, β_i, σ_i, ρ) are defined by the next formulas $\beta_i = \dfrac{1 + \overline{\beta}_i\lambda_i}{1 - s - \sum \overline{\beta}_j t_j}$, $\overline{\alpha_i} = \dfrac{-t_i\lambda_i + \lambda_i\alpha_i}{1 - s + \sum \alpha_j}$, $\overline{\rho} = \rho(1 - s - \sum \overline{\beta}_i t_i)$, $\overline{\sigma}_i = \sigma_i\lambda_i - \overline{\alpha}_i \sum \sigma_j$. We construct a map for the value $\zeta \triangleq \sum \alpha_i$ generated by the map T.The inequality for the image $\bar{\zeta}$ holds: $f_2(\zeta) < \bar{\zeta} < f_1(\zeta)$ where the functions $f_1(\zeta) = \dfrac{\lambda h + \lambda^-\zeta}{1 + ah + \zeta}$ and $f_2(\zeta) = \dfrac{-\lambda h + \lambda^-\zeta}{1 - ah + \zeta}$ are comparison functions. The graphs of the functions $\bar{\zeta} = f_1(\zeta)$ and $\bar{\zeta} = f_2(\zeta)$ are shown in the figure 5. This figure illustrate the existing invariant interval (ζ^-, ζ^+).
   (b) The existence of an invariant interval implies that there exists a set of intervals for each coordinate α_i. The latter finishes the proof of the existence and invariance of the cone (K_1^u).

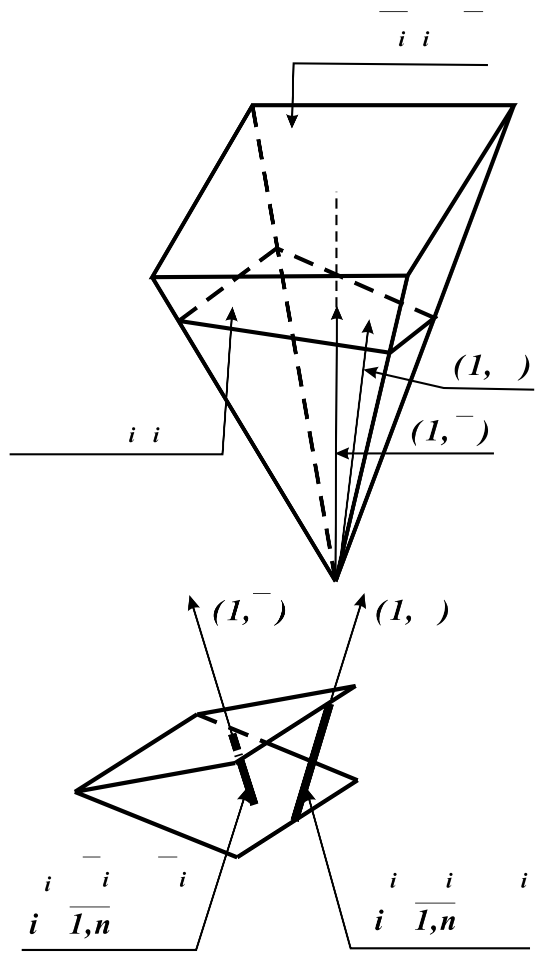

Figure 4. Transformation of linear manifolds.

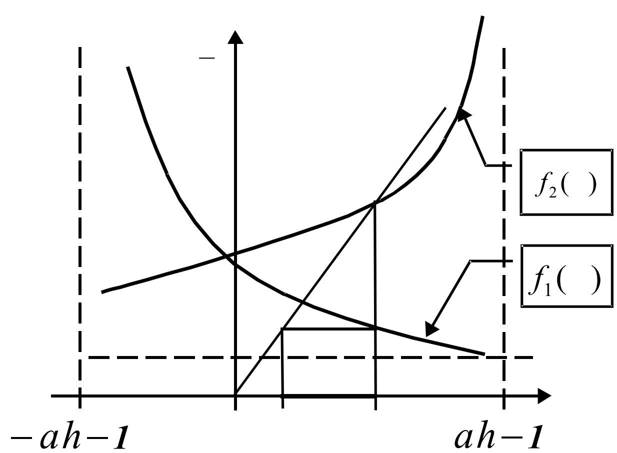

Figure 5. One-dimensional maps of a comparison.

2. Expansion:
   We prove the property of expansion of the variable $\rho \triangleq \xi + \sum \beta_i\eta_i$. This fact together with boundedness of coordinates of the vector β provides an expansion of any vector $(\xi,\eta) \in K_1^u$.

The proof for invariant cone K_n^s is similar to that for cone K_1^u.

Theorem 3. *Let the conditions of Theorem 2 hold. Then the map (3) has a strange hyperbolic nonstationary attractor.*

Proof. Let two different maps $F_1^{(1)}$ and $F_1^{(2)}$ satisfy the Theorem 2. Then the composition $F_1^{(1)}F_1^{(2)}$ satisfies the Theorem 2 as well. This statement follows from the property that both $F_1^{(1)}$ and $F_1^{(2)}$ have the same invariant domain and the same invariant cones K_1^u and K_n^s which are the same for any point. At each two neighbor iterate i and $i+1$ the integer z_i and z_{i+1} takes values within the interval $\mathbb{Z}_N$ and the maps $F\mid_{z=z_i}$ and $F\mid_{z=z_{i+1}}$ are representatives of the family F_1. Hence $F\mid_{z=z_i}\cdot F\mid_{z=z_{i+1}}$ satisfy the Theorem 2, the map F has strange hyperbolic nonstationary attractor with respect to integer z.

Example. For the function from the main example we obtain N different maps (3) for the sequence $f(x,1)$, $f(x,2)$,..., $f(x,N)$ from $\Re(h)$. Let the control rule for the integer z be given by the function $\psi(x,y,z)=k\in\mathbb{Z}_N$ with probability p_k, $\sum\limits_{k=1}^{N}p_k=1$.
Due to Theorem 3 the map (3) has a hyperbolic attractor which randomly changes its structure according to the probability distribution.

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