ASYMPTOTIC STABILIZATION OF THE DESIRED UNIFORM MOTION IN UNDERACTUATED HAMILTONIAN SYSTEMS BY LINEAR FEEDBACK

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Abstract: In some cases the desired uniform motion may be described by a pair of first integrals of the system with zero control input. These two integrals and the integration of nonlinear function of saturation are used to construct Lyapunov function The control is designed from the condition of decreasing Lyapunov function on the trajectories of the closed loop system. This control may be chosen a priori bounded and linear in a small viciniti of the desired motion. This method is applied to stabilize rotating body beam, for damping the oscillations of blades of an elastic propeller and for stabilization of the uniform transition of the pendulum on a cart.

Keywords: Lyapunov methods, nonlinear systems, saturation , cranes, robots, damping.

1. GENERAL IDEA

Consider the following controlled Hamiltonian system

$$\dot{q}_1 = \frac{\partial H}{\partial p_1}\ , \quad \dot{q}_2 = \frac{\partial H}{\partial p_2}\ ,$$
$$\dot{p}_1 = \tau\ , \quad \dot{p}_2 = -\frac{\partial H}{\partial q_2}\ , \tag{1}$$

where $q_1 \in R^1, \quad q_2 \in R^n$ are generalized coordinates, $p_1 \in R^1, \quad p_2 \in R^n$ are momenta (generalized impulses), $H = H(q_2, p_1, p_2)$ is Hamiltonian, and τ is a controlling generalized force (torque), q_1 is a cyclic variable. This system under the condition $\tau = 0$ has two classical first integrals H and p_1.

In some cases the desired motion may be described by the variables $q_1 = \omega_d t + q_{10}, q_2 = 0, ..., q_n = 0, \dot{q}_{1d} = \omega_d, \dot{q}_2 = 0, ..., \dot{q}_n = 0$ Let us consider the following Lyapunov function

$$V = H - \omega_d p_1 + \int_0^{q_1 - q_{1d}} F(s)ds \tag{2}$$

where F is a continuous, strictly increasing, possibly bounded function with the condition $F(0) = 0$, f.e. $F(s) = s$ or $F(s) = \arctan(s)$. The idea of this function is the combination of Chetaev bundle of integrals (Rouche, Habets and Laloy, 1977, ch.4) and the integration of nonlinearity (Dunskaya and Pyatnitskii,1988). It is easy to check that the following equality

$$\dot{V} = (\dot{q}_1 - \omega_d)(\tau + F(q_1 - q_{1d}))$$

holds. If the following expression

$$\tau = -F(\alpha(\dot{q}_1 - \omega_d) + q_1 - q_{1d}), \quad \alpha > 0, \tag{3}$$

is chosen as control, then $\dot{V} \leq 0$ and in some cases it may be applied Barbashin and Krasovsky theorem to prove asymptotic stability. The proposed method of stabilization is simpler than the nonlinear one considered earlier (Burkov, 2004).

In the following sections the examples of application of this general idea are considered.

2. ROTATING BODY BEAM

Consider the discrete model of the known system (Coron and d'Andrea-Novel, 1998), see Fig.1. The system consists of a disk with a rod elastically attached to its center and perpendicular to the disk plane. The rod is confined to another plane which is perpendicular to the disk and rotates with the disk. The moment of inertia of the disk is I, the length of the rod is l, the mass of a bob attached to the end of the rod is m, the coefficient of stiffness between the bob and the rotation axis is κ. Let θ denote the angle of rotation of the disk, u be corresponding control torque acting on the disk, z be the deviation between the rod and the rotation axis of the disk, e be the eccentricity of the bob, $z = e$ in nonstrained state. This system may be considered as simple model of the whirling shaft which is simpler than the standard model (Dimentberg,1959). The kinetic energy of the system can be expressed as

$$T = \frac{1}{2}(m(\dot{z}^2 + z^2\dot{\theta}^2) + I\dot{\theta}^2)$$

and the potential energy is expressed as $\Pi = \frac{\kappa}{2}(z-e)^2$. The full energy $E = T + \Pi$ and the kinetic moment

$$J = \frac{\partial E}{\partial \dot{\theta}} = (mz^2 + I)\dot{\theta}$$

are the first integrals of the system under the condition $u = 0$. By Lagrange formalism the explicit equations of motion may be obtained as follows

$$\ddot{\theta}(mz^2 + I) + 2mz\dot{\theta}\dot{z} = u \ ,$$

$$\ddot{z}m + \kappa(z - e) - mz\dot{\theta}^2 = 0 \ . \tag{4}$$

Suppose that the square of the desired angular velocity ω_d is less than the square of the so-called critical velocity κ/ml^2. Let $\dot{\theta}_d = \omega_d, \quad z_d = \kappa e/(\kappa - m\omega_d^2)$ be the desired motion. The simple stabilizing control can be proposed as follows

$$u = -F(\dot{\theta} - \omega_d) \ . \tag{5}$$

This control may be chosen a priori bounded. The boundness and simplicity are the advantage in comparison with the control proposed by Coron and d'Andrea-Novel (1998).

Proposition 1. The closed loop system (4), (5) has the desired rotation as the solution when control torque $u = 0$. Under the condition $\frac{\kappa}{ml^2} > \omega_d^2 \neq 0$ this system is asymptotically stable with respect to the variables $\dot{\theta} - \omega_d, \ z - z_d, \dot{z}$. If the eccentrity $e = 0$, then the case of asymptotic stability is critical in the sense of Lyapunov. In case $e \neq 0$ all the eigenvalues of the linear approximation of the closed loop system have negative real parts.

Proof. In case $e = 0$ the asymptotic stability follows from the application of Barbashin-Krasovsky theorem on asymptotic stability to the function $V = H - \omega_d p_1$. Using the variables $\tilde{\omega}, z, v = \dot{z}$ it may be rewritten as follows

$$V = \frac{1}{2}(I\tilde{\omega}^2 + mv^2 + (\kappa - m\omega_d^2)z^2 + mz^2\tilde{\omega}^2 - I\omega_d^2)$$

This function is positive definite if $\kappa > m\omega_d^2$. Computation of its derivative taking into account equations of the closed loop system results in the following equality

$$\dot{V} = -\tilde{\omega}F(\tilde{\omega})$$

Let us analyze the set of trajectotries of the closed loop system which is subject to the condition $\dot{V} = 0$. The last one implies $\omega = \omega_d$. It follows from this equality and first equation of (4) that $z = 0$ or $\dot{z} = 0$. If $\dot{z} = 0$ then the consideration of the second equation of (4) implies that $z = 0$.

In case of nonzero eccentricity the asymptotic stability is demonstrated by applying Routh and Hurwitz criterion. The linear approximation of the closed loop system may written as

$$\dot{X} = AX$$

where $X = (\tilde{\omega}, z - z_d, v)^t$ and matrix

$$A = \begin{pmatrix} \frac{-\delta}{I+mz_d^2} & 0 & \frac{-2z_d\omega_d}{I+mz_d^2} \\ 0 & 0 & 1 \\ 2z_d\omega_d & \omega_d - \kappa/m & 0 \end{pmatrix}$$

with $\delta = F'(0) > 0$. The characteristic polynom of this matrix is as follows

$$\lambda^3 + \frac{\lambda^2\delta}{I + mz_d^2} + \lambda(\frac{\kappa}{m} - \omega_d^2 + \frac{4z_d^2\omega_d^2}{I + mz_d^2}) + \frac{\delta}{I + mz_d^2}(\frac{\kappa}{m} - \omega_d^2)$$

It is evident that all the coefficients of this polynom are positive and the following inequality from Routh and Hurwitz criterion holds

$$(\frac{\kappa}{m} - \omega_d^2 + \frac{4z_d^2\omega_d^2}{I + mz_d^2}) - (\frac{\kappa}{m} - \omega_d^2) > 0$$

Modelling. The closed loop system described in Proposition 1 was integrated with the following parameters: $m = l = I = \kappa = 1$, $F(\tau) = \tau$, $e = 0$, $\omega_d = 1/2$ in the time interval [0,1500] with the initial data $z_0 = 1$, $\dot{z}_0 = 0$, $\dot{\theta}_0 = 0.4$. The amplitude of oscillations of the deviation z slowly decreased from 1 to 1/10. Such slowness is not surprising because the control acting on the deviation z is indirect. In Fig.2 there is a graph of the deviation $z(t)$.

The influence of the friction on asymptotic stability of the whirling shaft was investigated (Dimentberg,1959, Tondl,1965). Here it have been shown that linear angular feedback guaranties asymptotic stability even in absense of friction.

3. ELASTIC PROPELLER

In many cases the propellers are essentially elastic, for example, helicopter propellers. Sometimes the oscillating blades touch a helicopter fuselage and cause the catastrophe of the aircraft. So, it is interesting to solve the following problem: how to rotate the propeller with the desired velocity ω_d and simultaneously to damp the oscillations of the blades.

Consider the following simple discrete model of propeller with two blades (Fig.3). Let two equal rods with the bobs at free ends are elastically attached to the axis of propeller rotation. These rods may oscillate in the imaginary plane which rotates around the propeller axis. In non-strained state of elastic elements the rods are perpendicular to the propeller axis. Let θ denote the angle of rotation and φ_i $(i = 1, 2)$ denote the angles of deviations of rods from perpendicular state. For simplicity it is assumed that the lengths of rods, bob masses, stiffness coefficients, and the moment of inertia of propeller axis are equals to 1. The controlling torque u acts on the propeller axis.

The kinetic energy of the system may be expressed as

$$T = \frac{1}{2}(\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + (\cos^2\varphi_1 + \cos^2\varphi_2)\dot{\theta}^2 + \dot{\theta}^2)$$

and the potential energy is expressed as $\Pi = \frac{\varphi_1^2}{2} + \frac{\varphi_2^2}{2}$. The full energy $E = T + \Pi$ and kinetic moment

$$J = \frac{\partial E}{\partial \dot{\theta}} = (1 + \cos^2\varphi_1 + \cos^2\varphi_2)\dot{\theta}$$

are first integrals of the system under the condition $u = 0$. With aid of the Lagrangian $L = T - \Pi$ the explicit equations of motion can be obtained as follows

$$\begin{gathered} \ddot{\theta}(1 + \cos^2\varphi_1 + \cos^2\varphi_2) - 2\sin\varphi_1\cos\varphi_1\dot{\theta}\dot{\varphi}_1 \\ -2\sin\varphi_2\cos\varphi_2\dot{\theta}\dot{\varphi}_2 = u\ , \\ \ddot{\varphi}_i + \varphi_i + \sin\varphi_i\cos\varphi_i\dot{\theta}^2 = 0 \quad (i = 1, 2)\ . \end{gathered} \tag{6}$$

By means of Lyapunov function described in the first section the stabilizing control can be proposed as

$$u = -F(\dot{\theta} - \omega_d)\ . \tag{7}$$

Proposition 2. The closed loop system (6), (7) has the desired rotation as the solution when control torque $u = 0$. This system is asymptotically stable with respect to the variables $\dot{\theta} - \omega_d$ and $\varphi_i, \dot{\varphi}_i$.

Idea of proof. The asymptotic stability follows from the application of Barbashin-Krasovsky theorem on asymptotic stability to the function $V = H - \omega_d p_1$.

Modelling. The closed loop system described in Proposition 2 was integrated with the following parameters: $F(s) = s$, $\omega_d = 2$ in the time interval [0,1000] with the initial data $\varphi_{0i} = 1$, $\dot{\varphi}_{i0} = 0$, $\dot{\theta}_0 = 0.4$. The amplitude of oscillations of φ_i slowly decreased from 1 rad to 0.2 rad. Such slowness is natural due to the indirect control action on the angles φ_i . In Fig. 4 there is the graph of the angular velocity $\dot{\theta}(t)$.

4. A PENDULUM ON A CART

The planar pendulum on a cart is the well-known physical device (see, f.e., Mazenc and Bowong, 2003). Its dynamics obtained by Lagrange formulation are

$$\begin{gathered} (M+m)\dot{v} + ml\cos(\theta-\gamma)\dot{\omega} - ml\sin(\theta-\gamma)\omega^2 + \\ g\sin\gamma(M+m) = f, \quad \dot{z} = v, \\ \cos(\theta-\gamma)\dot{v} + l\dot{\omega} + g\sin\theta = 0, \quad \dot{\theta} = \omega \end{gathered} \tag{8}$$

where (M, z) are mass and position of the cart moving along a strait line, which has the angle γ with respect to the horizontal line, (m, l, θ) are mass, length and angular deviation from the downward vertical position for the pendulum which is pivoting around a point fixed on the cart, f is a control force acting on the cart, g is a gravitational constant, see Fig.5. The system has the kinetic energy

$$K = \frac{1}{2}(M+m)v^2 + mlv\omega\cos(\theta-\gamma) + \frac{1}{2}ml^2\omega^2$$

the potential energy

$$\Pi = \Pi_z + \Pi_\theta,$$

with $\Pi_z = Mgz\sin\gamma + mgz\sin\gamma$, $\Pi_\theta = -gml\cos\theta$ and the kinetic moment

$$J = (M+m)v + ml\omega\cos(\theta-\gamma)$$

The pendulum on the cart is a simple model for overhead crane moving the load (d'Andrea-Novel and Coron, 2000). If the angle $\gamma = \pi/2$ than this model describes lifting a load by the rope of a crane.

By means of Lyapunov function mentioned in sect. 1 the stabilizing control

$$f = g\sin\gamma(M+m) - F(\alpha(v-v_d) + z - z_d), \quad \alpha > 0. \tag{9}$$

where $z_d = v_d t + z_{d0}$ may be obtained.

Proposition 4. The closed loop system (8), (9) has the desired uniform transition as the solution. This system is asymptotically stable with respect to the variables $z - z_d, v - v_d, \theta, \omega$. The closed loop system has also another equilibrium $z = z_d, v = v_d, \theta = \pi, \omega = 0$, which is unstable.

Proof. Applying Sylvester criterion the positive definetness of Lyapunov function

$$V = K + \Pi_\theta - v_d J + \int_0^{z-z_d} F(s)ds$$

with respect to the variables $z - z_d, v - v_d, \theta, \omega$ can be proven. Consideration of the condition $\dot{V} = 0$ shows that the closed loop system has only two equilibria. The down equilibrium $v = v_d, \theta = 0, \omega = 0$ is asymptotically stable in accordance with Barbashin theorem . The instability of the upper equilibrium $v = v_d, \theta = \pi, \omega = 0$ can be shown by applying necessary condition of Routh and Hurwitz criterion.

By means of Lyapunov function mentioned in sect. 2 obtain the stabilizing control

$$f = g \sin \gamma (M + m) - F(v - v_d), \qquad (10)$$

may be obtained.

Proposition 5. The closed loop system (8), (10) has the desired uniform transition as the solution. This system is asymptotically stable with respect to the variables $v - v_d, \theta, \omega$. The closed loop system has also another equilibrium $v = v_d, \theta = \pi, \omega = 0$, which is unstable.

Modelling. In Fig. 6,7 there are graphs of the velocity $v(t)$ and the angle $\theta(t)$ for the following parameters $M = m = l = g = \alpha = v_d = 1, F(\tau) = \tau, \gamma = \pi/2$ and initial data $z(0) = 0, v(0) = 1.2, \theta(0) = 0.5, \omega(0) = 0$.

5. INERTIA WHEEL PENDULUM

The inertia wheel pendulum is shown schematically in Fig.8. It is a physical pendulum with a symmetric disk attached to the end which is free to spin about an axis parallel to the axis of rotation of the pendulum. The disk is actuated by a motor and the coupling torque generated by the angular acceleration of the disk can be used to actively control the system. The equations of motion are as follows

$$(J + J_r)\ddot{\theta} + J_r \ddot{\varphi} + mgl \sin \theta = 0 \qquad (11)$$

$$J_r \ddot{\theta} + J_r \ddot{\varphi} = u,$$

where θ is the pendulum angle, φ is the disk angle, u is the motor torque input and J, J_r, m, g, l are positive parameters. Let $\theta = 0, \dot{\varphi} = \omega_d$ be the desired motion. The system has the kinetic energy

$$K = \frac{1}{2}((J + J_r)\dot{\theta}^2 + J_r \dot{\varphi}^2 + 2J_r \dot{\theta}\dot{\varphi})$$

and potential energy $\Pi = mgl(1 - \cos \theta)$. Consider the following simple control input

$$u = -F(\dot{\varphi} - \omega_d). \qquad (12)$$

Kolesnichenko e.a., 2002 proposed another control input which needs information on $\dot{\theta}$.

Proposition 6. The closed loop system (11), (12) has the desired down position which is asymptotically stable with respect to the variables $\theta, \dot{\theta}$, $\dot{\varphi} - \omega_d$ and unstable upper position $\theta = \pi, \dot{\theta} = 0$, $\dot{\varphi} = \omega_d$.

Idea of proof. Consider Lyapunov function

$$K + \Pi - \omega_d \frac{\partial (K + \Pi)}{\partial \dot{\varphi}}$$

Modelling. In Fig. 9 there is the graph of the angle $\theta(t)$ for the following parameters $J = J_r = m = l = g = \omega_d = 1$, $F(\tau) = \arctan \tau$, and initial data $\theta(0) = 1, \dot{\theta}(0) = 0, \dot{\varphi}(0) = 1$.

6. PENDUBOT

Consider the two link underactuated planar robot, called pendubot, see Fig.10. Introduce the following notations: m_1 is the mass of the first link, m_2 the mass of the second link, q_1 the angle that the first link makes with the axis OX, q_2 the angle that the second link makes with the first link, l_1 and l_2 the lengths of links, l_{c1} and l_{c2} the distances to center of masses, I_1 and I_2 the moments of inertia of he links about their centroids. Introduce the following parameters: $n_1 = m_1 l_{c1}^2 + m_2 l_c^2 + I_1$, $n_2 = m_2 l_{c2}^2 + I_2$, $n_3 = m_2 l_1 l_{c2}$, $n_4 = m_1 l_{c1} + m_2 l_1$, $n_5 = m_2 l_{c2}$. The kinetic energy is expressed as follows $K = \frac{1}{2}(v_1, v_2) D(q_2)(v_1, v_2)^t$, where $v_i = \dot{q}_i$ and

$$D = \begin{pmatrix} n_1 + n_2 + 2n_3 \cos q_2 & n_2 + n_3 \cos q_2 \\ n_2 + n_3 \cos q_2 & n_2 \end{pmatrix}$$

Using Lagrange procedure the equations of dynamics are obtained (Fantoni e.a. 2000)

$$d_{11}\dot{v}_1 + d_{12}\dot{v}_2 + n_3 \sin q_2 (-2v_1 v_2 - v_2^2) = u, \quad (13)$$

$$d_{21}\dot{v}_1 + d_{22}\dot{v}_2 + n_3 \sin q_2 v_1^2 = 0,$$

where u is controlling torque acting on the first link. Let $v_1 = \omega_d$, $q_2 = 0, v_2 = 0$ be the desired motion. Consider the following simple control input

$$u = -F(\dot{q}_1 - \omega_d). \qquad (14)$$

Proposition 7. The closed loop system (13), (14) has the desired rotation which is asymptotically stable with respect to the variables $v_1 - \omega_d, q_2, v_2$ and unstable rotation $v_1 = \omega_d, q_2 = \pi, v_2 = 0$.

Idea of proof. Consider Lyapunov function

$$K - \omega_d \frac{\partial K}{\partial \dot{q}_2}$$

CONCLUSION

The combination of Chetaev method and the integration of nonlinearity for constructing Lyapunov functions is effective to solve difficult problems of nonlinear stabilization. Some questions are still open in the considered problems; for example, robustness of stabilization with respect to noise in measurements and external deterministic or stochastic disturbances. It is necessary to investigate the domain of attractions. This is the theme for future research.

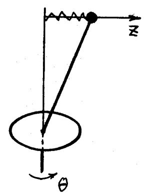

Figure 1: Rotating body and beam

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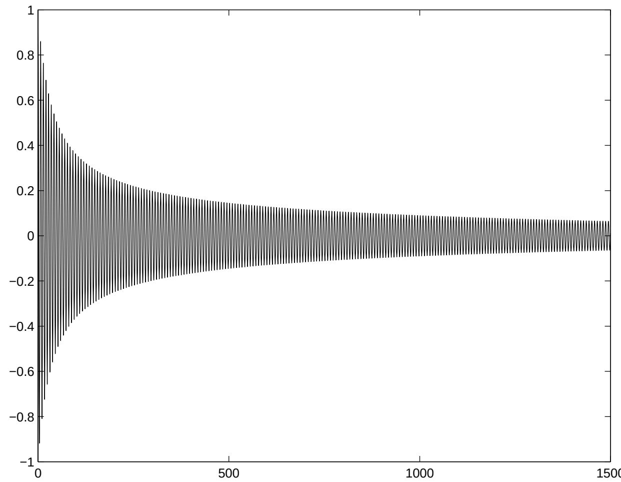

Figure 2: Deviation $z(t)$

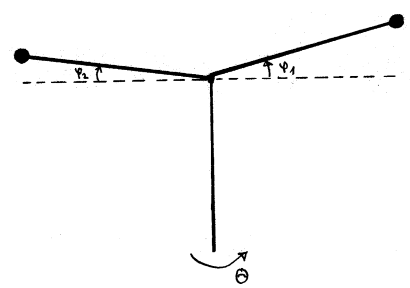

Figure 3: Elastic propeller

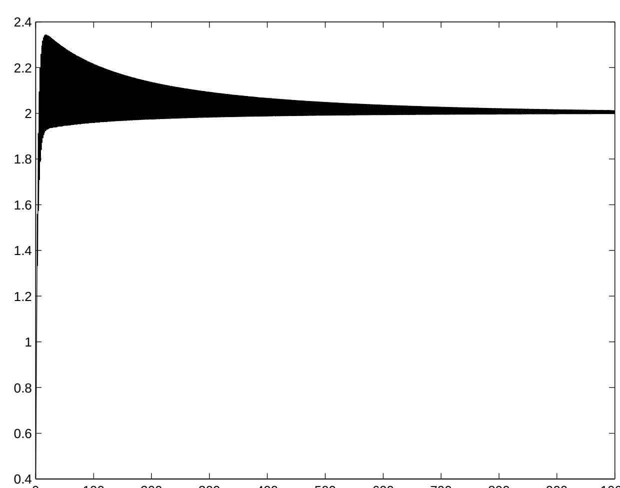

Figure 4: Angular velocity $\dot{\theta}(t)$

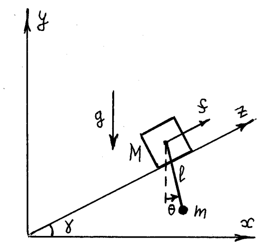

Figure 5: Cart and pendulum

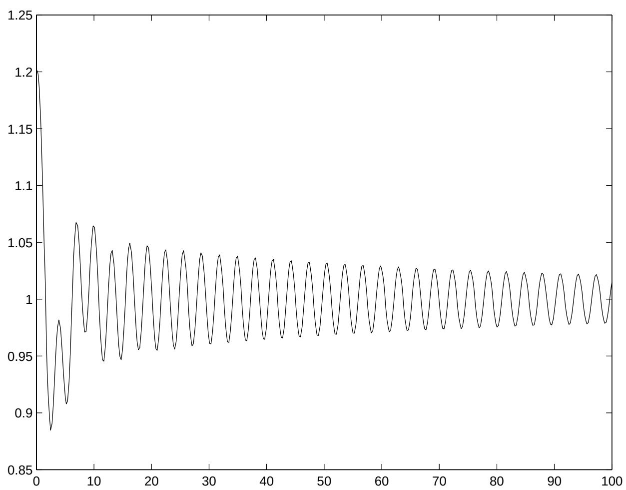

Figure 6: Velocity $v(t)$

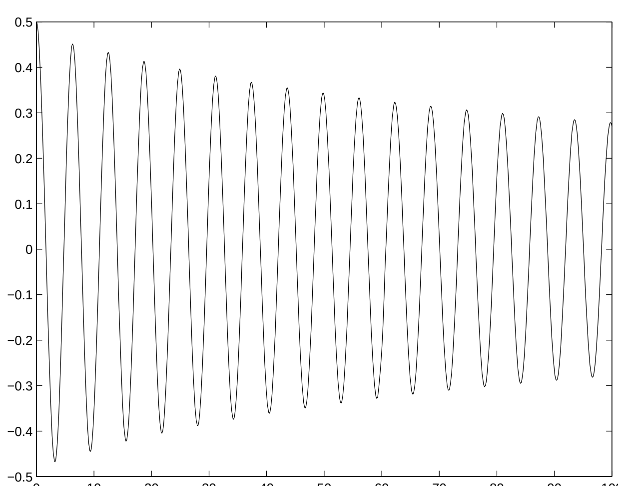

Figure 7: Angle $\theta(t)$

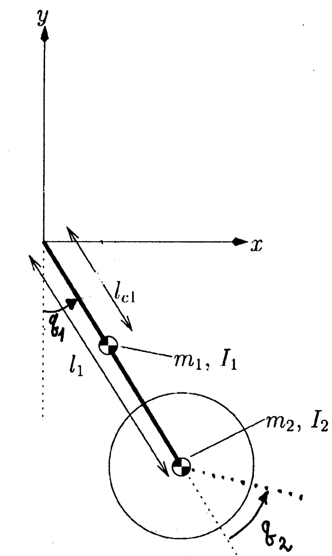

Figure 8: Inertia wheel pendulum

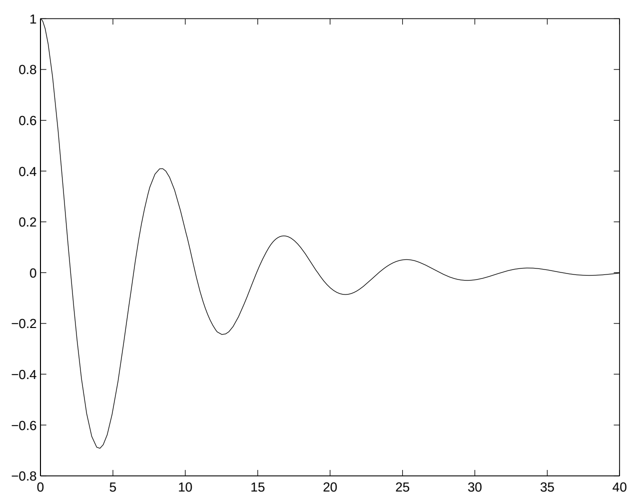

Figure 9: Angle $\theta(t)$

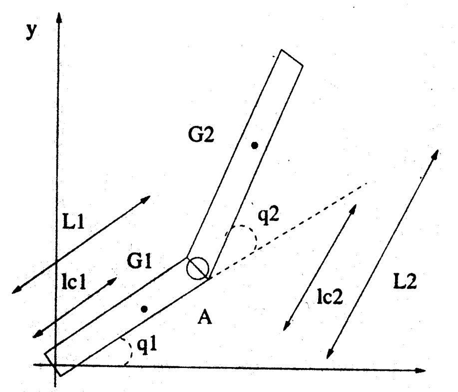

Figure 10: Pendubot