REACHABLE SETS OF NONLINEAR CONTROL SYSTEMS WITH STATE CONSTRAINTS AND UNCERTAINTY AND THEIR ESTIMATES

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Abstract

The state estimating problems for nonlinear control system with uncertainty in the initial data and parameters are studied. It is assumed that the matrix presented in differential equations of the control system is not exactly known, but belongs to the given compact set in the corresponding space. The right-hand sides of differential equations of a dynamical system may contain nonlinearities defined by functions quadratic in state coordinates. The emphasis in this paper is on the problem of estimating the states of such systems in the presence of additional state constraints. We describe here new approaches and algorithms allowing finding outer (external with respect to the operation of including sets) ellipsoidal estimates of reachable sets of studied nonlinear control system. The approaches and results presented here may be used in many applied areas, when the description of the model contains uncertainty and nonlinearity.

Key words

Control systems, Nonlinear dynamics, Estimation problem, Set-membership uncertainty, Ellipsoidal calculus.

1 Introduction

The paper is a further contribution to the study of estimation problems for uncertain systems in the case when a probabilistic description of noise and errors is not available, but only bounds on them are known [Kurzhanski and Valyi, 1997; Kurzhanski and Varaiya, 2014; Chernousko, 1994; Chernousko, 1996; Schweppe, 1973; Bertsekas, 1995; Walter and Pronzato, 1997; Milanese, Norton, Piet-Lahanier, and Walter, 1996; Milanese and Vicino, 1991].

The research presents a number of useful tools for the mathematical modeling of many natural systems in physics, cybernetics, astrophysics, climatology, chemistry and biology, human systems in economics, psychology and other applied areas when a stochastic nature of the errors of modeling is questionable and only related bounds of uncertain items are available.

The key issue in the set-membership estimation theory is to find suitable techniques, which produce some bounds on the set of unknown system states without being too computationally demanding. Some of such approaches may be found in [Baier, Gerdts, and Xausa, 2013; Dontchev and Lempio, 1992; Kurzhanski and Filippova, 1993; Kishida and Braatz, 2015; Mazurenko, 2012; Filippova and Lisin, 2000; Matviychuk, 2016; Polyak, Nazin, Durieu and Walter, 2004; Sinyakov, 2015].

In this paper the modified state estimation approaches which use the special nonlinear structure of a control system and also take into account state constraints are presented. We assume here that the system nonlinearity is generated by the combination of two types of functions in related differential equations, one of which is bilinear and the other one is quadratic. The additional state constraints appear in such mathematical models in a very natural way when we consider concrete applications in cybernetics, physics, robotics, aeronautics, medicine and other branches [Fradkov, 2007; Apreutesei, 2009; August and Koeppl, 2012; Ceccarelli, Di Marco, Garulli, and Giannitrapani, 2004].

We find here the set-valued estimates of related reachable sets of such nonlinear uncertain control system under an additional complication when we assume that unknown states of the system should belong to a prescribed region in the state space (we consider here the case when this region is defined by an ellipsoid in related space). The paper is devoted to further developments of results of [Filippova, 2017b], namely the continuous-time version of the estimates of reachable sets of the control system with uncertainty and nonlinearity is given here, a new theorem concerning this estimation procedure is proven. Also some additional numerical examples related to the considered problems are included.

Therefore the results of the paper may be of interest not only for specialists in the mathematical control theory, but also may be useful for researchers studying corresponding physical models and other applications in the areas mentioned above.

2 Preliminaries and Problem Formulation

We need to define some auxiliary constructions and results which will be used in the following.

2.1 Basic Notations and Definitions

We will start by introducing the following basic notations. Let $\mathbb{R}^n$ be the n–dimensional Euclidean space and $x'y$ be the usual inner product of $x, y \in \mathbb{R}^n$ with the prime as a transpose, with $\|x\| = (x'x)^{1/2}$. Denote comp $\mathbb{R}^n$ to be the variety of all compact subsets $A \subset \mathbb{R}^n$ and conv $\mathbb{R}^n$ to be the variety of all compact convex subsets $A \subset \mathbb{R}^n$. Let us denote the variety of all closed convex subsets $A \subseteq \mathbb{R}^n$ by the symbol clconv $\mathbb{R}^n$. Let $\mathbb{R}^{n\times m}$ stands for the set of all real $n \times m$-matrices, $\mathrm{diag}\, v$ denotes a diagonal matrix with the elements of vector v on the main diagonal. Let $I \in \mathbb{R}^{n\times n}$ be the identity matrix, $\mathrm{tr}\,(A)$ be the trace of $n \times n$-matrix A (the sum of its diagonal elements). We denote by $B(a, r) = \{x \in \mathbb{R}^n : \|x - a\| \le r\}$ the ball in $\mathbb{R}^n$ with a center $a \in \mathbb{R}^n$ and a radius $r > 0$ and by

$$E(a, Q) = \{x \in \mathbb{R}^n : (Q^{-1}(x - a), (x - a)) \le 1\}$$

the *ellipsoid* in $\mathbb{R}^n$ with a center $a \in \mathbb{R}^n$ and with a symmetric positive definite $n \times n$-matrix Q.

Consider the ordinary differential equation

$$\dot{x} = f(t, x, u(t)) \tag{1}$$

with function $f : T \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ measurable in t and continuous in other variables. Here x stands for the state space vector, t stands for time ($t \in T = [t_0, t_1]$) and $u(t)$ is a control function,

$$u(t) \in Q(t) \tag{2}$$

where $Q(t)$ is a set-valued map ($Q : T \to \mathrm{comp}\, \mathbb{R}^m$) measurable in t. The given data allows to consider a set-valued function

$$\mathcal{F}(t, x) = \bigcup \{ f(t, x, u) \ | \ u \in Q(t) \} \tag{3}$$

and further on, a differential inclusion [Aubin and Frankowska, 1990; Filippov, 1985]

$$\dot{x} \in \mathcal{F}(t, x) \tag{4}$$

that reflects the variety of all models of type (1)-(2).

Let us assume that the initial condition to the system (1) (or to the differential inclusion (4)) is unknown but bounded

$$x(t_0) = x_0, \ \ x_0 \in X_0 \in \mathrm{comp}\, \mathbb{R}^n \tag{5}$$

One of the principal points of interest of the theory of control under uncertainty conditions [Kurzhanski and Valyi, 1997; Kurzhanski and Varaiya, 2014] is to study the set of all solutions $x[t] = x(t, t_0, x_0)$ to (1)-(5) (respectively, (4)-(5)) and furthermore the subset of those trajectories $x[t] = x(t, t_0, x_0)$ that satisfy both (4)-(5) and a restriction on the state vector (the "viability" constraint [Aubin and Frankowska, 1990; Kurzhanski and Filippova, 1993])

$$x[s] \in Y(s), \quad s \in [t_0, t] \tag{6}$$

where $Y(t) \in \mathrm{conv}\, \mathbb{R}^p$ for $t \in [t_0, t_1]$.

The viability constraint (6) may be induced by state constraints defined for a given plant model or by the so-called measurement equation (details of the problem setting may be found in [Kurzhanski and Filippova, 1993])

$$y(t) = G(t)x + w, \tag{7}$$

where y is the measurement, $G(t)$ is a matrix function, w is an unknown but bounded "noise" with a given bound,

$$w \in Q^*(t), \ \ Q^*(t) \in \mathrm{comp}\, \mathbb{R}^p,$$

(here $Q^*(t)$ is a given set-valued function).

The problem consists in describing the set $X[\cdot] = \{x[\cdot] = x(\cdot, t_0, x_0)\}$ of solutions to the system (4)-(6) (the viable solution bundle or "viability bundle"). The point of special interest is to describe the t – cross-section $X[t]$ of this set that is actually the attainability domain of system (4)-(6) at the moment t. Unfortunately, the exact determination of the reachable set $X[t]$ is a difficult problem and hence the problem of finding its estimating sets of a canonical type (e.g., ellipsoids, parallelotopes, polyhedrons etc.) is of interest.

2.2 First Order Approximation

In this section we formulate necessary techniques and results. We assume that the notions of continuity and measurability of set-valued maps are taken in the sense of [Filippov, 1985; Aubin and Frankowska, 1990].

Consider the differential inclusion (4), where $x \in \mathbb{R}^n$, $\mathcal{F}$ is a continuous set-valued map ($\mathcal{F} : [t_0, t_1] \times \mathbb{R}^n \to \text{conv}\mathbb{R}^n$) that satisfies the Lipschitz condition with constant $L > 0$, namely

$$h(\mathcal{F}(t,x), \mathcal{F}(t,y)) \leq L \parallel x - y \parallel, \ \ \forall x, y \in \mathbb{R}^n$$

where $h(A, B)$ is the Hausdorff distance for $A, B \subseteq \mathbb{R}^n$, i.e.

$$h(A, B) = \max \{h^+(A, B), h^-(A, B)\},$$

with $h^+(A, B), h^-(A, B)$ being the Hausdorff semidistances between the sets A, B,

$$h^+(A, B) = \sup\{d(x, B) \mid x \in A\},$$

$$h^-(A, B) = h^+(B, A),$$

$$d(x, A) = \inf \{\parallel x - y \parallel \ \mid y \in A\}.$$

Assuming a set $X_0 \in \text{comp}\,\mathbb{R}^n$ to be given, denote $x[t] = x(t, t_0, x_0)$ ($t \in T = [t_0, t_1]$) to be a solution to (4) (an isolated trajectory) that starts at point $x[t_0] = x_0 \in X_0$.

We take here the Caratheodory–type trajectory $x[\cdot]$, i.e. as an absolutely continuous function $x[t]$ $(t \in T)$ that satisfies the inclusion

$$\frac{d}{dt}\, x[t] = \dot{x}[t] \in \mathcal{F}(t, x[t]) \tag{8}$$

for almost every $t \in T$.

We require all the solutions $\{x[t] = x(t, t_0, x_0) \mid x_0 \in X_0\}$ to be extendable up to the instant t_1 that is possible under some additional assumptions [Filippova and Berezina, 2008].

Let $Y(t)$ be a continuous set-valued map ($Y : T \to \text{conv}\,\mathbb{R}^n$), $X_0 \subseteq Y(t_0)$.

Definition 1. [Kurzhanski and Filippova, 1993] *A trajectory $x[t] = x(t, t_0, x_0)$ ($x_0 \in X_0,\ t \in T$) of the differential inclusion (8) will be called viable on $[t_0, \tau]$ if*

$$x[t] \in Y(t) \ \ \textit{for all} \ \ t \in [t_0, \tau]. \tag{9}$$

We will assume that there exists at least one solution $x^*[t] = x^*(t, t_0, x_0^*)$ of (8) (together with a starting point $x^*[t_0] = x_0^* \in X_0$) that satisfies the condition (9) with $\tau = t_1$.

Let $\mathcal{X}(\cdot, t_0, X_0)$ be the set of all solutions to the inclusion (8) that emerge from X_0 (the "trajectory bundle"). Denote $\mathcal{X}[t] = \mathcal{X}(t, t_0, X_0)$ its crossection at instant t.

The subset of $\mathcal{X}(\cdot, t_0, X_0)$ that consists of all solutions to (8) viable on $[t_0, \tau]$ will be further denoted as $X(\cdot, \tau, t_0, X_0)$ (the "viable trajectory bundle") with its s – cross-sections as $X(s, \tau, t_0, X_0)$, $s \in [t_0, \tau]$. We introduce symbol $X[\tau]$ for these cross-sections at instant τ, namely

$$X[\tau] = X(\tau, t_0, X_0) = X(\tau, \tau, t_0, X_0).$$

The set-valued functions $\mathcal{X}[t]$ and $X[t]$ ($t \in T$) will be referred to as the *trajectory tube* and *viable trajectory tube* (or *viability tube*) respectively. They may be considered as the set-valued analogies of the classical isolated trajectories constructed now under uncertainty conditions.

Let us consider the so-called funnel equation

$$\lim_{\sigma \to +0} \sigma^{-1} h\big(\mathcal{X}[t + \sigma], \bigcup_{x \in \mathcal{X}[t]} (x + \sigma\mathcal{F}(t, x)) \big) = 0, \ \ t_0 \leq t \leq t_1, \ \mathcal{X}[t_0] = X_0. \tag{10}$$

Theorem 1. [Panasyuk, 1990; Kurzhanski and Filippova, 1993] *The multifunction $\mathcal{X}[t] = \mathcal{X}(t, t_0, X_0)$ is the unique set–valued solution to the evolution equation (10).*

Now consider the analogy of the funnel equation (10) but now for the viable trajectory tubes $X[t] = X(t, t_0, X_0)$:

$$\lim_{\sigma \to +0} \sigma^{-1} h\Big(X[t + \sigma], \bigcup_{x \in X[t]} (x + \sigma\mathcal{F}(t, x)) \bigcap Y(t + \sigma) \Big) = 0, \ t \in T, \ X[t_0] = X_0. \tag{11}$$

The following result is valid (under some additional assumptions on $\mathcal{F}(t, x)$ and $Y(t)$ [Kurzhanski and Filippova, 1993; Filippova, 2001]).

Theorem 2. [Kurzhanski and Filippova, 1993] *The set-valued function $X[t] = X(t, t_0, X_0)$ is the unique solution to the evolution equation (11).*

2.3 Second Order Approximations

The above theorems produce the first order approximation of the solution tubes $X[t]$, $\mathcal{X}[t]$. The second order approximations of reachable sets for differential inclusions and for control systems under uncertainty were studied in [Dontchev and Lempio, 1992; Baier, Gerdts, and Xausa, 2013] (but without a viability condition of type (9)).

We shortly formulate here two results which yield the second order approximation schemes (the set-valued analogies of the Runge-Kutta schemes) for $\mathcal{X}[t]$. This section is included in the paper because it also complements the previous results and can stimulate new further researches of numerical methods (of second-order accuracy) for estimating the reachable sets of control systems under consideration. Moreover, some of the results of this kind are used below in Section 2.4 where the examples illustrating the complicated geometric structure of reachable sets are given.

Consider the evolution equation

$$\lim_{\sigma\to+0}\sigma^{-2}h\Big(\mathcal{X}[t+\sigma],\big(\bigcup_{x\in X[t]}(x\ +\ 0.5\,\sigma\times \times\bigcup_{z\in\mathcal{F}(t,\mathcal{X}[t])}(z+\mathcal{F}(t+\sigma,x+\sigma z)))\big)\Big)=0, \qquad (12)$$

$$t_0\le t\le t_1,\ \mathcal{X}[t_0]=X_0.$$

Certainly the higher order approximations require more assumptions on the data. We will assume in addition that the map $\mathcal{F}$ has strongly convex values $\mathcal{F}(t,x)$ and that the support function

$$f(l,t,x)=\max_{u\in\mathcal{F}(t,x)} l'u$$

and the (unique) support vector-function $y(l,t,x)$ defined as

$$l'y(l,t,x)=f(l,t,x)$$

are both continuously differentiable in l,t,x (for $l\neq 0$).

Theorem 3. [Filippova, 2001] *The multifunction $\mathcal{X}[t]=\mathcal{X}(t,t_0,X_0)$ is the unique set–valued solution to the evolution equation (12).*

Find now the equation that produces the second order approximation for the viability tubes $X[\tau]=X(\tau,t_0,X_0)$ of (8)-(9).

Define now an auxiliary notion.

Definition 2. *Given two set-valued functions $W_1(\cdot)$, $W_2(\cdot)$, a symbol $\oint_\alpha^\beta W_1(s)*W_2(s)ds$ denotes the set-valued convolution integral of $W_1(\cdot)$ and $W_2(\cdot)$ where*

$$\oint_\alpha^\beta W_1(s)*W_2(s)ds=\bigcap_{M(\cdot)}\Big\{\int_\alpha^\beta\Big((I-M(s))W_1(s)+M(s)W_2(s)\Big)ds\Big\} \qquad (13)$$

where the intersection in (13) is taken over all continuous $n\times n$-matrix-functions $M(s)$ defined on $[\alpha,\beta]$ and the integral is understood as the Aumann integral.

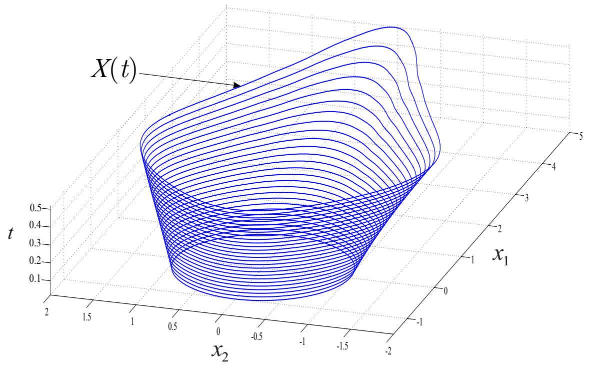

Figure 1. Trajectory tube $X(\cdot)$ (example 1).

Consider the equation

$$\lim_{\sigma\to+0}\sigma^{-2}h\Big(\ X[t+\sigma],\ \oint_t^{t+\sigma}\big(\bigcup_{x\in X[t]}(x+0.5\ s\times\bigcup_{z\in\mathcal{F}(t,X[t])}(z+\mathcal{F}(t+s,x+sz)))\big)*Y(s)ds\Big)=0,\ \ X[t_0]=X_0,\ \ t_0\le t\le t_1. \qquad (14)$$

Theorem 4. [Filippova, 2001] *The viability tube $X[t]=X(t,t_0,X_0)$ is the unique set–valued solution to the evolution equation (14).*

Remark 1. Results of this section may be used as background for computer simulations for finding the reachable sets of uncertain dynamical systems with (or without) state constraints. Unfortunately related computer simulations require a large amount of memory and a lot of time, in fact they are grid methods, see, for example [Baier, Gerdts, and Xausa, 2013]. Therefore, the question arises how to construct external (and if possible, internal) estimating sets for reachable sets, the calculation of which could turn out to be more rapid.

2.4 Numerical Simulations

Consider some examples which show that in the studied nonlinear case the reachable sets may lose their convexity with increasing time $t>t_0$ and may be of very complicated structure.

Example 1. Consider the following control system

$$\begin{cases}\dot{x}_1 = a_1x_1+x_1^2+x_2^2+u_1,\\ \dot{x}_2 = a_2x_2+u_2.\end{cases} \qquad (15)$$

Here we take $x_0\in\mathcal{X}_0=B(0,1)$, $0\le t\le T=0.6$ and $\mathcal{U}=B(0,0.1)$, unknown parameters $a=(a_1,a_2)$ satisfy the constraint $a\in B(0,1)$. The trajectory tube $\mathcal{X}(t)$ is shown in Figure 1, here time moments are taken as $t=0.01;0.3;0.45;0.55;0.6$.

The projection of the above tube onto the plane of state variables $\{x_1,x_2\}$ is shown at Figure 2. We see

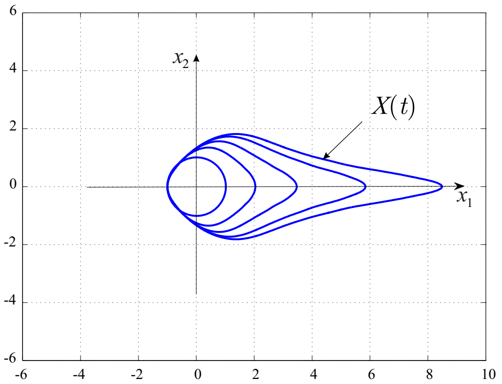

Figure 2. Reachable sets $X(t)$ (example 1).

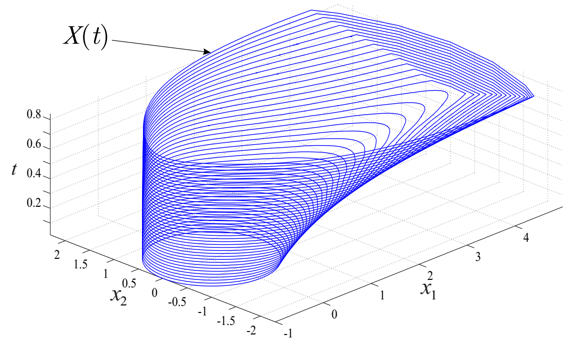

Figure 3. Trajectory tube $X(\cdot)$: the case of state constraints (example 1).

here that the property of convexity of $X(t)$ begins to be violated with increasing time t.

The next Figure 3 shows the trajectory tube $\mathcal{X}(t)$ of the system (15) for time instants $t = 0.01; 0.3; 0.45; 0.55; 0.6$ but with additional state constraint defined by a circle of radius 5 with center at zero.

Example 2. Consider the following control system

$$\begin{cases} \dot{x}_1 = a_1x_1 + x_1^2 + x_2^2 + u_1, \\ \dot{x}_2 = a_2x_2 + u_2. \end{cases} \tag{16}$$

Here we take $x_0 \in \mathcal{X}_0 = B(0,1)$, $0 \le t \le T = 0.85$ and $\mathcal{U} = B(0, 0.1)$. Unknown parameters $a = (a_1, a_2)$ satisfy the ellipsoidal constraint

$$0.01 \cdot a_1^2 + a_2^2 \le 1. \tag{17}$$

The reachable sets $\mathcal{X}(t)$ are shown in Figure 4 for $t = 0.45; 0.75; 0.82; 0.85$.

The trajectory tube $\mathcal{X}(\cdot)$ is shown in Figure 5.

Figure 6 shows the trajectory tube $\mathcal{X}(t)$ of the system

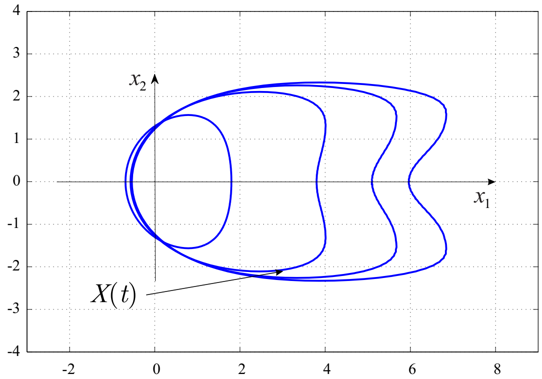

Figure 4. Reachable sets $X(t)$ (example 2).

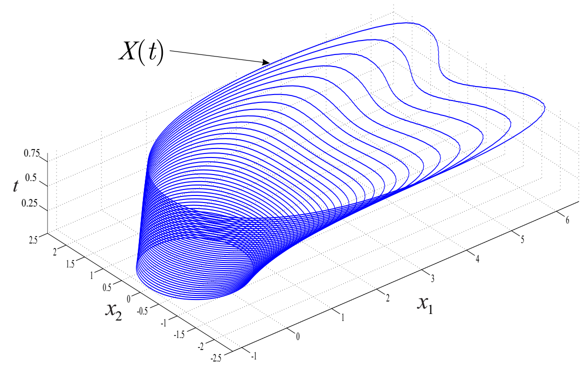

Figure 5. Trajectory tube $X(\cdot)$ (example 2).

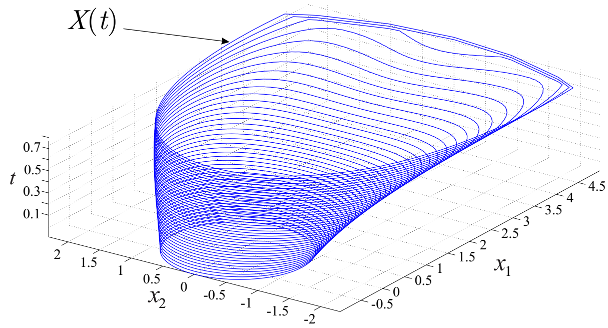

Figure 6. Trajectory tube $X(\cdot)$: the case of state constraints (example 2).

(16) with additional state constraint defined by a circle of radius 5 with center at zero.

It can be noted that the effect of nonconvexity of attainability sets is expressed in the second example even more strongly, possibly because of the special type of ellipsoidal constraint (33) on unknown parameters $a = (a_1, a_2)$.

3 Main Results

Here we consider the general case and we develop and apply the modified state estimation approaches which use the special structure of nonlinearity of studied control system and also take into account state constraints are presented. The studies of systems of this class are motivated, in particular, by the applied problems of controlling the movement of objects under conditions of uncertainty, including optimization of maneuvers of an artificial Earth satellite with thrusters in a strong gravitational field [Kuntsevich and Kurzhanski, 2010; Kuntsevich and Volosov, 2015; Malyshev and Tychinskii, 2005].

3.1 Systems with State Constraints: Discrete-time Case

Consider the following system

$$\dot{x} = A(t)x + f(x)d + u(t),\ t_0 \le t \le t_1,$$
$$x_0 \in X_0 = E(a_0, Q_0),\ \ u(t) \in \mathcal{U} = E(\hat{a}, \hat{Q}), \tag{18}$$

where $x, d, x_0 \in \mathbb{R}^n$, $\|x\| \le K$ ($K > 0$), $f(x)$ is the nonlinear function, which is quadratic in x, $f(x) = x'Bx$, here we assume that the $n \times n$-matrices B, Q_0 and $\hat{Q}$ are symmetric and positive definite.

The $n \times n$-matrix function $A(t)$ in (18) is of the form

$$A(t) = A^0 + A^1(t), \tag{19}$$

where the $n \times n$-matrix A^0 is given and the measurable and $n \times n$-matrix $A^1(t)$ is unknown but bounded, $A^1(t) \in \mathcal{A}^1$ ($t \in [t_0, t_1]$),

$$A(t) \in \mathcal{A} = A^0 + \mathcal{A}^1. \tag{20}$$

Here

$$\mathcal{A}^1 = \big\{A{=}\{a_{ij}\}{\in}R^{n\times n} : a_{ij} = 0 \ \text{ for } \ i \ne j,$$
$$\text{and } \ a_{ii} = a_i,\ \ i = 1, \ldots, n, \tag{21}$$
$$a = (a_1, \ldots, a_n),\ \ a'Da \le 1\big\},$$

where $D \in \mathbb{R}^{n\times n}$ is a symmetric and positive definite matrix.

We assume also that we have the additional state constraint on trajectories of the system (18), namely the following inclusion should be satisfied

$$x[t] \in Y = E(\tilde{a}, \tilde{Q}),\ \ t_0 \le t \le t_1, \tag{22}$$

where the ellipsoid $E(\tilde{a}, \tilde{Q})$ is given (with the center $\tilde{a} \in \mathbb{R}^n$ and the positive definite $n \times n$–matrix $\tilde{Q}$).

Let the absolutely continuous function $x[t] = x\big(t; u(\cdot), A(\cdot), x_0\big)$ be a solution to dynamical system (18)–(34) with initial state $x_0 \in X_0$, with admissible control $u(\cdot)$ and with a matrix $A(\cdot)$. The reachable set $X[t]$ at time t ($t_0 < t \le t_1$) of the system (18)–(34) (under *viability* constraint (34) of type (9)) is defined as

$$X[t] = \{x \in \mathbb{R}^n : \exists\, x_0{\in}X_0, \exists\, u(\cdot){\in}\mathcal{U}, \exists\, A(\cdot){\in}\mathcal{A},$$
$$x = x[t] = x\big(t; u(\cdot), A(\cdot), x_0\big), \tag{23}$$
$$x[\tau] = x\big(\tau; u(\cdot), A(\cdot), x_0\big) \in Y,\ \forall \tau \in [t_0, t]\}.$$

Using the analysis of the special bilinear-quadratic type of nonlinearity of control systems with uncertain initial data and with ellipsoidal state constraints we find here the external ellipsoidal estimate $E(a^+(t), Q^+(t))$ (with respect to the inclusion of sets) of the reachable set $X[t]$ ($t_0 < t \le t_1$).

We will need further the following Minkowski (gauge) functional of the star-shaped sets $M \subseteq \mathbb{R}^n$ ($0 \in M$) [Demyanov and Rubinov, 1986; Filippova and Lisin, 2000],

$$h_M(z) = \inf\{t > 0 : z \in tM, x \in \mathbb{R}^n\}.$$

The following result presents the external estimate of reachable sets of system under viability (state) constraints. First we need to formulate the following auxiliary result.

Lemma 1. ([Filippova and Lisin, 2000; Matviychuk, 2016]) *For $X_0 = E(0, Q_0)$ and $\mathcal{A}^1$ defined in (21) the Minkowski function of the set $(I + \sigma\mathcal{A}^1) * X_0$ has the form*

$$h_{(I+\sigma\mathcal{A}^1)*X_0}(z) = \Big(\,||Q_0^{-1/2}z||^2 -$$
$$2\sigma\Big(\sum_{i,j=1}^{n} w_i^2 (D^{-1/2})_{ij} \cdot w_j^2)^{1/2}\Big)^{1/2} + o(\sigma)||Q_0^{-1/2}z||,$$
$$w(z) = Q_0^{-1/2}z,\ \ \lim_{\sigma\to+0} \sigma^{-1}o(\sigma) = 0. \tag{24}$$

Theorem 5. *Let $X_0 = E(a_0, k^2B^{-1})$, $k \ne 0$. Then for any matrix $L \in \mathbb{R}^{n\times n}$ and for all $\sigma > 0$ the following external estimate is true*

$$X[t_0 + \sigma] \subseteq E(a_L^+(\sigma), Q_L^+(\sigma))\ + \tag{25}$$
$$o(\sigma)B(0,1),\ \ \lim_{\sigma\to+0} \sigma^{-1}o(\sigma) = 0,$$

where

$$a_L^+(\sigma) = a_0 + \sigma(\hat{a} + k^2 d + a_0 B a_0 \cdot d \; + (A^0 - L)a_0 + L\tilde{a}),$$

$$Q_L^+(\sigma) = (p^{-1}+1)Q_1(\sigma) + (p+1)\sigma^2 \hat{Q}^*,$$

$$Q_1(\sigma) = \mathrm{diag}\{(p^{-1}+1)\sigma^2 a_{0i}^2 \; + (p+1)r^2(\sigma) \,|\, i = 1,\ldots,n\},$$

$$r(\sigma) = \max_z ||z|| \cdot (h_{(I+\sigma\mathcal{A})*X_0}(z))^{-1},$$

p is the unique positive root of the equation $\sum_{i=1}^{n} \frac{1}{p+\alpha_i} = \frac{n}{p(p+1)}$ with $\alpha_i \geq 0$ $(i = 1,...,n)$ being the roots of the following equation $|Q_1(\sigma) - \alpha\sigma^2\hat{Q}^*| = 0$, and $E(\hat{a},\hat{Q}^*)$ is the ellipsoid with minimal volume such that

$$E(\hat{a},\hat{Q}) + L\cdot E(0,\tilde{Q}) + (2d\cdot a_0' B + A^0)\cdot E(0, k^2 B^{-1}) \subseteq E(\hat{a},\hat{Q}^*). \tag{26}$$

Proof. We use here the idea of [Kurzhanski and Filippova, 1993] for elimination of state constraints in the construction of reachable sets (see also related results in [Bettiol, Bressan, and Vinter, 2010; Gusev, 2016]). Consider the following differential inclusion with $n\times n$–matrix parameter L,

$$\dot{z} \in (A_0 - L + \mathcal{A}^1)z \; + \; f(z)\cdot d + E(\hat{a},\hat{Q}) + L\cdot E(\tilde{a},\tilde{Q}), \;\; t_0 \leq t \leq T, \quad z_0 \in X_0 = E(a_0, Q_0). \tag{27}$$

Denote by $Z(t; t_0, X_0, L)$ $(t \in [t_0, t_1])$ the trajectory tube to (27) for a fixed matrix parameter L. We have the following estimate [Kurzhanski and Filippova, 1993]

$$X[t] \subseteq \bigcap_L Z(t; t_0, X_0, L), \;\; t_0 \leq t \leq t_1. \tag{28}$$

Using results of Theorems 1-2 and also taking into account the approaches and results of [Filippova, 2012; Filippova, 2016; Filippova and Lisin, 2000; Filippova and Matviychuk, 2012; Filippova and Matviychuk, 2015; Matviychuk, 2016; Filippova, 2017a; Filippova, 2017b] we can find the upper ellipsoidal estimates for reachable sets $Z[t] = Z(t; t_0, X_0, L)$ of the nonlinear system (27) (we underline here that after the above elimination this new system does not have state constraints and therefore we may use some estimation results mentioned in Section 2.2). Resulting estimate (25) follows from (28) and from the above remark. □

The following algorithm is based on Theorem 5 and may be used to produce the external ellipsoidal estimates for the reachable sets of the system (18)-(34).

Fix a finite number of matrices L_s, $s = 1,\ldots,r$ (r is an arbitrary integer, $r > 0$).

Algorithm 1. Subdivide the time segment $[t_0, t_1]$ into subsegments $[\tau_i, \tau_{i+1}]$, where $\tau_i = t_0 + i\sigma$ $(i = 1,\ldots,m)$, $\sigma = (t_1 - t_0)/m$.

1. For given $X_0 = E(a_0, Q_0)$ define the smallest $k_0 > 0$ such that $E(a_0,Q_0) \subseteq E(a_0, k_0^2 B^{-1})$ (k_0^2 is the maximal eigenvalue of the matrix $B^{1/2}Q_0B^{1/2}$, [Filippova, 2012; Filippova and Matviychuk, 2015]).
2. For $X_0 = E(a_0, k_0^2 B^{-1})$ as an initial set define by Theorem 5 the upper estimate $E(a_{L_s}^+(\sigma), Q_{L_s}^+(\sigma))$ of the set $X(t_0+\sigma)$, $s = 1,\ldots,r$.
3. Take a compact and convex set X_0^* such that $\bigcap_{1\leq s\leq r} E(a_{L_s}^+(\sigma), Q_{L_s}^+(\sigma)) \subseteq X_0^*$.
4. Consider the system on the next subsegment $[\tau_1,\tau_2]$ with the initial (at time instant τ_1) set X_0^* and with initial ellipsoid $E(a_1, k_1^2 B^{-1})$ found as in step 1.
5. The next step repeats the previous iteration beginning with new initial data.

At the end of the process we will get the external estimate tube $E(a^+(t), Q^+(t))$ of the reachable sets $X(t)$ $(t_0 \leq t \leq t_1)$ of the system (18)-(34).

3.2 Systems with State Constraints: Continuous-time Case

The following result describes the dynamics of the external ellipsoidal estimates of the reachable set $X(t) = X(t; t_0, X_0)$ $(t_0 \leq t \leq T)$.

Theorem 6. *Let $X_0 = E(a_0, k^2B^{-1})$, $k \neq 0$. Then for any matrix $L \in \mathbb{R}^{n\times n}$ and for all $t \in [t_0, T]$ the following external estimate is true*

$$X(t; t_0, X_0) \subseteq E(a_L^+(t), r_L^+(t)B^{-1}), \tag{29}$$

where functions $a^+(t)$, $r^+(t)$ are the solutions of the following system of ordinary differential equations

$$\dot{a}_L^+(t) = (A^0 - L)a_L^+(t) + ((a_L^+(t))'Ba_L^+(t) + r_L^+(t))d + \hat{a} + L\tilde{a}, \quad t_0 \leq t \leq T,$$
$$\dot{r}_L^+(t) = \max_{||l||=1} \Big\{ l'\big(2r_L^+(t)B^{1/2}(A_0 + 2d(a^+(t))'B)B^{-1/2} + q^{-1}(r_L^+(t))B^{1/2}\hat{Q}^*B^{1/2}\big)l \Big\} + q(r_L^+(t))r_L^+(t),$$
$$q(r) = ((nr)^{-1}\mathrm{Tr}(B\hat{Q}_L^*))^{1/2}, \tag{30}$$

where the positive definite matrix $\hat{Q}^*_L$ is such that the following inclusion holds

$$\mathcal{A}^1 a_0 + E(0,\hat{Q}) + LE(0,\tilde{q}) + \\ k_0 D^{1/2} B^{1/2} B(0,1) + E(0,\tilde{Q}) \subseteq E(0,\hat{Q}^*_L), \quad (31)$$

with initial state

$$a^+_L(t_0) = a_0, \quad r^+_L(t_0) = k^2.$$

Proof. The above estimates are derived from Theorem 5 with the necessary corrections done according to new constraints on unknown parameters and functions included in the system description and following the general schemes of the paper [Filippova, 2010]. □

Remark 2. If the initial set X_0 in (5) is an arbitrary ellipsoid of the form $X_0 = E(a_0, Q_0)$ we can define the smallest $k_0 > 0$ such that $E(a_0, Q_0) \subseteq E(a_0, k_0^2 B^{-1})$ (k_0^2 is the maximal eigenvalue of the matrix $B^{1/2} Q_0 B^{1/2}$, [Filippova, 2012; Filippova and Matviychuk, 2015]) and apply after that the Theorem 6 to get the upper estimates of the reachable sets $X(t) = X(t; t_0, X_0)$.

Remark 3. It may be noted that the matrix $\hat{Q}^*_L$ (and therefore the estimating ellipsoid $E(0,\hat{Q}^*_L)$) in (31) depends on a_0, it is a new feature appeared here due to more complicated bilinear structure of uncertainties in the system dynamics.

Remark 4. The numerical scheme and related algorithm for constructing upper estimates of reachable sets of the system under consideration may be also formulated similar to Algorithm 1. It seems that in the future it will be possible to obtain more accurate upper estimates of reachable sets $X(t) = X(t; t_0, X_0)$ using the results of Theorems 3-4.

3.3 Illustrative Example

Consider an example showing the main idea and the estimation capacity of the Algorithm 1.

Example 3. Consider the following control system

$$\begin{cases} \dot{x}_1 = a_1 x_1 + x_1^2 + x_2^2 + u_1, \\ \dot{x}_2 = a_2 x_2 + u_2. \end{cases} \quad (32)$$

Here we take $x_0 \in \mathcal{X}_0 = B(0,1)$, $0 \le t \le T = 0.8$ and $\mathcal{U} = B(0, 0.1)$. We assume that unknown parameters $\{a_1, a_2\}$ satisfy the constraint

$$a_1^2 + a_2^2 \le 1. \quad (33)$$

We assume also that we have the additional constraint on trajectories of the system (32), namely the following inclusion should be satisfied

$$x[t] \in Y = E(\tilde{a}, \tilde{Q}), \quad t_0 \le t \le T, \quad (34)$$

where we take the center $\tilde{a} = (0.5, 0)$ and the positive definite 2×2–matrix $\tilde{Q} = \mathrm{diag}\{0.64, 4\}$.

The reachable set $\mathcal{X}(t)$ of the system (32)-(34) for $t = 0.2$ is shown in Figure 7 in blue color. Here the estimating ellipsoids $E(a_{L_i}, Q_{L_i})$, $i = 1,2,3$ found according to results of Theorems 5-6 are indicated in red, with

$$L_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0.01 \end{pmatrix},$$

and the state constraint defined by the ellipsoid $Y = E(\tilde{a}, \tilde{Q})$ is shown in Figure 7 in black color (as a dotted line). We see here that each ellipsoid $E(a^+_{L_i}, r^+_{L_i})$ produces the outer (external) estimate of reachable set $\mathcal{X}(t)$ and crossing together they make the estimate of $\mathcal{X}(t)$ more accurate.

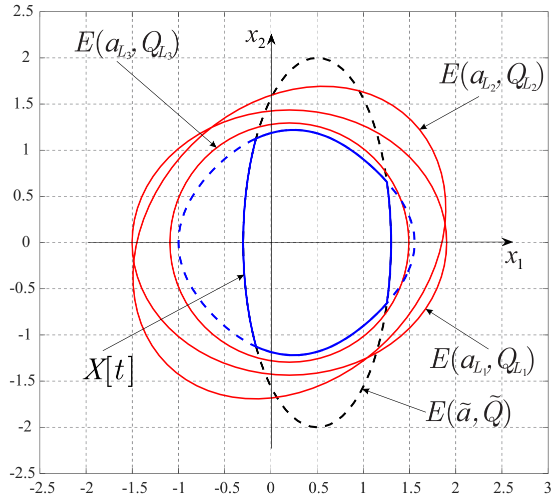

Figure 7. Reachable sets $X(t)$ and its estimating ellipsoids $E(a^+_{L_i}, r^+_{L_i})$, $i = 1,2,3$ (example 3).

4 Conclusions

The paper deals with the problems of state estimation for a dynamical control system described by nonlinear differential equations with unknown but bounded initial states. Nonlinearity in dynamics is described by a combination of bilinear and quadratic functions of the state of the system. Also the bilinear terms presented in the system may be considered as uncertain disturbances in matrix coefficients of the linear elements in the system state velocities. The case of quadratic constraints on bilinear uncertain parameters are studied here.

We present in this paper the modified state estimation approaches which use the special structure of the nonlinear control system and we investigate the complicated dynamical properties of reachable sets under uncertainty.

The applications of the problems studied in this paper are in guaranteed state estimation for nonlinear systems with unknown but bounded errors and in nonlinear control theory including numerous applications in physics and mechanics, in particular, in robotics and also in other branches with uncertainty and nonlinearity in related dynamical models including problems in biology, medicine and economics.

Acknowledgements

The research was supported by the Russian Foundation for Basic Researches (RFBR Projects No.16-29-04191-ofi-m and No.15-01-02368-a).

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