CYBERNETICS AND PHYSICS, VOL. 3, NO. 4, 2014 , 166–173

AUTORESONANCE IN A PAIR OF COUPLED OSCILLATORS

Agnessa Kovaleva
Space Research Institute
Russian Academy of Sciences
Russia
agnessa_kovaleva@hotmail.com

Abstract
We investigate passage through resonance in a two-degree of freedom system consisting of a linear oscillator weakly coupled to a nonlinear forced actuator. Two classes of problems are studied analytically and numerically: (1) a periodic force with constant frequency is applied to the nonlinear actuator (the Duffing oscillator) with slowly time-decreasing linear stiffness; (2) the time-invariant nonlinear oscillator is excited by a force with slowly increasing frequency. In both cases, the attached linear oscillator and linear coupling remain time-invariant, and the system is initially engaged in resonance. This paper demonstrates that in the systems of the first type autoresonance (AR) occurs in both oscillators. In the system of the second type AR occurs only in the excited nonlinear oscillator but the coupled linear oscillator exhibits small bounded oscillations. Assuming a slow change of detuning rate, we obtain explicit asymptotic approximations for the amplitudes and the phases of oscillations close to exact (numerical) results.

Key words
Nonlinear oscillations, asymptotic methods, autoresonance.

1 Introduction

Resonance energy transfer from a source of energy to a receiver represents one of the most effective methods of excitation and control of oscillations for a broad range of natural, physical and engineering systems. Theoretical approaches and applications of this effect in physical and engineering systems have been widely discussed, see, e.g., [May and Kühn, 2011; Vakakis, et.al., 2008, Vazquez, MacKay, and Zorzano, 2003], etc.

High-energy resonance motion can be achieved, e.g., with the help of feedback intended to sustain "a resonance under action of the force produced by the system's itself" [Andronov, Vitt,and Khaikin, 1966]. The theory of feedback resonance has been developed in a series of works, e.g., [Andrievsky and Fradkov, 1999; Fradkov, 1999; Kovaleva, 1999]. Feedback control building on this idea and using self-sustained oscillations with predefined energy as a working process has been employed in a number of engineering systems, see, e.g., [Astashev and Babitsky, 2007]. Control schemes in these systems have included electronic and electromechanical positive feedback and a synchronous type actuator for self-excitation of resonant vibration in combination with negative feedback for its stabilization.

Note that feedback does not need an additional source of energy. However, its practical realization requires careful diagnostics of nonlinear states and may become extremely complicated and costly in a multi-degree of freedom system. A large class of systems can avoid feedback, still producing the required state with the help of a properly controlled resonant excitation. Resonance control employs an intrinsic property of a nonlinear oscillator to change both its amplitude and natural frequency when the driving frequency changes. This means that the oscillator may be captured into resonance with its drive if the driving frequency varies slowly in time to be consistent with the frequency of the oscillator. The ability of a nonlinear oscillator to stay captured into resonance due to variance of its structural or excitation parameters is known as autoresonance (AR).

Autoresonance was first used in applications to particle acceleration and reported as "the phase stability principle" [Veksler, 1944; McMillan, 1945]. Building on that works, a large number of theoretical studies, experimental results and applications of AR in different fields of natural science have been reported in literature, see, e.g., [Blekhman, 2012; Chapman, 2011; Charman, 2007; Friedland, 2014]. The analysis was first concentrated on AR in the basic single-degree of freedom model but then the developed methods and approaches were extended to two- and three-degree of freedom systems. Examples in this category are interactions of the plasma waves with laser beams [Charman, 2007; Barth and Friedland, 2007; Chapman, et al., 2010], particle transport in a weak external field with slowly varying frequency [Galow, 2013; Zelenyi, 2013], control of diatomic molecules [Marcus, Friedland, and Zigler, 2005], etc.

Some particular results (e.g., [Barth and Friedland, 2007]) suggest that external forcing with a slowly varying frequency applied to a pair of coupled nonlinear oscillators generates AR in both oscillators. However, this conclusion cannot be applied universally, because the dynamics of an oscillator in a coupled system can drastically differ from the dynamics of a single oscillator. We illustrate this effect by considering a mechanical model consisting of a linear oscillator weakly coupled to a nonlinear actuator (the Duffing oscillator). Two types of autoresonant problems are studied: (1) a periodic force with constant frequency is applied to the Duffing oscillator with slowly time-decreasing stiffness; (2) a time-invariant nonlinear actuator is excited by a force with slowly increasing frequency. In both cases the system the attached linear oscillator and linear coupling remain time-invariant, and the system is initially engaged in resonance. It is obvious that oscillations with growing energy in the linear attachment may arise only in the presence of AR in the nonlinear actuator. The purpose of this paper is to find the conditions under which AR in the nonlinear actuator brings about growing oscillations in the linear attachment. We demonstrate that periodic forcing with constant (resonant) frequency may cause AR in both oscillators but the drive with the slowly-varying frequency gives rise to AR only in the excited nonlinear oscillator while the attached oscillator exhibits small bounded oscillations.

The paper is organized as follows. In Sec. 2, the system of the first type is considered. Given small detuning rate, we derive approximate solutions describing growing oscillations in both oscillators. In Sec. 3 we show that AR in the time-invariant nonlinear actuator is unable to sustain oscillations with increasing amplitudes in the attachment but an additional slow change of the actuator parameters may entail growing oscillations of the coupled oscillator. Escape from resonance is investigated in Sec. 4. Section 5 contains a brief summary and conclusions.

It is important to note that leading-order asymptotic equations derived in Sec.2 and Sec. 3 are similar to inhomogeneous Schrödinger equations described a wide variety of physical models. This similarity suggests a potential extension of the results obtained for a mechanical model to the study of energy transfer in systems of different physical nature.

2 Autoresonance in a System with a Constant Forcing Frequency

The model studied in this section consists of a time-invariant linear oscillator weakly coupled to a time-dependent nonlinear actuator (the Duffing oscillator) subjected to a periodic excitation with constant frequency. Our purpose is to demonstrate that AR occurring in the nonlinear oscillator entails oscillations with gradually increasing amplitude in the linear attachment.

The equations of motion are given by

$$m_1\frac{d^2u_1}{dt^2} + c_1u_1 + c_{10}(u_1 - u_0) = 0, \quad (1)$$
$$m_0\frac{d^2u_0}{dt^2} + C(t)u_0 + ku_0^3 + c_{10}(u_0 - u_1) = A\cos\omega t,$$

where u_0 and u_1 denote absolute displacements of the nonlinear and linear oscillators, respectively; m_0 and m_1 are their masses; c_1 and k are the coefficients of linear stiffness and cubic nonlinearity; c_{10} is the linear coupling coefficient; $C(t) = c_0 - (k_1 + k_2t), k_{1,2} > 0$; A and ω denote the amplitude and the frequency of the periodic force. The system is initially at rest, that is, $u_r = 0, v_r = du_r/dt = 0$ at $t = 0; r = 0, 1$.

We introduce the small parameter of the system $2\varepsilon = c_{10}/c_1 << 1$, which represents a dimensionless coefficient of weak coupling. Considering weak nonlinearity and taking into account resonance properties of the system, we redefine the parameters as follows:

$$\tau_0 = \omega t, \tau_1 = \varepsilon\tau_0, A = \varepsilon m\omega^2 F, c_r/m_r = \omega^2, \quad (2)$$
$$k_1/c_0 = 2\varepsilon s, k_2/c_0 = 2\varepsilon^2 b\omega, k/c_0 = 8\varepsilon\alpha, c_{10}/c_r = 2\varepsilon\lambda_r,$$
$$\zeta(\tau_1) = s + b\tau_1, \quad (1)$$

and then rewrite (1) as:

$$\frac{d^2u_1}{d\tau_0^2} + u_1 + 2\varepsilon\lambda_1(u_1 - u_0) = 0, \quad (3)$$
$$\frac{d^2u_0}{d\tau_0^2} + (1 - 2\zeta(\tau_1))u_0 + 8\varepsilon\alpha u_0^3 + 2\varepsilon\lambda_0(u_0 - u_1) = 2\varepsilon F\cos\tau_0.$$

In the next step, we introduce the new variables Y_r by formulas

$$Y_r = (v_r + iu_r)e^{-i\tau_0}, i = \sqrt{-1}; r = 0, 1. \quad (4)$$

It follows from (3), (4) that energy E_r of each of the oscillators (3) can be asymptotically evaluated as $E_r = \frac{1}{2}|Y_r|^2 + \varepsilon \dots (r = 0, 1)$.

Inserting (4) into (3), we obtain the following firstorder equations for the complex-valued amplitudes Y_r :

$$\frac{dY_1}{d\tau_0} = i\varepsilon[\lambda_1(Y_1 - Y_0) + G_1], Y_1(0) = 0, \quad (5)$$
$$\frac{dY_0}{d\tau_0} = -i\varepsilon\{[\zeta(\tau_1) - 3\alpha|Y_0|^2]Y_0 - \lambda_0(Y_0 - Y_1) + F + G_0\}, Y_0(0) = 0$$

and similar equations for the complex conjugate variablesY_0^*, Y_1^*. The terms G_0, G_1 involve fast (in τ_0) harmonics with coefficients depending on Y_r and Y_r^* $(r = 0, 1)$, but explicit expressions of G_0, G_1 are insignificant for further analysis.

Explicit analytical approximations for the complex amplitudes Y_r are constructed in the form of the multiple scales expansions with the slow main terms:

$$Y_r(\tau_0, \tau_1, \varepsilon) = \varphi_r(\tau_1) + \varepsilon\varphi_r^{(1)}(\tau_1, \tau_1) + \varepsilon^2 ... \quad (6)$$

Next, we introduce the new independent time-scale $\tau = s\tau_1$ and perform the following rescaling of the variables and the parameters:

$$\Lambda = (s/3\alpha)^{1/2}, f = F/s\Lambda, \beta = b/s^2, \quad (7)$$
$$\zeta_0(\tau) = 1 + \beta\tau, \psi_r = \varphi_r/\Lambda, \mu_r = \lambda_r/s.$$

Inserting (6), (7) into (5) and applying the multiple scales formalism [Nayfeh and Mook, 2004], we obtain the dimensionless equations for the slow variables ψ_r:

$$\frac{d\psi_1}{d\tau} - i\mu_1(\psi_1 - \psi_0) = 0, \psi_1(0) = 0, \quad (8)$$
$$\frac{d\psi_0}{d\tau} - i\mu_0(\psi_0 - \psi_1) + i(\zeta_0(\tau) - |\psi_0|^2)\psi_0 = -if,$$
$$\psi_0(0) = 0.$$

The real-valued amplitudes $a_r > 0$ and phases Δ_r of oscillations are calculated as

$$a_r = |\psi_r|, \Delta_r = \arg(\psi_r). \quad (9)$$

Details of the derivation of asymptotic solutions for similar systems can be found in [Kovaleva and Manevitch, 2012]. It is important to note that equations (8) are identical to inhomogeneous Schrödinger equations. This suggests that, in analogy with nonlinear tunneling [Manevitch and Kovaleva, 2013], the results concerning AR in a mechanical chain can be extended to a wide variety of systems of different physical nature. Note that AR in a one-dimensional Schrödinger equation was investigated earlier [Friedland, 1998].

We show that in some special cases the equation of the excited oscillator can be solved independently. Let us consider an asymmetric system in which $m_1 = \varepsilon\delta m_0, \delta = O(1)$. In this case, $\mu_0 = \mu_1 c_1/c_0 = \mu_1 m_1/m_0 = \varepsilon\delta\mu_1$, and thus the term proportional to μ_0 may be removed from (8) in the considered approximation. The resulting truncated system is given by

$$\frac{d\psi_1^{(0)}}{d\tau} - i\mu_1(\psi_1^{(0)} - \psi_0^{(0)}) = 0, \psi_1^{(0)}(0) = 0, \quad (10)$$
$$\frac{d\psi_0^{(0)}}{d\tau} + i(\zeta_0(\tau) - |\psi_0^{(0)}|^2)\psi_0^{(0)} = -if, \psi_0^{(0)}(0) = 0.$$

The nonlinear equation in (10) can be investigated separately. Therefore, if approximations $\psi_r^{(0)}$ are close to exact solutions ψ_r, then the condition of the occurrence of AR in a single Duffing oscillator derived in [Kovaleva and Manevitch, 2013a, b] can be extended to the weakly coupled system (8). The effect of weak coupling may be taken into account in subsequent iterations (see, e.g., [Kovaleva and Manevitch, 2012]). It is important to note that the assumption $\mu_0/\mu_1 << 1$ is used to simplify the analysis but the qualitative features of the dynamical behavior hold true for a wide range of parameters such that $\mu_0 < 1$ and $\mu_1 < 1$.

Now we recall earlier obtained results [Kovaleva and Manevitch, 2013a, b] needed for our analysis. It was shown that AR in the Duffing oscillator may occur at $f > f_1 = \sqrt{2/27}$, while the values $f < f_1$ corresponds to bounded oscillations at any rate β. In the domain $f > f_1$ the Duffing oscillator admits AR at $\beta < \beta^*$ and bounded oscillations at $\beta > \beta^*$. The critical rate β^* is defined as $\beta^* = [(f/f_1)^{2/3} - 1]/T^*$, where $\tau = T^*$ corresponds to the first minimum of the phase $\Delta_0(\tau)$ in the time-independent Duffing oscillator with $\beta = 0$. The values of T^* and β^* were found both analytically and numerically in [Kovaleva and Manevitch, 2013b].

In analogy with a single oscillator, the solution $\psi_0(\tau) = \bar{\psi}_0(\tau) + \tilde{\psi}_0(\tau)$ represents small fast fluctuations $\psi_0(\tau)$ near a quasi-steady state $\bar{\psi}_0(\tau)$ calculated as a stationary point of (8) with the "frozen" parameter ζ_0. Assuming $\mu_0 = O(\varepsilon)$, we obtain the following equation for $\bar{\psi}_0$:

$$(\zeta_0 - |\bar{\psi}_0|^2)\bar{\psi}_0 = -f. \quad (11)$$

The quasi-stationary value of the amplitude $a(\tau)$ is defined as $\bar{a}_0 = |\bar{\psi}_0|$; it can be interpreted as the backbone curve, which expresses a relationship between the amplitude and the frequency of free oscillations. If $|f/2\zeta_0| << 1$, then the quasi-steady state $\bar{\psi}_0$ and the corresponding amplitude $\bar{a}_0$ can be approximated as

$$\bar{\psi}_0 \approx \pm\sqrt{\zeta_0}, \bar{a}_0 \approx \sqrt{\zeta_0} \to \sqrt{\beta\tau}, \text{as } \tau \to \infty, \quad (12)$$

with the phases of oscillations $\Delta = 0$ or $\Delta = \pi$.Asymptotic approximations for fast fluctuations can be computed by linearizing nonlinear equation in (9) near $\bar{\psi}_0$ (see [Manevitch, Kovaleva, and Shepelev, 2011]).

If the solution $\psi_0(\tau)$ is known, then the response $\psi_1(\tau)$ is calculated from (8) by formula

$$\psi_1 = -i\mu_1 \int_0^\tau e^{i\mu_1(\tau-s)}\psi_0(s)ds. \tag{13}$$

Since the effect of small fast fluctuations $\tilde{\psi}_0$ on the value of integral (13) is negligibly small compared to the contribution of the slowly-varying function $\bar{\psi}_0$, the following approximation is valid:

$$\psi_1(\tau) \approx -i\mu_1 e^{i\mu_1\tau}J(\tau), \tag{14}$$
$$J(\tau) = \int_0^\tau e^{-i\mu_1 s}\bar{\psi}_0(s)ds,$$

where $\bar{\psi}_0 = \sqrt{1+\beta\tau}$.Integration by parts gives

$$J(\tau) = -i\mu_1^{-1}[e^{i\mu_1\tau}\bar{\psi}_0(\tau) - \bar{\psi}_0(0)] - \Phi(\tau),$$
$$\Phi(\tau) = \frac{\beta}{2}\int_0^\tau \frac{e^{-i\mu_1 s}}{\sqrt{1+\beta s}}ds.$$

It is easy to deduce that $\Phi(\tau)$ is a Fresnel-type integral bounded at any $\tau > 0$. Hence, $\psi_1(\tau) = \bar{\psi}_1(\tau) + \tilde{\psi}_1(\tau) + O(\sqrt{\beta})$, where

$$\bar{\psi}_1(\tau) = \bar{\psi}_0(\tau), \tilde{\psi}_1(\tau) \approx -\bar{\psi}_0(\tau)e^{i\mu_1\tau}. \tag{15}$$

It follows from (15) that $a_1 = |\bar{\psi}_1| = a_0$. Note that the equalities $\bar{\psi}_1(\tau) = \bar{\psi}_0(\tau)$, $a_1 = a_0$ can be directly obtained from (8) but the performed transformations provides the formal demonstration of the occurrence of growing oscillations in the linear attachment.

Theoretical results are illustrated in Fig. 1. The following parameters are used for numerical computations:

$$\beta = 0.05, \mu_0 = 0.02, \mu_1 = 0.25, f = 0.34. \tag{16}$$

We recall that a single oscillator with parameters $\mu_0 = 0, f = 0.34, \beta = 0.05 < \beta^* = 0.06$, admits AR [Kovaleva and Manevitch, 2013b].

Figures $1(a)$ and $1(b)$ show that exact amplitudes a_r $(r = 0, 1)$ calculated by formulas (8),(9) (solid lines) and their approximations $a_r^{(0)}$ obtained from approximate (dotted lines) amplitudes are close to each other. This implies that the conditions (10) of the occurrence of AR in a single nonlinear oscillator can be extended to the weakly coupled system (8). As seen in Fig.$1(a)$, fast fluctuations become negligibly small compared to the quasi-stationary amplitude $\bar{a}_0$. This implies that asymptotic approximations $\bar{\psi}_0(\tau)$ can be calculated from a linearized equation. Details are omitted for brevity.

Figure $1(b)$ demonstrates initially irregular oscillations of the linear attachment but at later times forcing with increasing amplitude dominates and motion is transformed into regular oscillations near the backbone curve $\bar{a}_1(\tau)$. The amplitude and the period of fast fluctuations near $\bar{a}_1(\tau)$ are calculated as $\tilde{a}_1(\tau) = |\tilde{\psi}_1(\tau) = |\bar{\psi}_0(0)| = 1$, $T_1 = 2\pi/\mu_1 \approx 25.12$. These values are close to the corresponding parameters in Fig. $1(b)$. Figures $1(c)$ and $1(d)$ illustrate phase locking typical for AR oscillations.

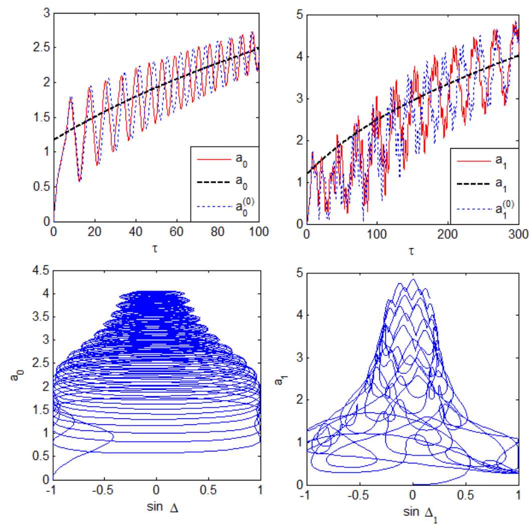

Figure 1. Amplitudes and phases of AR

As noted above, the required state can be maintained by terminating the change of the parameter $\zeta_0(\tau)$ at a prescribed energy level. Since the amplitude (and energy) of oscillations can be represented as small fast fluctuations near the monotonically increasing backbone curve, it is convenient to define the terminal time T^* as an instant at which energy of the attached oscillator achieves a required value E_1^*. It follows from (9), (12) that T^* can be evaluated as $T^* \approx (2E_1^* - 1)/\beta$. As an illustrative example, we consider a system with parameters (16) and $E_1^* = 5.5$; in this case, $a_1(T^*) \approx 3.3, T^* \approx 200$. It is seen in Fig. 2 that at $\tau > T^*$ AR turns into oscillations with almost constant amplitudes, and the theoretical value $a_1(T^*) \approx 3.3$ is close to the result presented in Fig. 2.

3 Energy Localization and Transfer in a System with a Slowly-Changing Forcing Frequency

We now investigate energy transfer in a system of the second type. First, we consider the time-invariant sys-

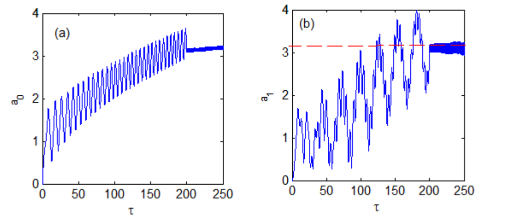

Figure 2. Transitions from AR to oscillations with prescribed terminal energy in the nonlinear (a) and linear (b) oscillators

tem with a slowly-changing forcing frequency. The dimensionless equations of motion are reduced to the form similar to (3):

$$\frac{d^2u_1}{d\tau_0^2} + u_1 + 2\varepsilon\lambda_1(u_1 - u_0) = 0, \tag{17}$$
$$\frac{d^2u_0}{d\tau_0^2} + (1 - 2\zeta(\tau_1))u_0 + 8\varepsilon\alpha u_0^3 + 2\varepsilon\lambda_0(u_0 - u_1) = 2\varepsilon F\cos\tau_0,$$

where $\frac{d\theta}{d\tau_1} = \zeta(\tau_1)$, $\zeta(\tau_1) = s + b\tau_1$, all other coefficients are defined by formulas (2). As in Sec. 2, the system is assumed to be initially at rest, and $\theta(0) = 0$. Transformations (4)–(7), applied together with the change of variables

$$\tau = s\tau_1, \zeta_0(\tau) = 1 + \beta\tau,$$
$$d\theta_0/d\tau = \zeta_0(\tau), \psi_r = \phi_r(\exp(i\theta_0), r = 0, 1.$$

yield the following dimensionless equations for the slow complex amplitudes $\psi_r(\tau)$:

$$\frac{d\phi_1}{d\tau} - i\mu_1(\phi_1 - \phi_0) + i\zeta_0(\tau)\phi_1 = 0, \phi_1(0) = 0, \tag{18}$$
$$\frac{d\phi_0}{d\tau} - i\mu_0(\phi_0 - \phi_1) + i(\zeta_0(\tau) - |\phi_0|^2)\phi_0 = -if,$$
$$\phi_0(0) = 0.$$

Note that system (18) has the constant right-hand side but the time-dependent coefficient $\zeta_0(\tau)$ is now involved in both equations. Similarly to (9), the real-valued amplitudes and phases of oscillations are defined as follows:

$$a_r = |\phi_r| > 0, \Delta_r = \arg\phi_r, r = 0.1. \tag{19}$$

As in Sec. 2, the response $\bar{\phi}_0(\tau)$ of the nonlinear oscillator is presented as $\phi_0(\tau) = \bar{\phi}_0(\tau) + \tilde{\phi}_0(\tau)$, where $\bar{\phi}_0(\tau)$ and $\tilde{\phi}_0(\tau)$ denote the quasi-steady state of system (18) and small fast fluctuations near this state, respectively. Assuming $\mu_0 << \mu_1$, we find that the state $\bar{\phi}_0$ satisfies the equations similar to (11) and (12). Fast fluctuations $\tilde{\phi}_0(\tau)$ can be calculated by linearizing Eqs. (18) and disregarding the terms proportional to μ_0. After calculating the nonlinear response $\phi_0(\tau)$, the response of the linear attachment $\phi_1(\tau)$ can be directly found from (18). Ignoring the effect of small fast fluctuations, we obtain after simple transformations that the linear response is expressed as:

$$\phi_1(\tau) = -i\frac{\mu_1}{2\beta}K(\tau)e^{-iS(\tau)/2\beta}, \tag{20}$$
$$K(\tau) = K_0(\tau) - K_0(1),$$
$$K_0(\tau) = \int_0^{S(\tau)} \frac{e^{iz/2\beta}}{z^{1/4}}dz, \; S(\tau) = (1 + \beta\tau)^2.$$

Although the expression for $K_0(\tau)$ cannot be analytically found, the limiting value $K_0(\infty)$ can be explicitly evaluated, and equals

$$K_0(\infty) = (2\beta)^{4/3}\Gamma(3/4)\exp(3i\pi/8),$$

where Γ is the gamma function [Gradshteyn and Ryzhik, 2000]. Hence,

$$a_1(\tau) = |\phi_1(\tau)| \to \mu_1(2\beta)^{4/3}\Gamma(3/4), \tau \to \infty. \tag{21}$$

Formula (21) indicates that AR in the nonlinear actuator is unable to generate oscillations with growing energy in the attached oscillator but it suffices to produce linear oscillations with bounded amplitude. The substitution of parameters (16) into (21) defines the limiting amplitude $a_{1\infty} = \lim_{\tau\to\infty} a(\tau) \approx 0.1$.

Figure 3 proves that the amplitude of nonlinear oscillations (Fig. 3(a)) is very close to its analogue in Fig. 1(a) but the amplitude of oscillations for the linear attachment (Fig. 3(b)) differs from that one in Fig. 1(b). The shape of the amplitude $a_1(\tau)$ is similar to the resonance curve with a noticeable resonance peak at an initial stage of motion, where the effect of time-dependent detuning is negligible, but then it turns into small oscillations with the limiting amplitude close to $a_{1\infty} \approx 0.1$.

A key conclusion from the obtained results is that in the system with constant excitation frequency the portion of energy transferred from the nonlinear actuator is insufficient to sustain oscillations with growing energy in the attached linear oscillator. The different dynamical behavior can be interpreted as a consequence of different resonance properties of the systems. In the system with a constant forcing frequency both oscillators are captured into resonance. If the forcing frequency slowly increases but the parameters of the system remain constant, AR in the nonlinear oscillator is still sustained by increasing amplitude, while the frequency of the linear oscillator falls into the domain beyond the resonance.

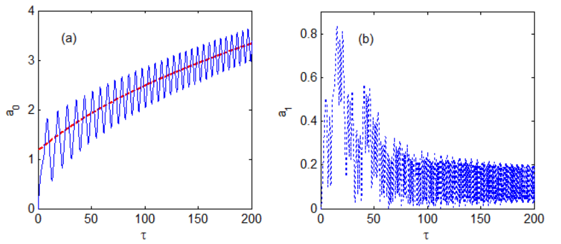

Figure 3. Amplitudes of oscillations of the nonlinear (a) and linear (b) oscillators of system (17)

It is important to note that the occurrence of decaying oscillations is by no means trivial, as the linear oscillator is actually driven by the coupling response with permanently increasing amplitude. As a counterexample, we consider a system with a slowly changing linear stiffness of the actuator. The system dynamics is described by the following equations:

$$\frac{d^2u_1}{d\tau_0^2} + u_1 + 2\varepsilon\lambda_1(u_1 - u_0) = 0, \quad (22)$$
$$\frac{d^2u_0}{d\tau_0^2} + (1 - 2\xi(\tau_1))u_0 + 2\varepsilon\lambda_0(u_0 - u_1) + 8\varepsilon\alpha u_0^3 = 2\varepsilon F\cos(\tau_0 + \theta(\tau_1)),$$

where $\frac{d\theta}{d\tau_1} = \zeta(\tau_1), \zeta(\tau_1) = s + b\tau_1, \xi(\tau_1) = b_3\tau_1^3$;all other coefficients are defined in (2). Our purpose is to show that slow changes in both the natural frequency and the forcing frequency of the actuator may sustain growing oscillations in the coupled linear oscillator.

As in the previous examples, transformations (4)–(7) are used to derive the following equations for the dimensionless complex amplitudes ψ_r :

$$\frac{d\psi_1}{d\tau} - i\mu_1(\psi_1 - \psi_0) = 0, \psi_1(0) = 0, \quad (23)$$
$$\frac{d\psi_0}{d\tau} - i\mu_0(\psi_0 - \psi_1) + i(\xi_1(\tau) - |\psi_0|^2)\psi_0 = -ife^{i\theta_0(\tau)},$$
$$\psi_0(0) = 0,$$

where $\tau = s\tau_1$, and

$$\frac{d\theta_0}{d\tau} = \zeta_1(\tau), \zeta_1(\tau) = 1 + \beta_1\tau, \xi_1(\tau) = \beta_3\tau^3, \quad (24)$$
$$\beta_1 = b/s^2, \beta_3 = b_3/s^4.$$

Finally, the change of variables $\psi_r = \phi_r \exp(i\theta_0), r = 0, 1,$ transforms (23) into the following system with a constant right-hand side and time-dependent coefficients:

$$\frac{d\phi_1}{d\tau} - i\mu_1(\phi_1 - \phi_0) + i\zeta_1(\tau)\phi_1 = 0, \phi_1(0) = 0, \quad (25)$$
$$\frac{d\phi_0}{d\tau} - i\mu_0(\phi_0 - \phi_1) + i(\zeta_0(\tau) - |\phi_0|^2)\phi_0 = -if,$$
$$\phi_0(0) = 0,$$

where $\zeta_0(\tau) = \zeta_1(\tau) + \xi_1(\tau)$.It follows from (12), (24), (25) that at large times the quasi-steady states $\bar{\phi}_r(\tau)$ and the correspoding backbone curves $\bar{a}_r$ can be evaluated as:

$$\bar{\phi}_0 \approx \zeta_0^{1/2}(\tau)\tilde{}O(\tau^{3/2}), \bar{a}_0 = |\bar{\phi}_0|, \quad (26)$$
$$\bar{\phi}_1 \approx \frac{\mu_1\zeta_0^{1/2}(\tau)}{\zeta_1(\tau) - \mu_1}\tilde{}O(\tau^{3/2}), \bar{a}_1 = |\bar{\phi}_1|.$$

Expressions (26) imply the simultaneous (but not equal) growth of backbone curves with time, thereby confirming the growth of energy of both oscillators. We illustrate this conclusion by numerical results for the system with the following parameter values:

$$\beta_1 = 10^{-3}, \beta_3 = 10^{-5}, \mu_0 = 0.01, \mu_1 = 0.15, f = 0.34. \quad (27)$$

Figure 4 compares the amplitudes of oscillations in the systems with and without additional time-dependent stiffness of the actuator. The results presented in Fig. 4 demonstrate that an additional slow change of the actuator frequency may increase the nonlinear response, thereby enhancing energy transfer and making it sufficient to sustain growing oscillations of the linear attachment.

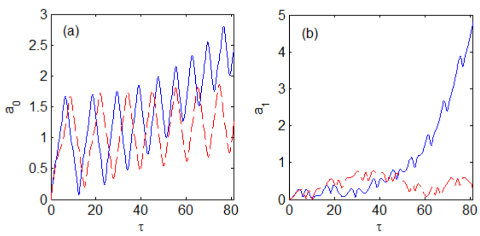

Figure 4. Amplitudes of oscillations of the actuator (a) and the linear attachment (b); solid lines corresponds to system (25) with parameters (27); dashed lines correspond to the time-independent actuator (β_3 = 0)

4 Escape from Resonance in the 2DOF System

In this section we briefly discuss the occurrence of bounded oscillations in the coupled system. It was demonstrated in earlier works [Kovaleva and Manevitch, 2013a, b] that the transition from AR to bounded oscillations in a single Duffing oscillator occurs at rate $\beta > \beta^*$. Figure 5 demonstrates a similar effect in system (3) with detuning rate $\beta = 0.065 > \beta^*$.

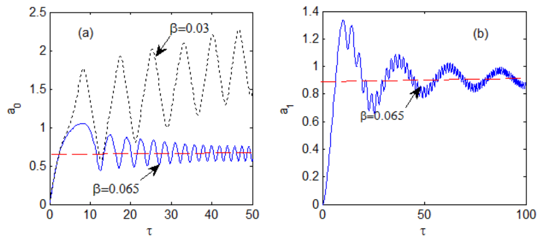

Figure 5. Amplitudes of bounded oscillations of the nonlinear (a) and linear (b) oscillators (solid lines); straight lines depict the limiting levels $\bar{a}_0$ and $\bar{a}_1$; dotted line corresponds to autoresonance at β = 0.03

It was shown [Kovaleva and Manevitch, 2013a, b] that the transition from AR to oscillations with relatively small amplitudes in a single Duffing oscillator is of the same nature as the transition from large to small oscillations in the system with constant parameters [Manevitch, Kovaleva, and Shepelev, 2011]. Figure 5 demonstrates a similar process in the coupled system. We underline that the limit values $\bar{a}_r = \lim_{\tau\to\infty} a_r(\tau)$, $r = 0, 1$ (straight lines in Fig. 5) cannot be interpreted as the quasi-steady states of system (8) at the "frozen" ζ_0.

The slow complex amplitudes of bounded oscillations can be approximately calculated with the help of the iteration procedure. In the first step, the initial iterations Ψ_r to the amplitudes ψ_r are computed as the solutions of the linear system

$$\frac{d\Psi_1}{d\tau} - i\mu_1(\Psi_1 - \Psi_0) = 0, \Psi_1(0) = 0, \qquad (28)$$
$$\frac{d\Psi_0}{d\tau} + i\zeta_0(\tau)\Psi_0 = -if, \Psi_0(0) = 0.$$

The solution $\Psi_0(\tau)$ is expressed through the Fresnel integral. The calculation of $\Psi_1(\tau)$ from the first equation (28) requires integration of a complicated combination of the Fresnel integrals and exponential functions. So, although explicit closed-form approximations are formally available, the functions $\Psi_r(\tau)$, as well as the amplitudes $a_r(\tau) = \mid \Psi_r(\tau)$ and the limit values $\bar{a}_r = \lim_{\tau\to\infty} a_r(\tau), r = 0, 1$, require numerical computation. Higher-order iterations at $n \geq 1$ can be found from the equations

$$\frac{d\Psi_1^{(n)}}{d\tau} - i\mu_1(\Psi_1^{(n)} - \Psi_0^{(n)}) = 0, \Psi_1^{(n)}(0) = 0, \quad (29)$$
$$\frac{d\Psi_0^{(n)}}{d\tau} + i\zeta_0(\tau)\Psi_0^{(n)} = -if + |\Psi_0^{(n-1)}|^2\Psi_0^{(n-1)} +$$
$$i\mu_0(\Psi_0^{(n-1)} - \Psi_1^{(n-1)}), \Psi_0^{(n)}(0) = 0.$$

5 Conclusions

It was shown in early works on particle acceleration that autoresonance (AR) could potentially serve as a tool for excitation and control of the high-energy regime in a single oscillator. However, the behavior of coupled oscillators may drastically differ from the dynamics of a single oscillator. In particular, the capture into resonance may not exist or AR in one part of the system may be insufficient to enhance the response of other oscillators.

This paper has illustrated this effect by an example of an oscillator system consisting of a linear time-invariant oscillator weakly coupled to a nonlinear actuator. Two types of excitation have been considered in details: (1) a periodic force with constant (resonance) frequency is applied to the nonlinear (Duffing) oscillator with slowly time-decreasing linear stiffness; (2) the time-invariant nonlinear oscillator is excited by a force with a slowly increasing frequency. In both cases, the system is initially engaged in resonance. It has been shown that in the system of the first type AR occurs in both oscillators but in the system of the second type energy transfer from the forced nonlinear actuator is insufficient to excite high-energy motion in the attached oscillator. This implies that energy transfer from the nonlinear oscillator may generate a high-energy regime in the linear oscillator only in the system of the first type, while in the system of the second type energy remains localized on the excited nonlinear actuator.

It has been shown that the different dynamical behavior arises due to different resonant properties of the systems under consideration. In the system with a constant excitation frequency both oscillators are captured into resonance: the nonlinear oscillator remains captured into resonance due to an increase of the amplitude compensating the change of stiffness, while the partial frequency of the linear oscillators is always close to the excitation frequency. However, if the forcing frequency slowly increases, AR in the nonlinear oscillator is still sustained by the growing amplitude, while the linear oscillator escapes from resonance. Also, it has been noted that escape from resonance does not immediately result in decreasing energy of the coupled linear oscillator, as this oscillator is actually driven by the growing coupling response. This implies that the response of the linear oscillator depends on the relationship between the growth of incoming energy and the loss of energy due to escape from resonance. This effect is illustrated by an example.

Note that a mechanical model is chosen for clarity. An analogy of the derived equations for slow complex amplitudes to inhomogeneous Schrödinger equations suggests that the results obtained in this work may be potentially applied to a wide variety of physical systems.

Acknowledgement
Support for this work received from the Russian Foundation for Basic Research through the RFBR grant 14-01-00284 is gratefully acknowledged. The author is grateful to L.I. Manevitch and A.I. Neishtadt for valuable discussion.

References

Andrievsky, B. and Fradkov, A. (1999) Feedback resonance in single and coupled 1-DOF oscillators. *Int. J. Bifurcat. Chaos*, 9(10), pp. 2047–2057.

Andronov, A.A., Vitt, A.A., and Khaikin, S.E. (1966) *Theory of Oscillators.* Pergamon Press, Oxford.

Astashev, V.K. and Babitsky, V.I. (2007) *Ultrasonic Processes and Machines.* Springer, Berlin.

Barth, I. and Friedland, L. (2007) Multiresonant control of two-dimensional dynamical systems. *Phys. Rev. E*, 76(1), pp. 016211-1–016211-9.

Blekhman, I. (2012) Oscillatory strobodynamics - a new area in nonlinear oscillations theory, nonlinear dynamics and cybernetical physics. *Cybernetics and Physics*, 1(1), pp. 5–10.

Chapman, T. (2011) *Autoresonance in Stimulated Raman Scattering.* École Polytechnique, Paris.

Chapman T., Hüller S., Masson-Laborde P. E., Rozmus W., and Pesme D. (2010) Spatially autoresonant stimulated Raman scattering in inhomogeneous plasmas in the kinetic regime. *Phys. Plasmas*, 17(12), pp. 122317-1–122317-8.

Charman, A.E. (2007) *Random Aspects of Beam Physics and Laser-Plasma Interactions.* University of California, Berkeley.

Fradkov, A. (1999) Exploring nonlinearity by feedback. *Physica D*, 128(2), pp.159–168.

Friedland, L. (1998) Autoresonant solutions of the nonlinear Schrödinger equation. *Phys. Rev. E*, 58(3), pp. 3865–3875.

Friedland, L. (2014) http://www.phys.huji.ac.il /˜lazar.

Galow, B.J., Li, J.X., Salamin, Y.I., Harman, Z., and Keitel, C.H. (2013) High-quality multi-GeV electron bunches via cyclotron autoresonance. *Phys. Rev. STAB*, 16(8) 081302-1 - 081302-6.

Gradshteyn, I. S. and Ryzhik, I. M. (2000) *Tables of Integrals, Series, and Products*, 6th Ed. Academic Press, San Diego, CA.

Kovaleva, A. (1999) *Optimal Control of Mechanical Oscillations*. Springer, Berlin New York.

Kovaleva, A. and Manevitch, L. (2012) Classical analog of quasilinear Landau-Zener tunneling. *Phys. Rev. E*, 85(1), pp. 016202-1–016202-8.

Kovaleva, A. and Manevitch, L.I. (2013a) Limiting phase trajectories and emergence of autoresonance in nonlinear oscillators. *Phys. Rev. E*, 88(2), pp.02490-1–02490-6.

Kovaleva, A. and Manevitch, L.I. (2013b) Emergence and stability of autoresonance in nonlinear oscillators. *Cybernetics and Physics*, 2(1), pp.25–30.

Manevitch, L.I, Kovaleva, A., and Shepelev, D. (2011) Non-smooth approximations of the limiting phase trajectories for the Duffing oscillator near 1:1 resonance. *Physica D*, 240 (1), pp. 1–12.

Manevitch, L.I. and Kovaleva, A. (2013) Nonlinear energy transfer in classical and quantum systems. *Phys. Rev. E*, 87(2), pp. 022904-1–022904-12.

Marcus, G., Friedland, L., and Zigler, A. (2005) Autoresonant excitation and control of molecular degrees of freedom in 3D. *Phys. Rev. A*, 72(3), pp. 033404-1–033404-9.

May, V. and Kühn, O. (2011) *Charge and Energy Transfer Dynamics in Molecular Systems*. Wiley-VCH, Weinheim.

McMillan, E.M. (1945) The synchrotron—a proposed high energy particle accelerator. *Phys. Rev.* 68(5-6), pp. 143–144.

Nayfeh, A.H. and Mook, D.T. (2004) *Nonlinear Oscillations*. Wiley-VCH, Weinheim.

Vakakis, A.F., Gendelman, O., Bergman, L.A., McFarland, D.M., Kerschen, G., and Lee, Y.S. (2008) *Targeted Energy Transfer in Mechanical and Structural Systems*, Springer, Berlin New York.

Vazquez L., MacKay R., and Zorzano M.P. (2003) *Localization and Energy Transfer in Nonlinear Systems*. World Scientific, Singapore.

Veksler, V.I. (1944) Some new methods of acceleration of relativistic particles. *Comptes Rendus (Dokaldy) de l'Academie Sciences de l'URSS*, 43(8), pp. 329–331.

Yaakobi, O., and Friedland, L. (2010) Autoresonance of coupled nonlinear waves. *AIP Conference Proceedings*, 1320, pp. 97–103.

Zelenyi, L.M., Neishtadt, A.I., Artemyev, A.V., Vainchtein, D.L., and Malova, H.V. (2013) Quasiadiabatic dynamics of charged particles in a space plasma. *Phys. Usp.* 56(4), pp. 347–394.